Renormalization group in stochastic theory of developed turbulence 4

Low-dimensional fluctuations and improved ε expansion

Juha Honkonen

National Defence University, Helsinki, Finland

Outline

- Two-parameter expansion
- Improved ε expansion
- Two-loop results
  - Kolmogorov constant
  - Prandtl number
- Conclusion

Skewness factor in the inertial range

Large-scale pumping: $\varepsilon \to 2$, $m = \dfrac{1}{L} \to 0 \Rightarrow d_f(\mathbf{k}) \to \delta(\mathbf{k})$.

Skewness factor in the inertial range

Large-scale pumping: $\varepsilon \to 2$, $m = \dfrac{1}{L} \to 0 \Rightarrow d_f(\mathbf{k}) \to \delta(\mathbf{k})$.
Connect to experimental data through ($m = 0$)

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\ \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

Skewness factor in the inertial range

Large-scale pumping: $\varepsilon \to 2$, $m = \dfrac{1}{L} \to 0 \Rightarrow d_f(\mathbf{k}) \to \delta(\mathbf{k})$.
Connect to experimental data through ($m = 0$)

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\; \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

Experimental C_K at $\varepsilon \to 2$, but $C_K(\varepsilon)$ is ambiguous via order-of-magnitude estimate of Λ.

Skewness factor in the inertial range

Large-scale pumping: $\varepsilon \to 2$, $m = \frac{1}{L} \to 0 \Rightarrow d_f(\mathbf{k}) \to \delta(\mathbf{k})$.
Connect to experimental data through ($m = 0$)

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\; \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

Experimental C_K at $\varepsilon \to 2$, but $C_K(\varepsilon)$ is ambiguous via order-of-magnitude estimate of Λ.

Attempts to relate D_{10} and $\overline{\mathcal{E}}$ in the momentum-shell approach not flawless.

Skewness factor in the inertial range

Large-scale pumping: $\varepsilon \to 2$, $m = \frac{1}{L} \to 0 \Rightarrow d_f(\mathbf{k}) \to \delta(\mathbf{k})$.
Connect to experimental data through ($m = 0$)

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\; \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

Experimental C_K at $\varepsilon \to 2$, but $C_K(\varepsilon)$ is ambiguous via order-of-magnitude estimate of Λ.

Attempts to relate D_{10} and $\overline{\mathcal{E}}$ in the momentum-shell approach not flawless.

Use independent of D_{10} quantity - the skewness factor [Adzhemyan, Antonov, Kompaniets & Vasil'ev (2003)]:

$$\mathcal{S} = S_3/S_2^{3/2}\,.$$

Unambiguous Kolmogorov constant

For $\varepsilon \geq \frac{3}{2}$ the structure function $S_2(r) \sim$ const, replace in $\mathcal{S}$ by the function with powerlike asymptotics $r\partial_r S_2(r)$ and define:

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{[-S_3(r)]^{2/3}}.$$

Unambiguous Kolmogorov constant

For $\varepsilon \geq \frac{3}{2}$ the structure function $S_2(r) \sim$ const, replace in $\mathcal{S}$ by the function with powerlike asymptotics $r\partial_r S_2(r)$ and define:

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{[-S_3(r)]^{2/3}}.$$

Calculate Kolmogorov constant and skewness factor unambiguously as

$$C_K = \left[\frac{3Q(2)}{2}\right]\left[\frac{12}{d(d+2)}\right]^{2/3}, \quad \mathcal{S} = -\left[\frac{3Q(2)}{2}\right]^{-3/2}.$$

Effect of low-dimensional fluctuations

Two-loop corrections to C_K and $\mathcal{S}$ large: $\approx$ 100 % change for $d = 3$ but rapidly decreasing with growing d.

Drastic growth in the limit $d \to 2$ due to singular graphs.

Effect of low-dimensional fluctuations

Two-loop corrections to C_K and $\mathcal{S}$ large: $\approx$ 100 % change for $d = 3$ but rapidly decreasing with growing d.

Drastic growth in the limit $d \to 2$ due to singular graphs. Summing up singularities calls for additional renormalization near $d = 2$: to make it multiplicative, introduce ($m = 0$)

$$d_f(k) = D_{10} k^{4-d-2\varepsilon} + D_{20} k^2$$

(JH & Nalimov, 1996) with D_{20} to be renormalized.

Effect of low-dimensional fluctuations

Two-loop corrections to C_K and $\mathcal{S}$ large: $\approx$ 100 % change for $d = 3$ but rapidly decreasing with growing d.

Drastic growth in the limit $d \to 2$ due to singular graphs. Summing up singularities calls for additional renormalization near $d = 2$: to make it multiplicative, introduce ($m = 0$)

$$d_f(k) = D_{10} k^{4-d-2\varepsilon} + D_{20} k^2$$

(JH & Nalimov, 1996) with D_{20} to be renormalized.

Coarse-graining of finite band-width forcing always generates the local term (Forster, Nelson & Stephen, 1977).

2d vs. 3d turbulence

Why 2d fluctuations of importance for 3d? Fluctuations present in all d's, sum in low dimensions! Then extrapolate.

$2d$ vs. $3d$ turbulence

Why 2d fluctuations of importance for 3d? Fluctuations present in all d's, sum in low dimensions! Then extrapolate.

Different physics in $2d$ and 3d: is it legal to extrapolate?

$2d$ vs. $3d$ turbulence

Why $2d$ fluctuations of importance for $3d$? Fluctuations present in all d's, sum in low dimensions! Then extrapolate.

Different physics in $2d$ and $3d$: is it legal to extrapolate?

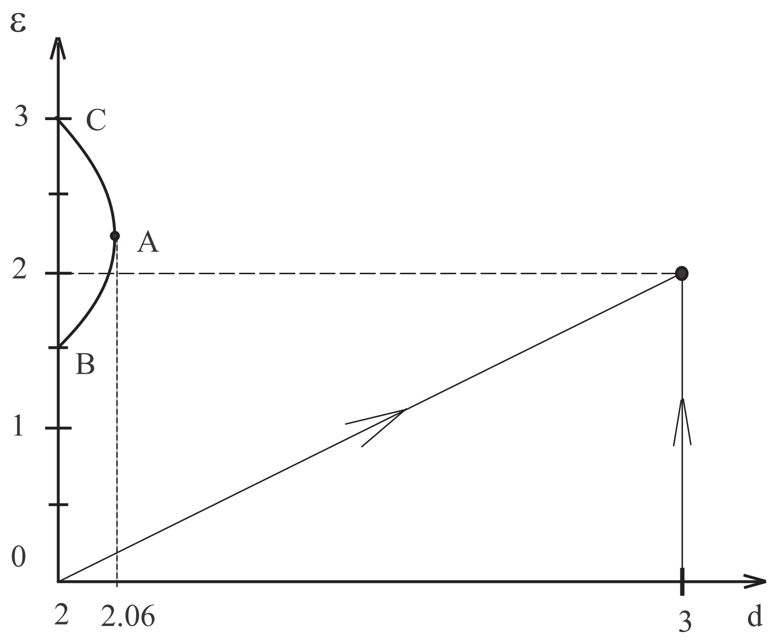
Borderline between direct and inverse cascades near the point (2,2) in the d, ε plane (Fournier & Frisch, 1977):

$2d$ vs. $3d$ turbulence

Why $2d$ fluctuations of importance for $3d$? Fluctuations present in all d's, sum in low dimensions! Then extrapolate.

Different physics in $2d$ and $3d$: is it legal to extrapolate?

Borderline between direct and inverse cascades near the point (2,2) in the d, ε plane (Fournier & Frisch, 1977):

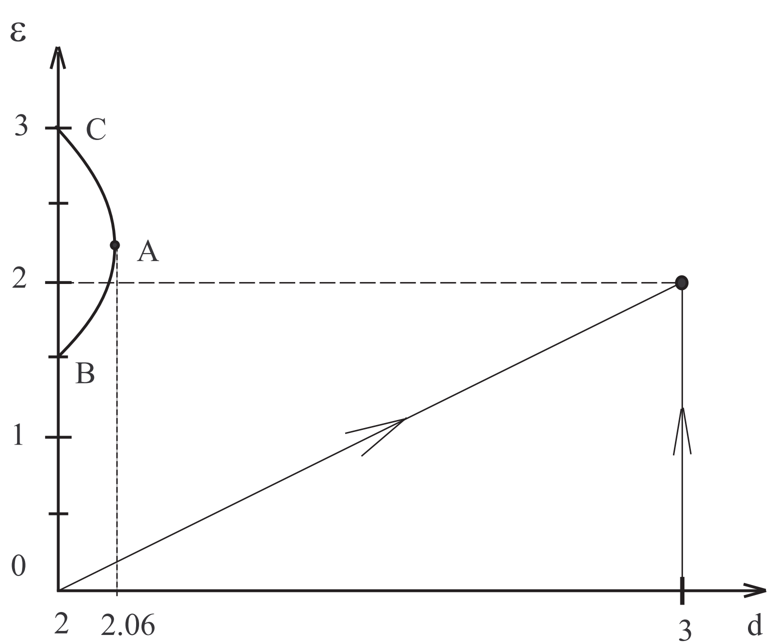


$2d$ vs. $3d$ turbulence

Why $2d$ fluctuations of importance for $3d$? Fluctuations present in all d's, sum in low dimensions! Then extrapolate.

Different physics in $2d$ and $3d$: is it legal to extrapolate?

Borderline between direct and inverse cascades near the point (2,2) in the d, ε plane (Fournier & Frisch, 1977):



Yes, inverse energy cascade far from the linear extrapolation path.

Two-parameter expansion

Additional UV-renormalization near $d=2$ required

$$S_{\mathrm{R}} = \frac{1}{2}\mathbf{v}'\left(D_1 k^{4-d-2\varepsilon} + D_2 Z_{D_2} k^2\right)\mathbf{v}' - \mathbf{v}'\left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right]$$

with $\nu_0 = \nu Z_\nu$ and

$$g_{01} = D_{10}\nu_0^{-3} = g_1 \mu^{2\varepsilon} Z_\nu^{-3}\,,\ g_{20} = D_{20}\nu_0^{-3} = g_2 \mu^{2-d} Z_{D_2} Z_\nu^{-3}\,.$$

Two-parameter expansion

Additional UV-renormalization near $d = 2$ required

$$S_{\mathrm{R}} = \frac{1}{2}\mathbf{v}'\left(D_1 k^{4-d-2\varepsilon} + D_2 Z_{D_2} k^2\right)\mathbf{v}' - \mathbf{v}'\left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right]$$

with $\nu_0 = \nu Z_\nu$ and

$$g_{01} = D_{10}\nu_0^{-3} = g_1 \mu^{2\varepsilon} Z_\nu^{-3}\,,\; g_{20} = D_{20}\nu_0^{-3} = g_2 \mu^{2-d} Z_{D_2} Z_\nu^{-3}\,.$$

The RG solution [$m = 0$, UV cutoff Λ imposed]

$$G(k, g_{10}, g_{20}, \nu_0, \Lambda) = (D_{10}/\bar{g}_1)^{2/3}\, k^{2-d-4\varepsilon/3} R_\Lambda\left(1, \bar{g}_1, \bar{g}_2, \Lambda/k\right)\,.$$

Two-parameter expansion

Additional UV-renormalization near $d=2$ required

$$S_{\rm R} = \frac{1}{2}\mathbf{v}'\left(D_1 k^{4-d-2\varepsilon} + D_2 Z_{D_2} k^2\right)\mathbf{v}' - \mathbf{v}'\left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right]$$

with $\nu_0 = \nu Z_\nu$ and

$$g_{01} = D_{10}\nu_0^{-3} = g_1 \mu^{2\varepsilon} Z_\nu^{-3}\,,\; g_{20} = D_{20}\nu_0^{-3} = g_2 \mu^{2-d} Z_{D_2} Z_\nu^{-3}\,.$$

The RG solution [$m=0$, UV cutoff Λ imposed]

$$G(k, g_{10}, g_{20}, \nu_0, \Lambda) = (D_{10}/\bar{g}_1)^{2/3}\, k^{2-d-4\varepsilon/3} R_\Lambda\left(1, \bar{g}_1, \bar{g}_2, \Lambda/k\right)\,.$$

Near $d=2$ $\exists$ IR-stable fixed point giving rise to double expansion in ε and $2\Delta = d-2$.

Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,

Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,
- stable nontrivial fixed point at $d > 2$: dimensional regularization insufficient for thermal fluctuations.

Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,
- stable nontrivial fixed point at $d > 2$: dimensional regularization insufficient for thermal fluctuations.

Renormalize on a ray with intermediate Λ renormalization (Adzhemyan, JH, Kompaniets, Vasil'ev 2005):

Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,
- stable nontrivial fixed point at $d > 2$: dimensional regularization insufficient for thermal fluctuations.

Renormalize on a ray with intermediate Λ renormalization (Adzhemyan, JH, Kompaniets, Vasil'ev 2005):

- fix the ratio $(d-2)/\varepsilon = 2\zeta$ to restore MS scheme,

Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,
- stable nontrivial fixed point at $d > 2$: dimensional regularization insufficient for thermal fluctuations.

Renormalize on a ray with intermediate Λ renormalization (Adzhemyan, JH, Kompaniets, Vasil'ev 2005):

- fix the ratio $(d-2)/\varepsilon = 2\zeta$ to restore MS scheme,
- introduce explicit cutoff Λ, renormalize out large Λ terms [replace primary (physical) bare parameters by secondary ones],

Minimal subtractions on rays

Two-parameter renormalization not entirely trivial; problems

- analytic renormalization: no MS scheme,
- stable nontrivial fixed point at $d > 2$: dimensional regularization insufficient for thermal fluctuations.

Renormalize on a ray with intermediate Λ renormalization (Adzhemyan, JH, Kompaniets, Vasil'ev 2005):

- fix the ratio $(d-2)/\varepsilon = 2\zeta$ to restore MS scheme,
- introduce explicit cutoff Λ, renormalize out large Λ terms [replace primary (physical) bare parameters by secondary ones],
- the remainder is analytic continuation from $d < 2$.

Two expansions for the skewness factor

Two complementary ways to calculate the universal ratio Q:

Two expansions for the skewness factor

Two complementary ways to calculate the universal ratio Q:

- In ε, Δ expansion on the ray $\zeta = (d-2)/2\varepsilon = \text{const}$:

$$Q(\varepsilon) = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}} = \varepsilon^{1/3} \sum_{k=0}^{\infty} \Psi_k\,(\zeta)\,\varepsilon^k\,.$$

Two expansions for the skewness factor

Two complementary ways to calculate the universal ratio Q:

- In ε, Δ expansion on the ray $\zeta = (d-2)/2\varepsilon = $ const:

$$Q(\varepsilon) = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}} = \varepsilon^{1/3} \sum_{k=0}^{\infty} \Psi_k\left(\zeta\right)\varepsilon^k\,.$$

- In ε expansion with coefficients singular, when $d \to 2$:

$$Q(\varepsilon) = \varepsilon^{1/3} \sum_{k=0}^{\infty} Q_k(d)\varepsilon^k\,.$$

Two expansions for the skewness factor

Two complementary ways to calculate the universal ratio Q:

- In ε, Δ expansion on the ray $\zeta = (d-2)/2\varepsilon = \text{const}$:

$$Q(\varepsilon) = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}} = \varepsilon^{1/3} \sum_{k=0}^{\infty} \Psi_k\left(\zeta\right)\varepsilon^k\,.$$

- In ε expansion with coefficients singular, when $d \to 2$:

$$Q(\varepsilon) = \varepsilon^{1/3} \sum_{k=0}^{\infty} Q_k(d)\varepsilon^k\,.$$

These are two different subsequences of the double series

$$Q(\varepsilon, d) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[2\varepsilon/(d-2)\right]^k q_{kl}\,\left[(d-2)/2\right]^l\,.$$

Improved ε expansion

Combine the information from both expansions

$$Q_{eff}^{(n)} = \varepsilon^{1/3} \left[\sum_{k=0}^{n-1} Q_k(d) \varepsilon^k \right.$$

$$+ \sum_{k=0}^{n-1} \Psi_k \left(\frac{d-2}{2\varepsilon} \right) \varepsilon^k$$

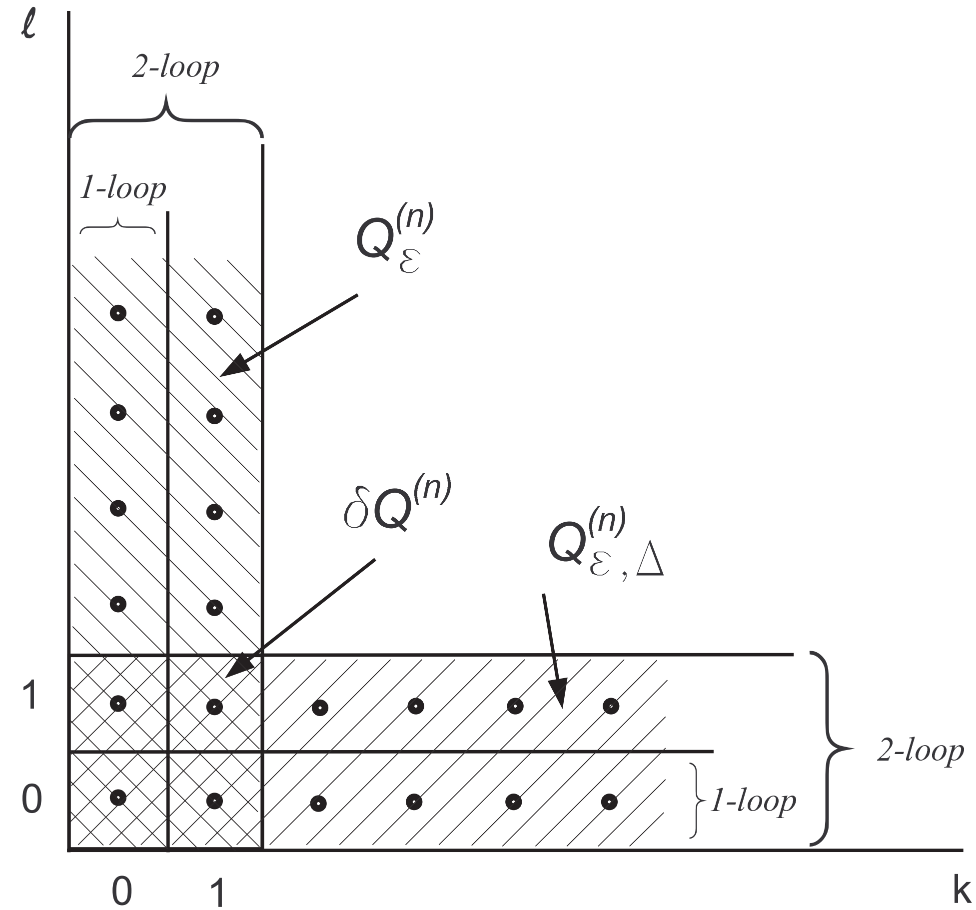
$$\left. - \sum_{k,l=0}^{n-1} \left(\frac{2\varepsilon}{d-2} \right)^k q_{kl} \left(\frac{d-2}{2} \right)^l \right].$$

Subtraction term to account for double counting in the overlap region.

Improved ε expansion

Combine the information from both expansions

$$Q^{(n)}_{eff} = \varepsilon^{1/3}\left[\sum_{k=0}^{n-1} Q_k(d)\varepsilon^k + \sum_{k=0}^{n-1}\Psi_k\left(\frac{d-2}{2\varepsilon}\right)\varepsilon^k - \sum_{k,l=0}^{n-1}\left(\frac{2\varepsilon}{d-2}\right)^k q_{kl}\left(\frac{d-2}{2}\right)^l\right].$$



Subtraction term to account for double counting in the overlap region.

Improved two-loop Kolmogorov constant

Comparison of one-loop and two-loop results for C_K:

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

- C_ε – ε expansion
- $C_{\varepsilon,\Delta}$ – double expansion
- C_δ – overlap correction
- C_{eff} – improved ε expansion

Improved two-loop Kolmogorov constant

Comparison of one-loop and two-loop results for C_K:

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

- C_ε – ε expansion
- $C_{\varepsilon,\Delta}$ – double expansion
- C_δ – overlap correction
- C_{eff} – improved ε expansion

Recommended experimental value: $C_K = 2.0$ (Sreenivasan, 1995).

Turbulent Prandtl number

Prandtl number for thermal conduction: $\mathrm{Pr} = \nu_0/\kappa_0 = 1/u$.

Turbulent Prandtl number

Prandtl number for thermal conduction: $\mathrm{Pr} = \nu_0/\kappa_0 = 1/u$.
Define turbulent (effective) inverse Prandtl number:

$$u_{eff} \equiv \Gamma_{\theta\theta'}(k, \omega = 0)/\Gamma_{vv'}(k, \omega = 0)\,.$$

Turbulent Prandtl number

Prandtl number for thermal conduction: $\mathrm{Pr} = \nu_0/\kappa_0 = 1/u$.
Define turbulent (effective) inverse Prandtl number:

$$u_{eff} \equiv \Gamma_{\theta\theta'}(k, \omega = 0)/\Gamma_{vv'}(k, \omega = 0)\,.$$

Singular in $d-2$ contributions cancel: two-loop correction small [Adzhemyan, JH, Kim & Sladkoff (2005)]:

$$u_{eff} = u_*^{(0)}(1 - 0.0358\varepsilon) + O(\varepsilon^2)\,,\ u_*^{(0)} = \frac{\sqrt{43/3} - 1}{2}\,,\ d = 3\,.$$

Turbulent Prandtl number

Prandtl number for thermal conduction: $\mathrm{Pr} = \nu_0/\kappa_0 = 1/u$. Define turbulent (effective) inverse Prandtl number:

$$u_{eff} \equiv \Gamma_{\theta\theta'}(k, \omega = 0)/\Gamma_{vv'}(k, \omega = 0)\,.$$

Singular in $d - 2$ contributions cancel: two-loop correction small [Adzhemyan, JH, Kim & Sladkoff (2005)]:

$$u_{eff} = u_*^{(0)}(1 - 0.0358\varepsilon) + O(\varepsilon^2)\,,\; u_*^{(0)} = \frac{\sqrt{43/3} - 1}{2}\,,\; d = 3\,.$$

At $\varepsilon = 2$ the turbulent Prandtl number Pr_t close to accepted experimental value $\mathrm{Pr}_t \approx 0.81$:

$$\mathrm{Pr}_t^{(0)} \simeq 0.72\,, \qquad \mathrm{Pr}_t \simeq 0.77\,.$$

Ramifications of the Navier-Stokes problem

- advection of passive scalar
  - hydrodynamic fluctuations, momentum-shell RG: Forster, Nelson & Stephen (1976),
  - LR correlated injection, field-theoretic RG: Adzhemyan, Vasil'ev & Pis'mak (1983),
  - decaying scalar, hydrodynamic fluctuations, LR correlated injection, field-theoretic RG: Hnatich (1990, reflecting boundary), Hnatich, JH (2000, absorbing boundary);

Ramifications of the Navier-Stokes problem

- advection of passive scalar
  - hydrodynamic fluctuations, momentum-shell RG: Forster, Nelson & Stephen (1976),
  - LR correlated injection, field-theoretic RG: Adzhemyan, Vasil'ev & Pis'mak (1983),
  - decaying scalar, hydrodynamic fluctuations, LR correlated injection, field-theoretic RG: Hnatich (1990, reflecting boundary), Hnatich, JH (2000, absorbing boundary);
- compressible fluid
  - LR correlated injection, momentum-shell RG: Staroselsky, Yakhot, Kida & Orszag (1990),
  - LR correlated injection, expansion in Mach number, FTRG, OPE: Adzhemyan, Nalimov & Stepanova (1995);

Ramifications of the Navier-Stokes problem

- advection of passive scalar
  - hydrodynamic fluctuations, momentum-shell RG: Forster, Nelson & Stephen (1976),
  - LR correlated injection, field-theoretic RG: Adzhemyan, Vasil'ev & Pis'mak (1983),
  - decaying scalar, hydrodynamic fluctuations, LR correlated injection, field-theoretic RG: Hnatich (1990, reflecting boundary), Hnatich, JH (2000, absorbing boundary);
- compressible fluid
  - LR correlated injection, momentum-shell RG: Staroselsky, Yakhot, Kida & Orszag (1990),
  - LR correlated injection, expansion in Mach number, FTRG, OPE: Adzhemyan, Nalimov & Stepanova (1995);
- anisotropic random forcing
  - LR, momentum-shell RG, weak anisotropy: Rubinstein & Barton (1987),
  - LR, FTRG, weak anisotropy: Adzhemyan, Hnatich, Horvath & Stehlik (1995); Kim & Serdukov (1995);
  - LR, FTRG, strong anisotropy: Buša, Hnatich, JH & Horvath (1997).