

# Dynamics of passive fields in turbulence

Dhrubaditya Mitra

Thesis Supervisor : Rahul Pandit

Indian Institute of Science, Bangalore, 560012

# Outline



- Introduction
- Dynamic multiscaling
- Summary of results
- Dynamics of Passive scalar
- Passive-vector

- Introduction
- Dynamic multiscaling  
Dynamic Multiscaling in fluid turbulence
  - Multifractal Picture
  - Bridge relations
  - Eulerian vs Lagrangian
  - Ongoing Work
- Summary of results
- Dynamics of Passive scalar
- Passive-vector

- Introduction
- Dynamic multiscaling
- Summary of results
- Dynamics of Passive scalar
  - Kraichnan model.
  - Simple dynamic scaling
  - Realistic advecting field
  - Dynamic multiscaling
- Passive-vector

# Outline



- Introduction
- Dynamic multiscaling
- Summary of results
- Dynamics of Passive scalar
- Passive-vector
  - "Anti-Dynamo"
  - Dynamic multiscaling
  - Passive-vector shell model

# Critical Phenomenon



Near the critical point

The spin-spin correlation function ( $\Gamma(r, T_r)$ ):

$$\langle \phi(x)\phi(x+r) - \phi(x)\phi(x) \rangle \approx \frac{1}{r^{d-2+\eta}} \mathcal{F}(T_r^\nu \xi); \quad (0)$$

- $T_r \equiv (T - T_c)/T_c$
- zero external magnetic field
- $\xi$  correlation length, diverges at the critical point;
- $\eta$  and  $\nu$  : static critical exponents;
- $\mathcal{F}$ : universal scaling function.

# Scaling and Universality



- log-log plot of  $\Gamma(r, T_r)$  vs  $r$  is a straight line slope  $-d + 2 - \eta$ .
- lattice-spacing  $\ll r \ll$  system-size.
- $\eta$   $\nu$  does not depend on details of the Hamiltonian.

# Experiment



Fig. 1.10. Wake behind two identical cylinders at  $R = 1800$ . Courtesy R. Dumas.

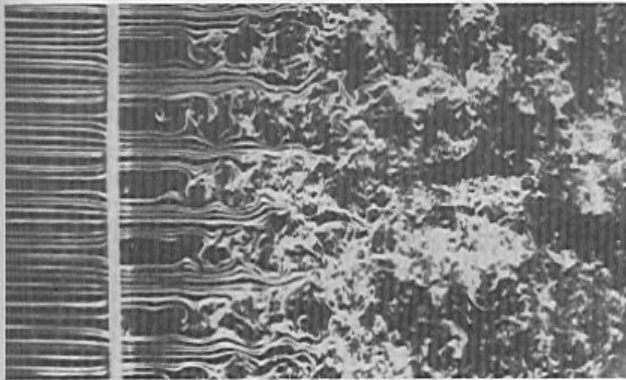
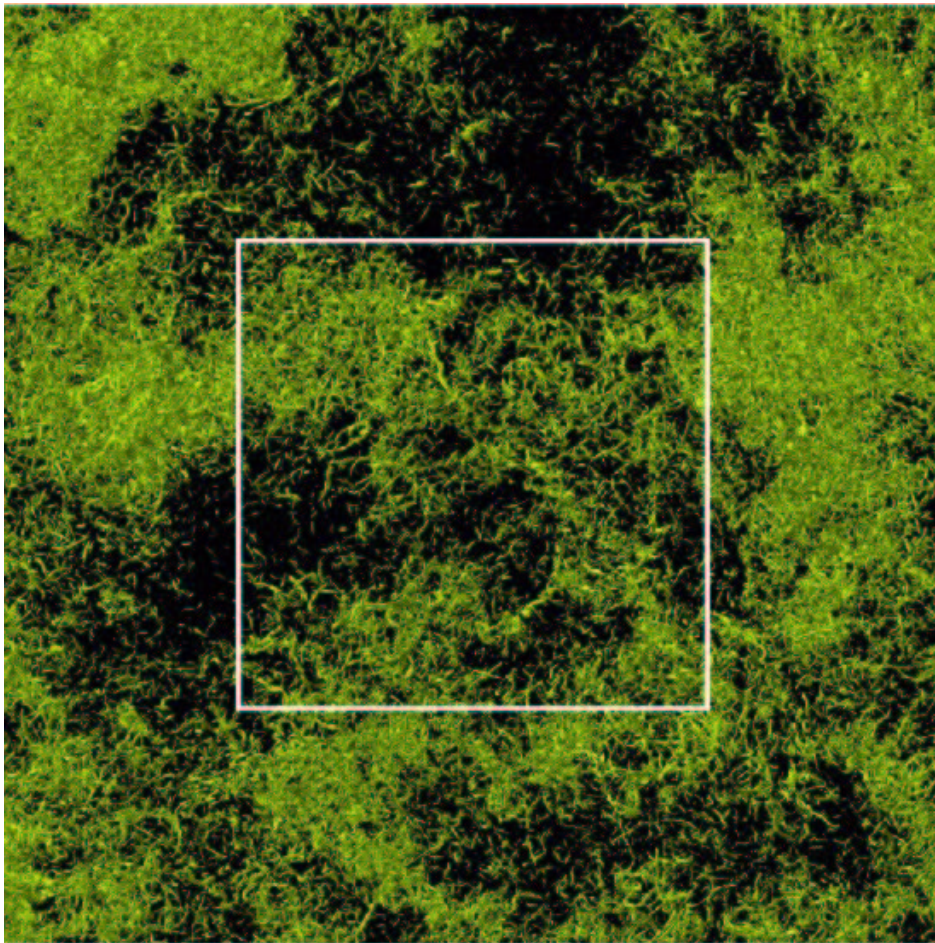
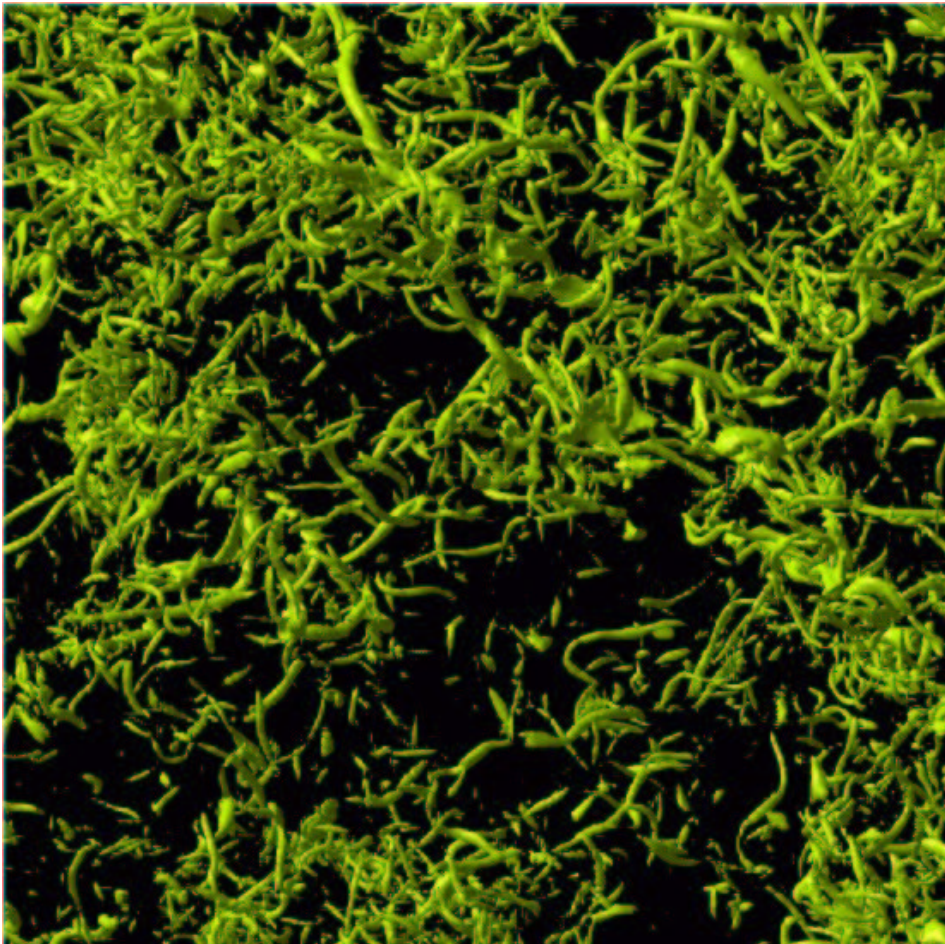


Fig. 1.11. Homogeneous turbulence behind a grid. Photograph T. Corke and H. Nagib.

# Direct Numerical Simulation



# Zoom-in



Homogeneous and Isotropic Turbulence:

- Navier-Stokes equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \nu \nabla^2 \vec{v} + \vec{\nabla} p / \rho + \vec{f} / \rho,$$

$\vec{v}$ ,  $p$ ,  $\nu$ , and  $\rho$  are, respectively: Eulerian velocity, pressure, kinematic viscosity, and density; and  $\vec{f}$  is the external forcing; incompressibility is imposed via  $\vec{\nabla} \cdot \vec{v} = 0$ .

# Introduction



Homogeneous and Isotropic Turbulence:

- Navier-Stokes equation:
- Energy is pumped in at lengths  $\sim L$  and dissipation is significant only for lengths  $\lesssim \eta_d$ .

## Homogeneous and Isotropic Turbulence:

- Navier-Stokes equation:
- Energy is pumped in at lengths  $\sim L$  and dissipation is significant only for lengths  $\lesssim \eta_d$ .
- In the *inertial range* ,  $\eta_d \ll \ell \ll L$ , scaling exponents are independent of both the forcing term and the viscosity. (**Universality**)

## Homogeneous and Isotropic Turbulence:

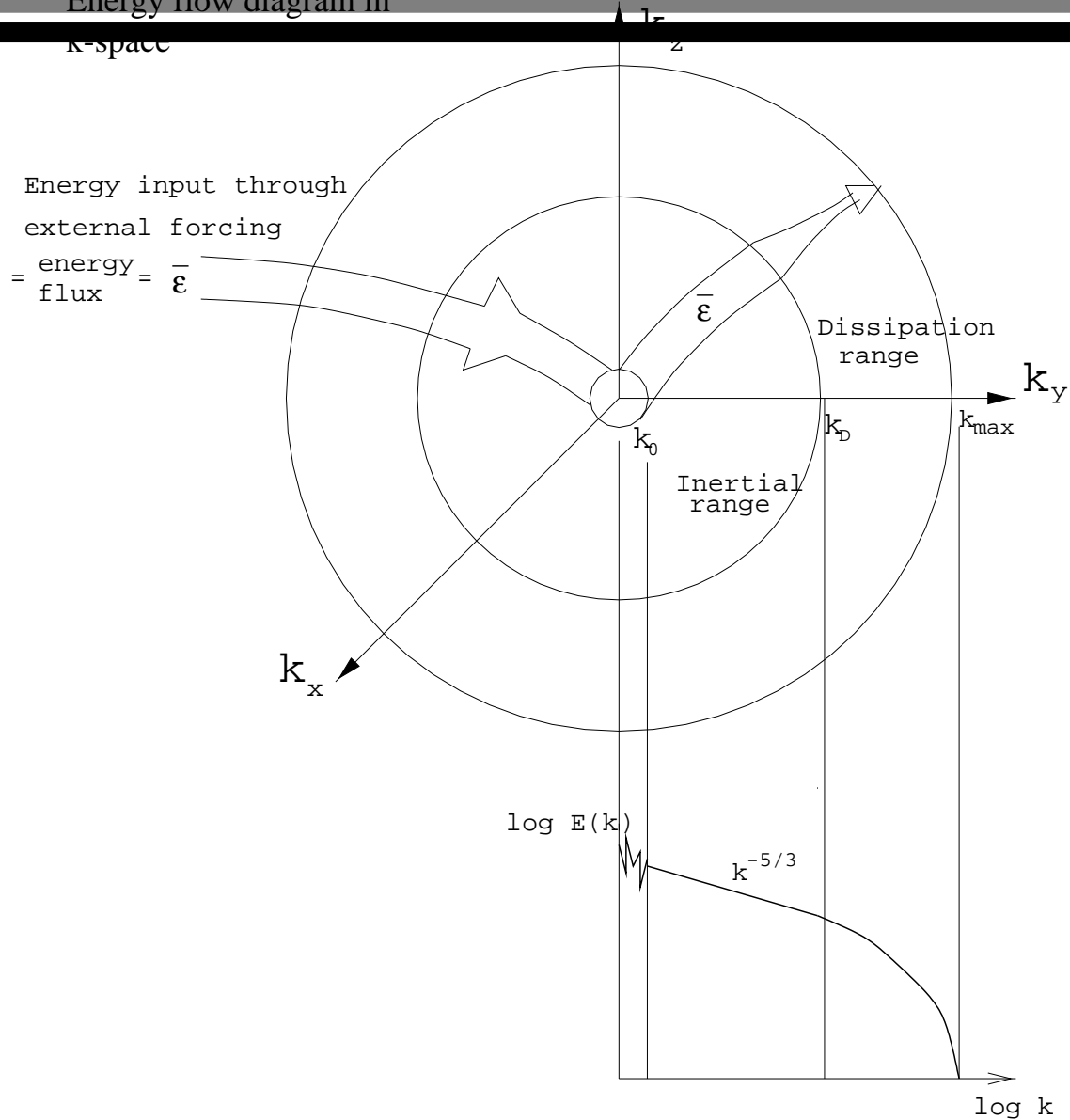
- Navier-Stokes equation:
- Energy is pumped in at lengths  $\sim L$  and dissipation is significant only for lengths  $\lesssim \eta_d$ .
- In the *inertial range* ,  $\eta_d \ll \ell \ll L$ , scaling exponents are independent of both the forcing term and the viscosity. (**Universality**)
- Energy cascades down the inertial range till it is dissipated in the dissipation range.

# Cascade



Energy flow diagram in

k-space



Energy spectrum in the steady state.

# Structure functions



Order- $p$  static, longitudinal velocity structure function.

$$S_p(\ell) \equiv \langle \delta v(\ell, t)^p \rangle,$$
$$\delta v(\ell, t) \equiv [\vec{v}(\vec{r} + \vec{\ell}, t) - \vec{v}(\vec{r}, t)] \cdot (\vec{\ell}/\ell),$$

# Multiscaling



- For  $\ell$  in the *inertial range* we have

$$S_p(\ell) \sim \ell^{\zeta_p}.$$

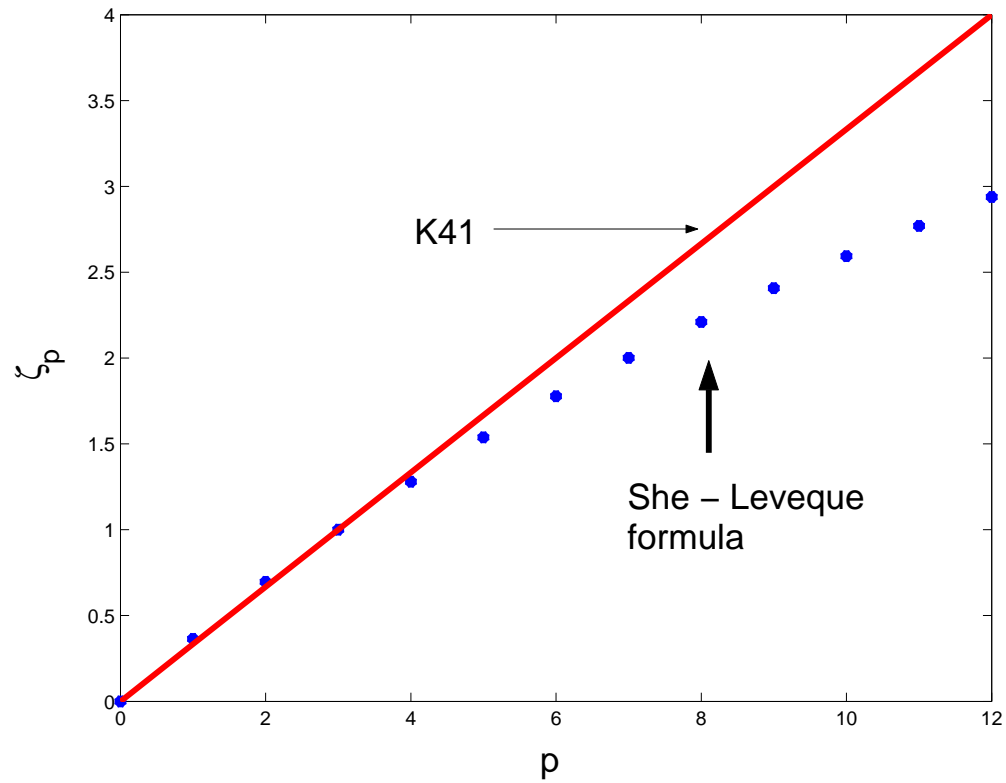
- $\zeta_p$  is a nonlinear, convex function of  $p$   
(Multiscaling) as in the She-Leveque formula

$$\zeta_p^{SL} = (p/9) + 2[1 - (2/3)^{p/3}],$$

which provides a reasonably good parametrization of experimental and numerical data for the multiscaling exponents

$\zeta_p$ .

# She-Leveque formula



# Exact relations



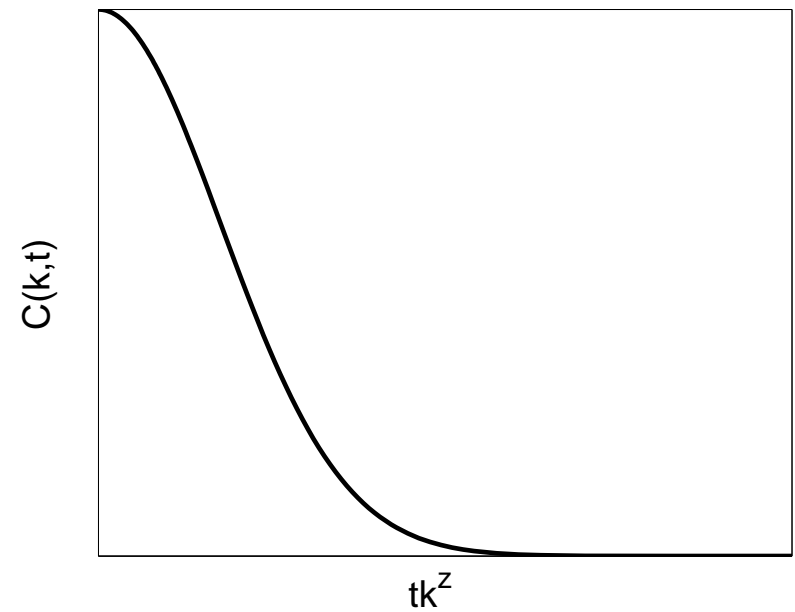
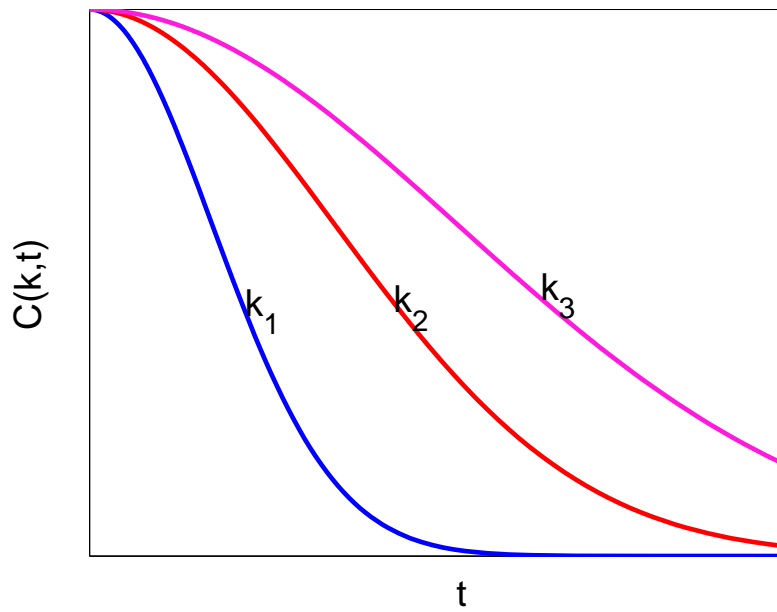
- $\zeta_3 = 1$  (von-Karman-Howarth)
- $\zeta_p$  versus  $p$  is a convex curve.

# Critical dynamics



- Near a critical point the relaxation time diverges.
- $\tau \sim \xi^z$
- $z$  dynamic scaling exponent.
- Collapse of the whole correlation function.
- $C(k, t) \equiv \langle \tilde{\phi}(k, 0) \tilde{\phi}(-k, t) \rangle$   
where  $\phi$  is the order parameter.

# Data-Collapse (Schematic)

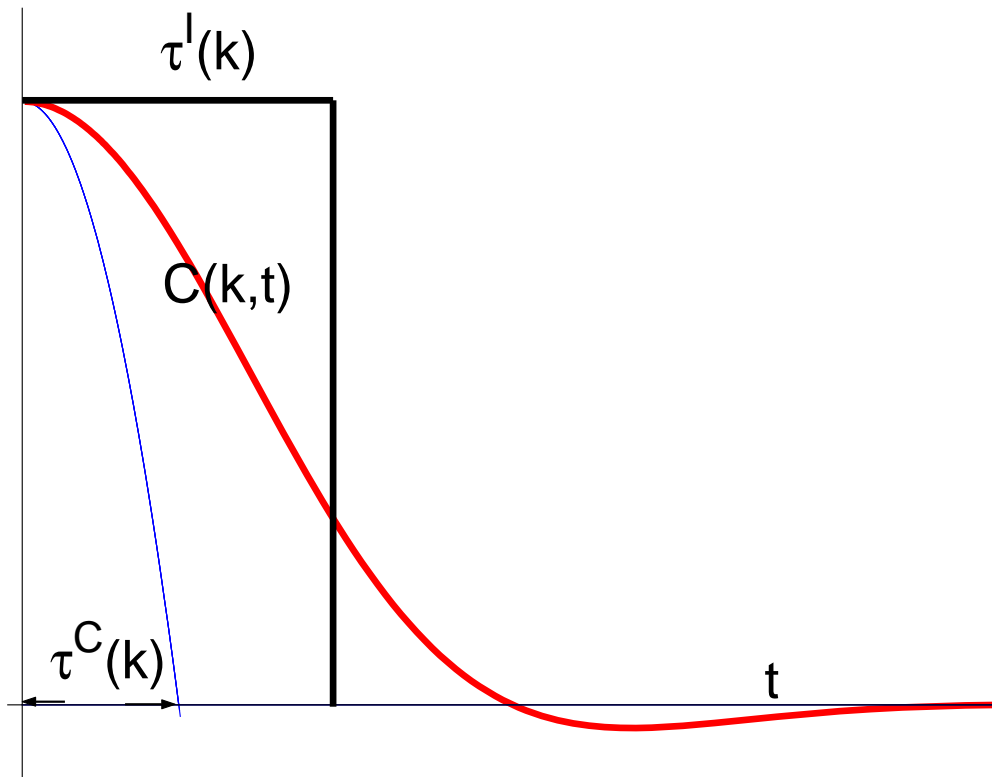


Each of the above curves in the first plot can be written in the form  $f(t/\tau)$  where  $\tau \sim k^{-z}$ . Hence we obtain a collapse if we plot on the horizontal axis  $tk^z$  (second plot) instead of  $t$  itself.

# Simple dynamic scaling



$$\tau^I(k) \sim \tau^C(k) \sim k^{-z}$$



# Dynamic Structure Functions



$$\mathcal{F}_p(\ell, \{t_1, \dots, t_{p-1}\}) \equiv \langle [\delta u_{\parallel}(\vec{x}, 0, \ell) \dots \delta u_{\parallel}(\vec{x}, t_{p-1}, \ell)] \rangle.$$

Clearly  $\mathcal{F}_p(\ell, \{t_1 = \dots = t_{p-1} = 0\}) = \mathcal{S}_p(\ell)$ . We normally restrict ourselves to the simple case  $t_1 = t_2 = \dots = t_q \equiv t$

# Dynamic scaling



$$T_p^I(\ell) \equiv \frac{1}{S_p(\ell)} \int_0^\infty \mathcal{F}(\ell, t) dt$$
$$\sim \ell^{z_p^I}$$

$$T_p^C(\ell) \equiv \left[ \frac{1}{S_p(\ell)} \frac{\partial^2}{\partial t^2} \mathcal{F}(\ell, t) \right]^{-1/2}$$
$$\sim \ell^{z_p^C}$$

# Dynamic multiscaling



- $z_p$  depends on exactly how  $\mathcal{T}_p(\ell)$  is extracted.
- No collapse of full correlation function.  
Breakdown of simple dynamic scaling.
- $z_p$  is a non-linear function of  $p$ .

# Summary of results

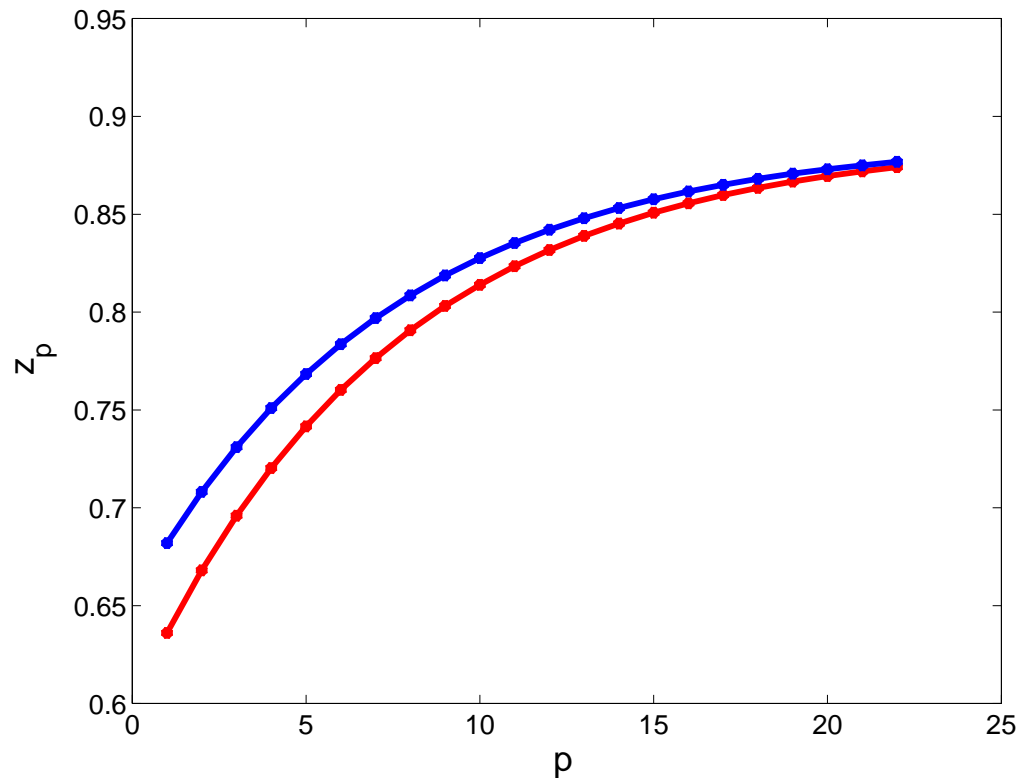


- Multifractal model predicts dynamic multiscaling

$$z_{p,1}^I = 1 + [\zeta_{p-1} - \zeta_p],$$

$$z_{p,2}^D = 1 + [\zeta_p - \zeta_{p+2}]/2.$$

# Comparing Bridge Relations



The two bridge relations, using the SL formula for  $\zeta_p$ , for  $p = 1 \dots 25$ . [red (integral); blue (curvature)]

# Summary of Results



- Fluid turbulence shows dynamic multiscaling.
- Breakdown of simple dynamic scaling
- Confirmed by numerical simulation in GOY shell model of turbulence.
- Can be generalised.
- No real analytical calculation is possible at present.
- If  $\zeta_p = p/3$  (K41), simple dynamic scaling is obtained,  $z_p = 2/3$ .

# Questions



1. Can dynamic scaling exponents be analytically extracted for other similar model ?
2. If answer to 1 is yes, then does multifractal model predicts correctly for 1.
3. Does equal-time multiscaling implies dynamic multiscaling ?
4. In general what are the ingredients of dynamic multiscaling ?

- Analytically tractable model exists, (Kraichnan model) but shows simple dynamic scaling, i.e. equal-time multiscaling does not necessarily imply dynamic multiscaling.
- Multifractal model gives correct result.
- If the advecting velocity is multiscaling, dynamic multiscaling is obtained.
- $z_p$  need not necessarily be a non-linear function of  $p$  for dynamic multiscaling.
- 3-d passive-vector should have same dynamic scaling like passive scalar.

# Kraichnan Model



$$\frac{\partial}{\partial t}\theta + \vec{u} \cdot \nabla\theta = \kappa\nabla^2\theta + f_\theta$$

$$\langle u_i(\vec{x}, t)u_j(\vec{x} + \vec{\ell}, t') \rangle = 2D_{ij}(\vec{\ell})\delta(t - t')$$

$$D_{ij}(\vec{\ell}) = D_o\delta_{ij} - \frac{1}{2}d_{ij}(\vec{\ell})$$

$L \rightarrow \infty$  and  $\eta \rightarrow 0$

$$d_{ij} = D_1\ell^\xi \left( (d - 1 + \xi)\delta_{ij} - \xi \frac{\ell_i\ell_j}{\ell^2} \right).$$

# Equal-time multiscaling



- Multiscaling can be analytically (but **perturbative**) demonstrated.
- $S_p(\ell) \sim \ell^{\zeta_p}$ .
- $0 < \xi < 2$ .
- Structure functions have good limits, not correlation functions.
- 2-nd order quantities show dimensional scaling.
- Numerical simulations support analytical results.

# Kraichnan shell model



$$\begin{aligned} \left[ \frac{d}{dt} + \kappa k_m^2 \right] \theta_m(t) = & i \left[ a_m (\theta_{m+1}^* u_{m-1}^* - \theta_{m-1}^* u_{m+1}^*) \right. \\ & + b_m (\theta_{m-1}^* u_{m-2}^* + \theta_{m-2}^* u_{m-1}) \\ & + c_m (\theta_{m+2}^* u_{m+1} + \theta_{m+1}^* u_{m+2}^*) \left. \right] \\ & + \delta_{m,1} f(t). \end{aligned}$$

$$\langle u_m(t) u_p^*(t') \rangle = D_m \delta_{p,m} \delta(t - t')$$

with  $D_m = k_m^{-\xi}$ .  $S_p^\theta(m) \equiv \langle [\theta_m \theta_m^*]^{p/2} \rangle \sim k_m^{-\zeta_p^\theta}$

# Dynamic structure functions



$$F_p(m, t) = \langle [\theta_m(0)\theta_m^*(t)]^{p/2} \rangle,$$

$$\begin{aligned} \frac{d}{dt} F_2(m, t) = & \langle \theta_m(0) [-\kappa k_m^2 \theta_m + \delta_{m,1} f] \rangle \\ & - \langle \theta_m(0) \{ i [ a_m (\theta_{m+1}^* u_{m-1}^* - \theta_{m-1}^* u_{m+1}^*) \\ & + b_m (\theta_{m-1}^* u_{m-2}^* + \theta_{m-2}^* u_{m-1}^*) \\ & + c_m (\theta_{m+2}^* u_{m+1}^* + \theta_{m+1}^* u_{m+2}^*) ] \} \rangle, \end{aligned}$$

$$\langle \theta_m(0) \theta_{m+1}^* u_{m-1}^* \rangle = \langle u_{m-1}^* u_q \rangle \left\langle \frac{\delta}{\delta u_q(t)} \theta_m(0) \theta_{m+1}^*(t) \right\rangle \quad (-15)$$

- $\langle u_m(t) u_p^*(t') \rangle = D_m \delta_{p,m} \delta(t - t')$
- Causality

# Exponential decay in time



$$F_2(m, t) = S_2(m) e^{-t/\tau(m)},$$
$$\tau(m) = 4k_m^{-(2-\xi)} A^{-1}(\xi)$$

$$A(\xi) = [2^{(2\xi-2)} + 2^{-(2\xi-2)}] + [2^\xi + 2^{-\xi}]$$
$$+ [2^{(\xi-2)} + 2^{-(\xi-2)}].$$

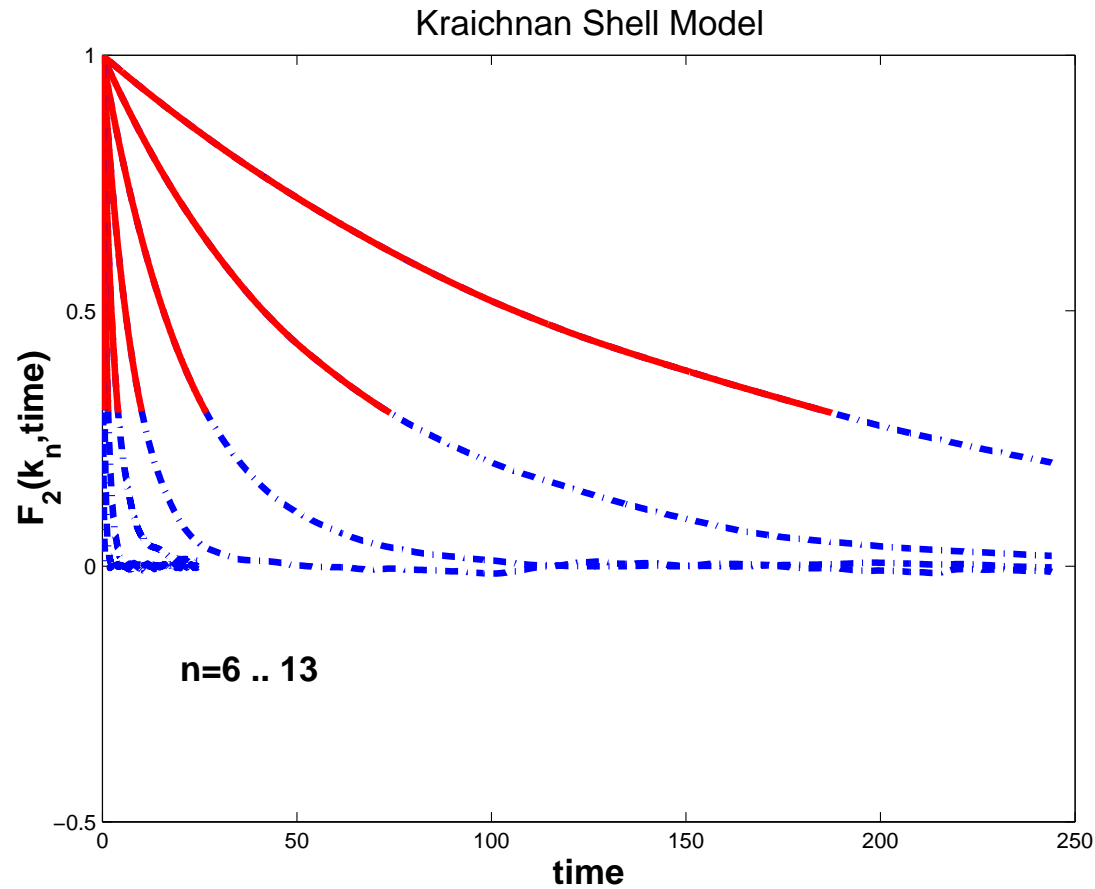
- Similar relations can be obtained for higher values of  $p$  but the process is more cumbersome.

# Numerical Simulation

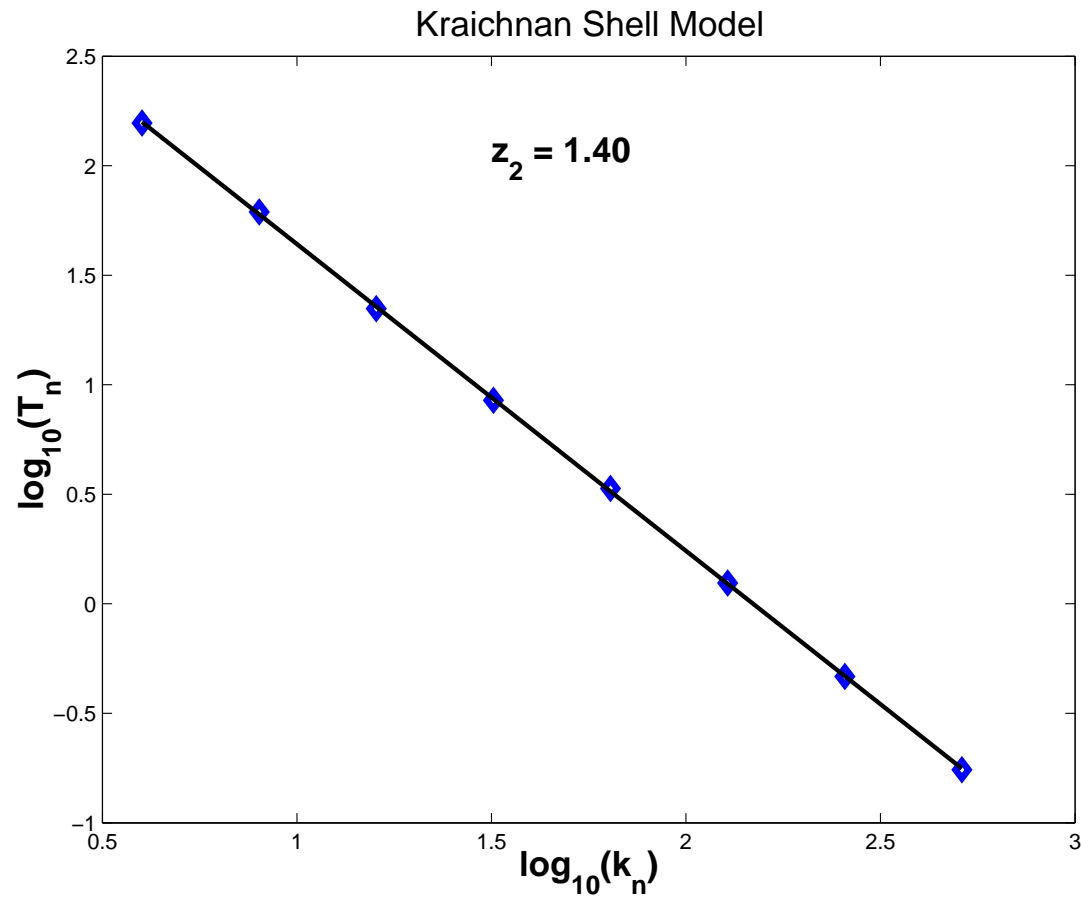


- Viscous term integrated exactly.
- Careful numerical scheme for the white-in-time velocity.

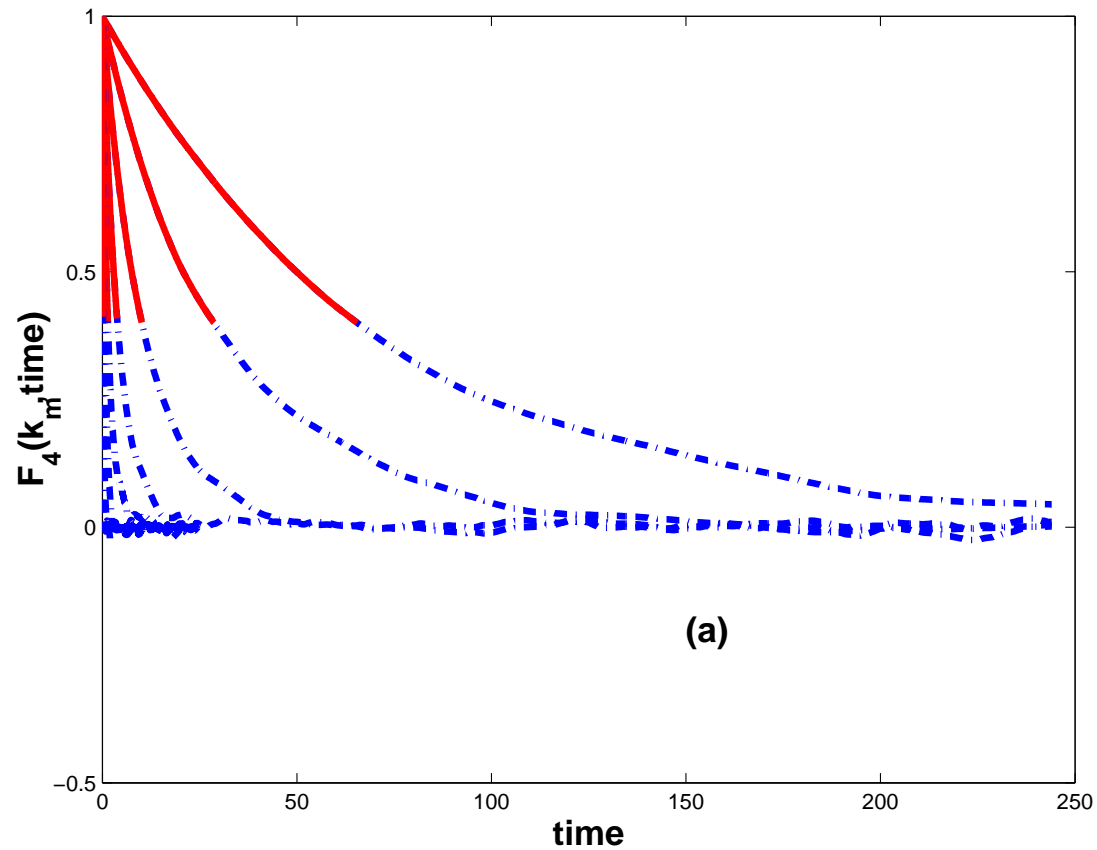
# $F_2(m, t) (\xi = 0.6)$



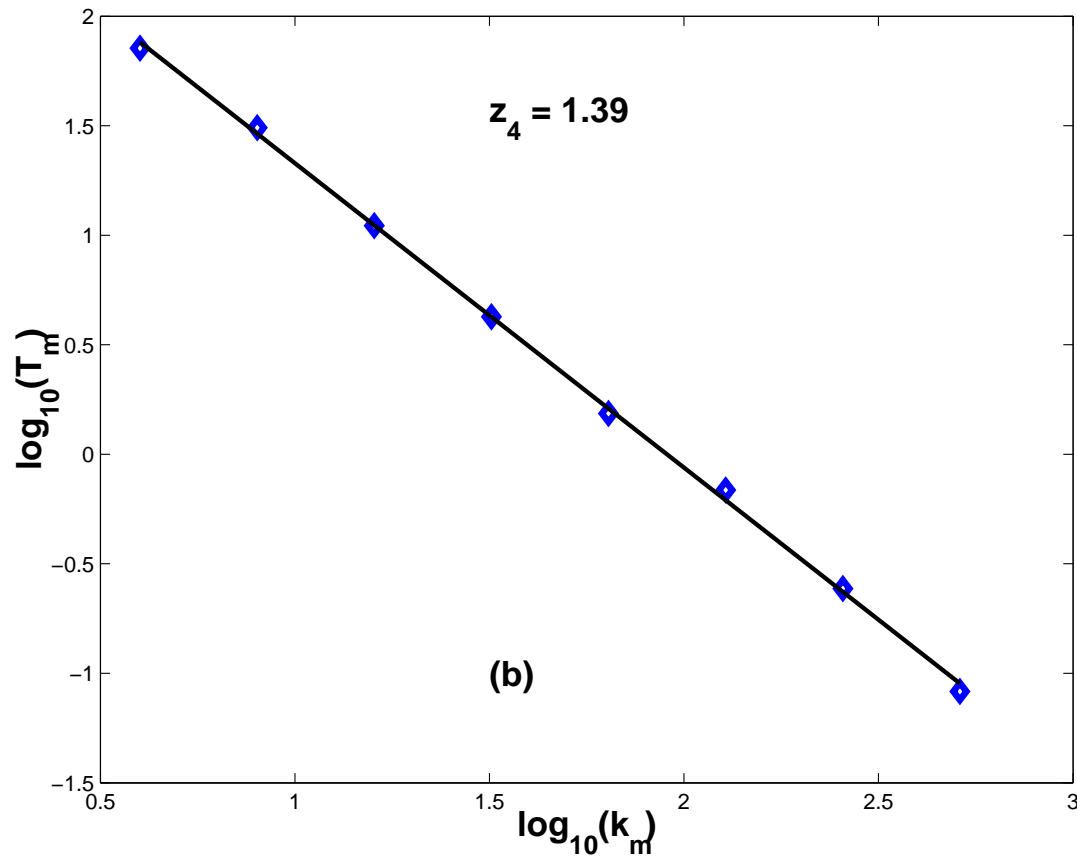
# $z_2(\xi = 0.6)$



# $F_4(m, t) (\xi = 0.6)$



# $z_4(\xi = 0.6)$



# Summary



- $z = 2 - \xi$
- Collapse for full correlation function.
- White-in-time nature of velocity plays major role.
- Equal-time multiscaling does not necessarily imply dynamic multiscaling.
- Agrees with multifractal model predictions.
- Same analytical calculation applies to the full 3-d Kraichnan model.

# More realistic velocity field



1. For velocity field with simple scaling but not white-in-time  $z = 2 - \xi$  (multifractal model).
2. 1 should be true for passive-vector too.  
(Kinematic dynamo model)
3. What happens if velocity field is multiscaling ?

# GOY Shell Model



$$\left[ \frac{d}{dt} + \nu k_n^2 \right] u_n = i(a_n u_{n+1} u_{n+2} + b_n u_{n-1} u_{n+1} + c_n u_{n-1} u_{n-2})^* + f_n;$$

- dynamical variables: complex, scalar velocities  $u_n$ , for the shells  $n$ .
- one-dimensional, logarithmically spaced wavevectors  $k_n$ , i.e.,  $k_n = k_0 2^n$ , and c.c is denoted by  $*$
- $a_n = k_n$ ,  $b_n = -\delta k_{n-1}$ ,  $c_n = -(1 - \delta) k_{n-2}$ ; chosen to conserve the shell-model analogues of energy and helicity in the

# Passive scalar shell model



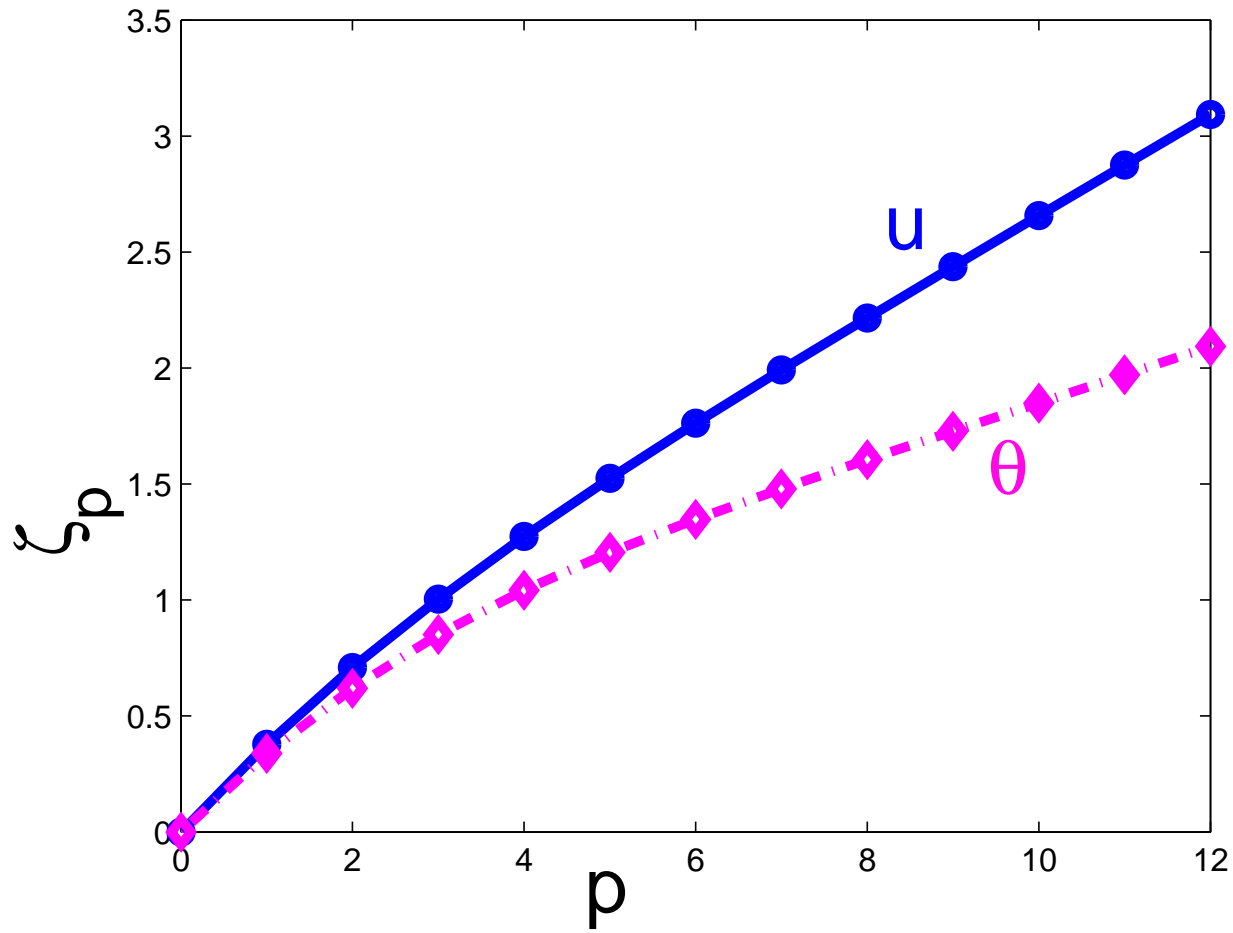
$$\begin{aligned} \left[ \frac{d}{dt} + \kappa k_m^2 \right] \theta_m(t) = & i \left[ a_m (\theta_{m+1}^* u_{m-1}^* - \theta_{m-1}^* u_{m+1}^*) \right. \\ & + b_m (\theta_{m-1}^* u_{m-2}^* + \theta_{m-2}^* u_{m-1}^*) \\ & + c_m (\theta_{m+2}^* u_{m+1}^* + \theta_{m+1}^* u_{m+2}^*) \left. \right] \\ & + \delta_{m,1} f(t), \end{aligned} \tag{-18}$$

# Equal-time properties



- Equal-time multiscaling.
- More intermittent than fluid.
- Agrees well with experiments.

# $\zeta_p^u$ and $\zeta_p^\theta$



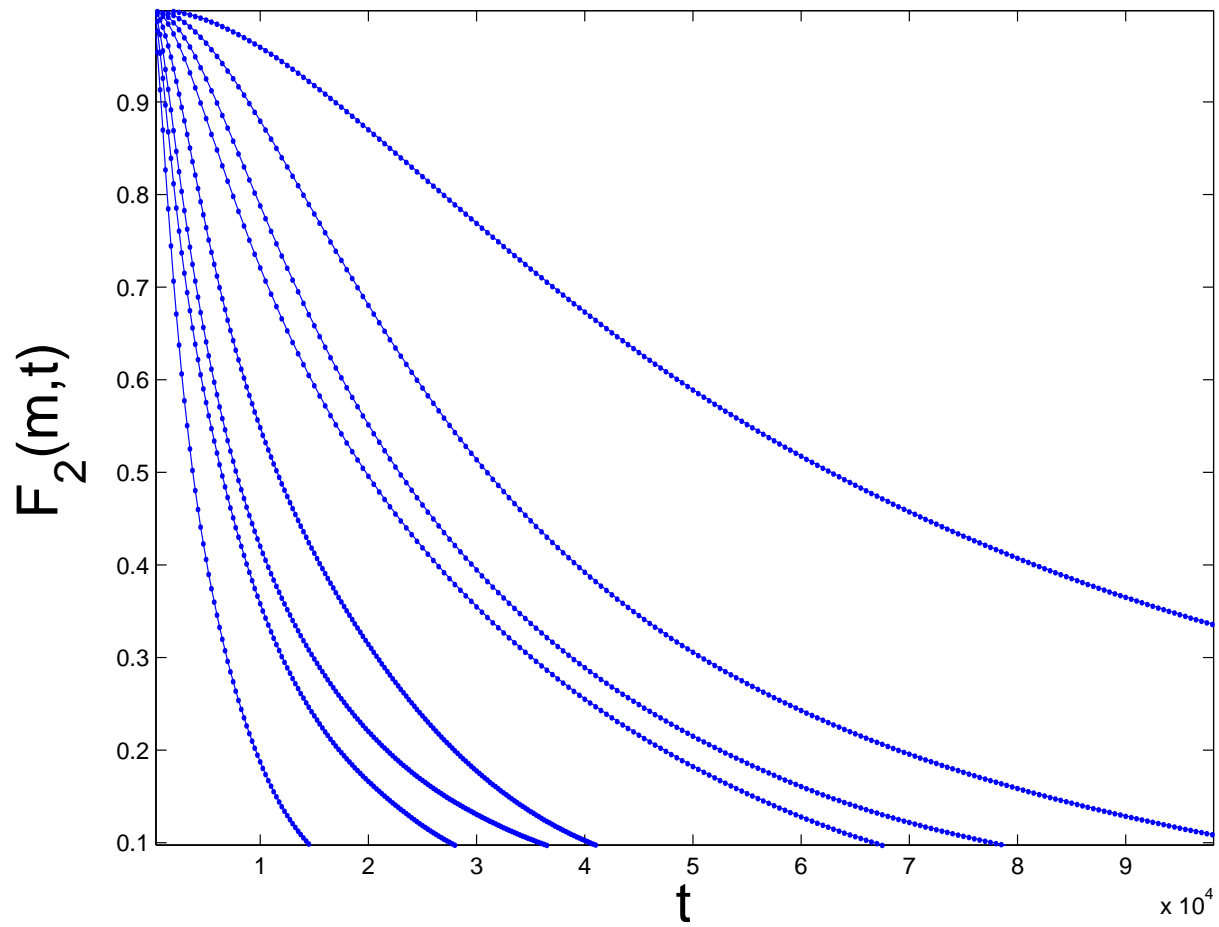
# Time-series of $u$



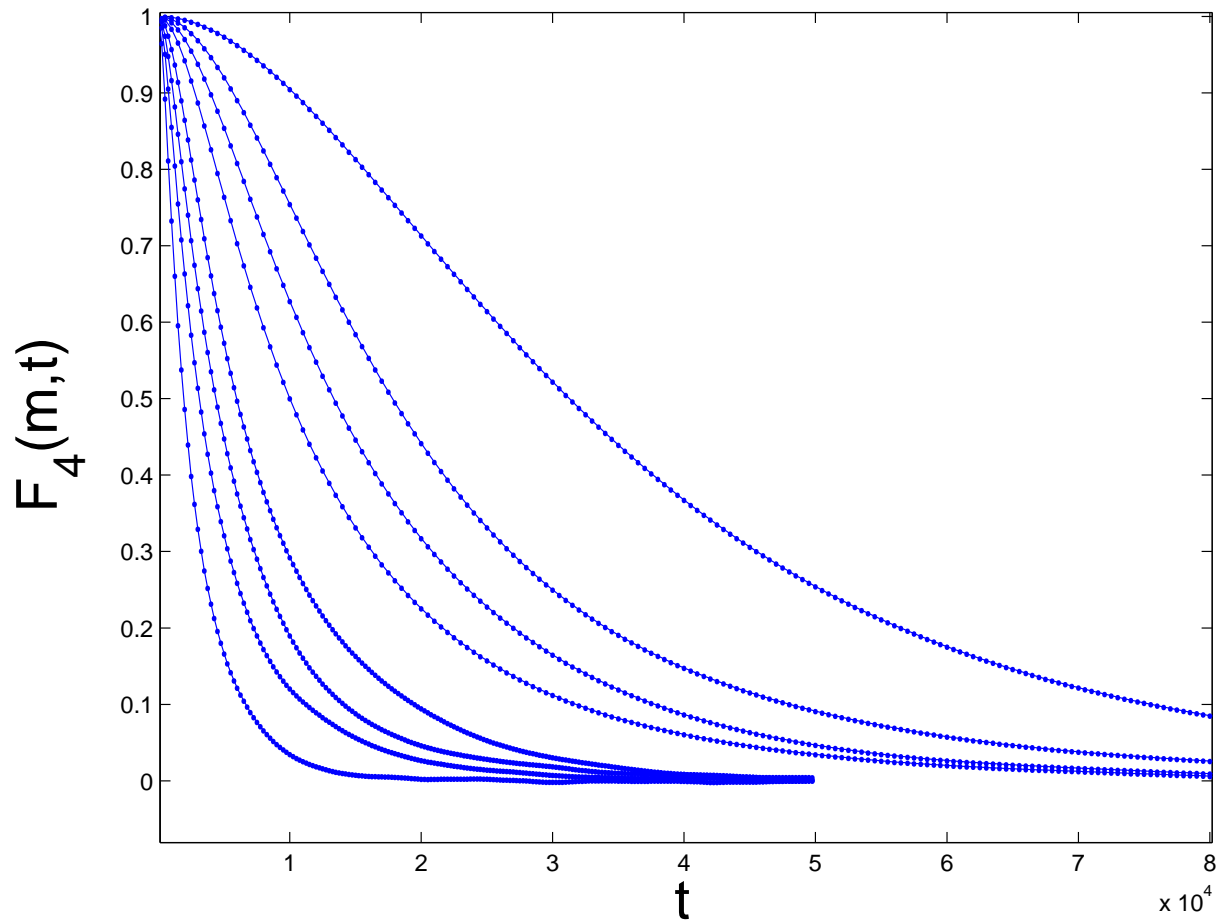
# Time-series of $\theta$

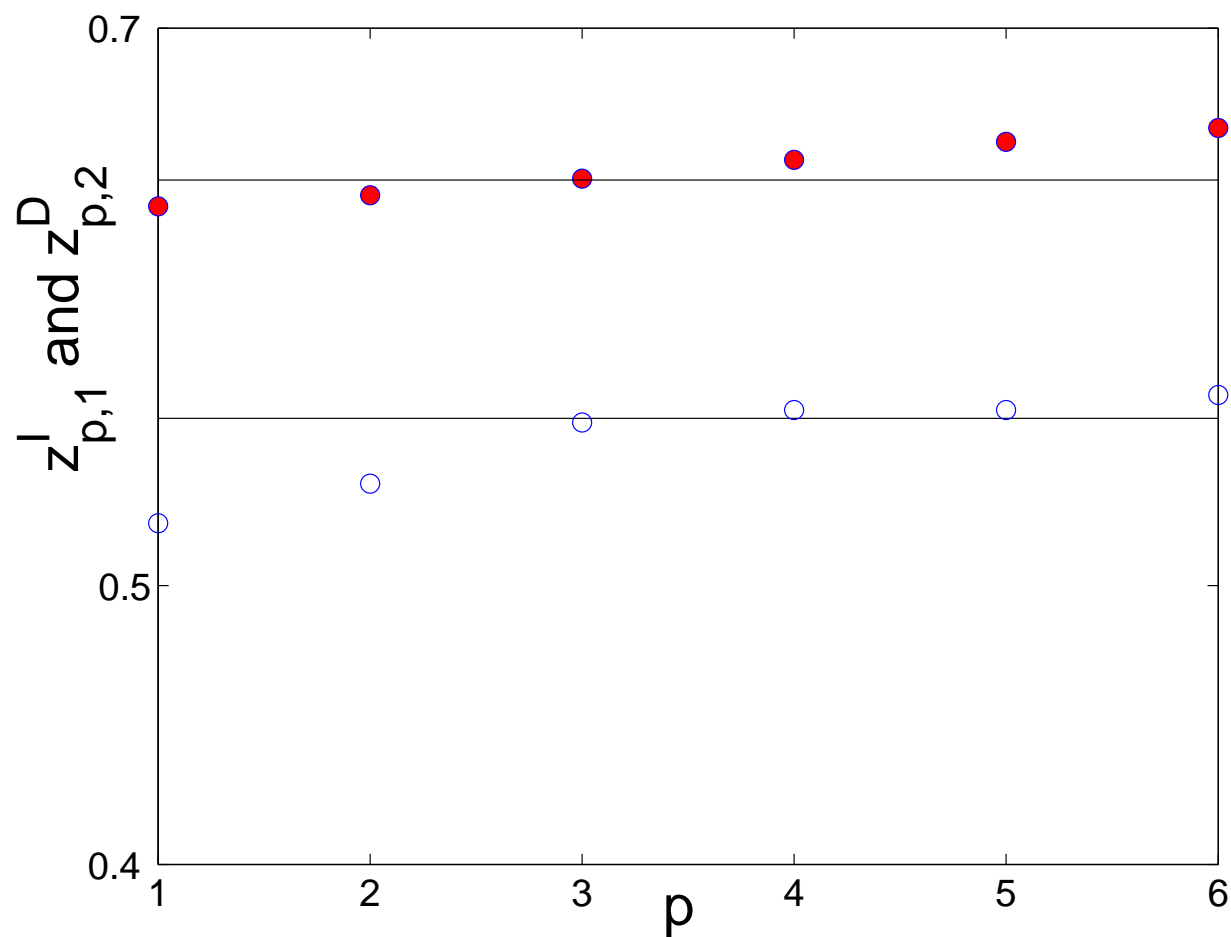


# $F_2(m, t)$



# $F_4(m, t)$

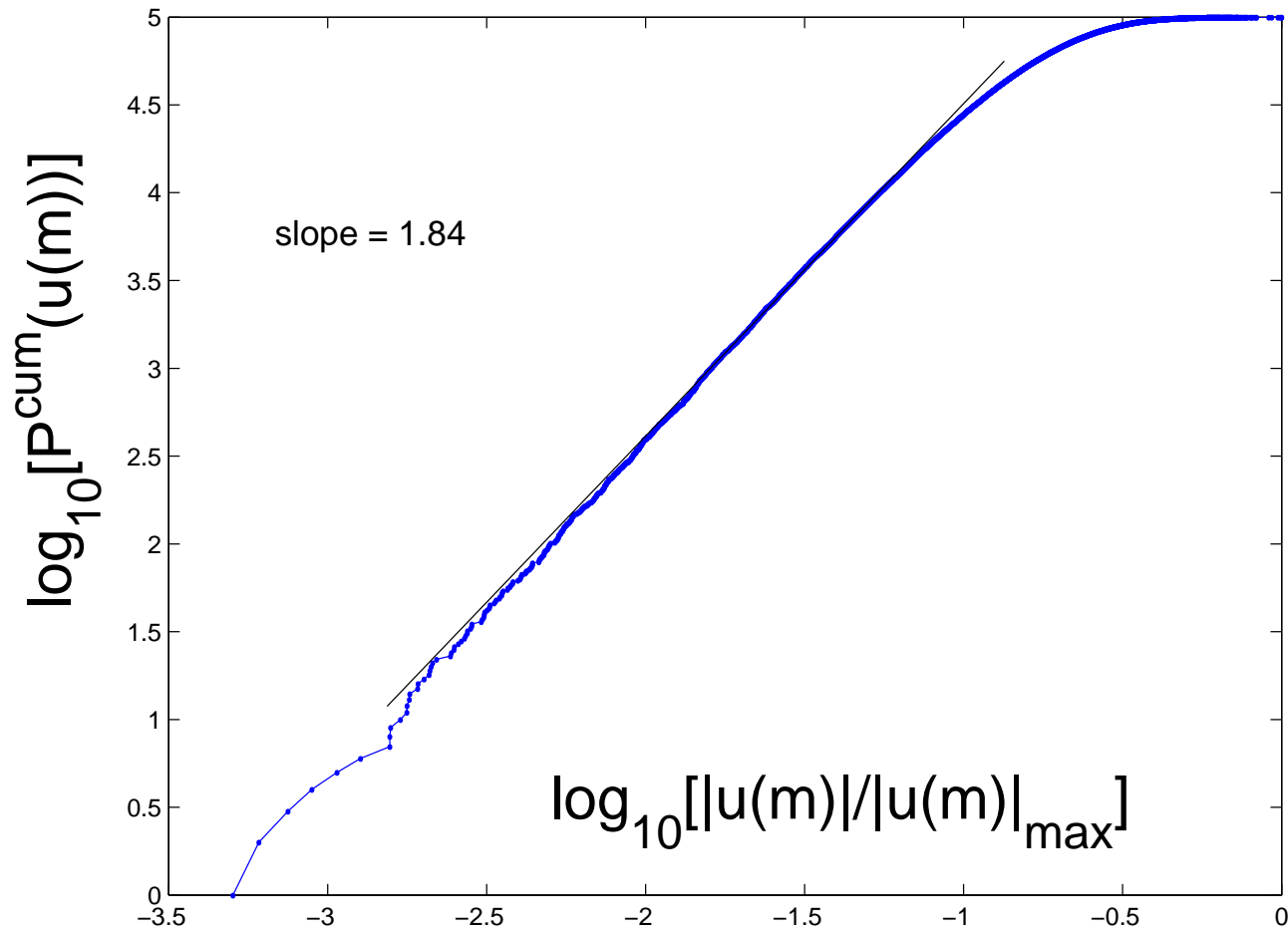




Multifractal model predicts:

- $z_{p,2}^D = 1 - \zeta_2^u / 2$
- $z_{p,1}^I = 1 - \zeta_{-1}^u$
- Breakdown of simple scaling.
- Does structure functions with negative exponents exists ?

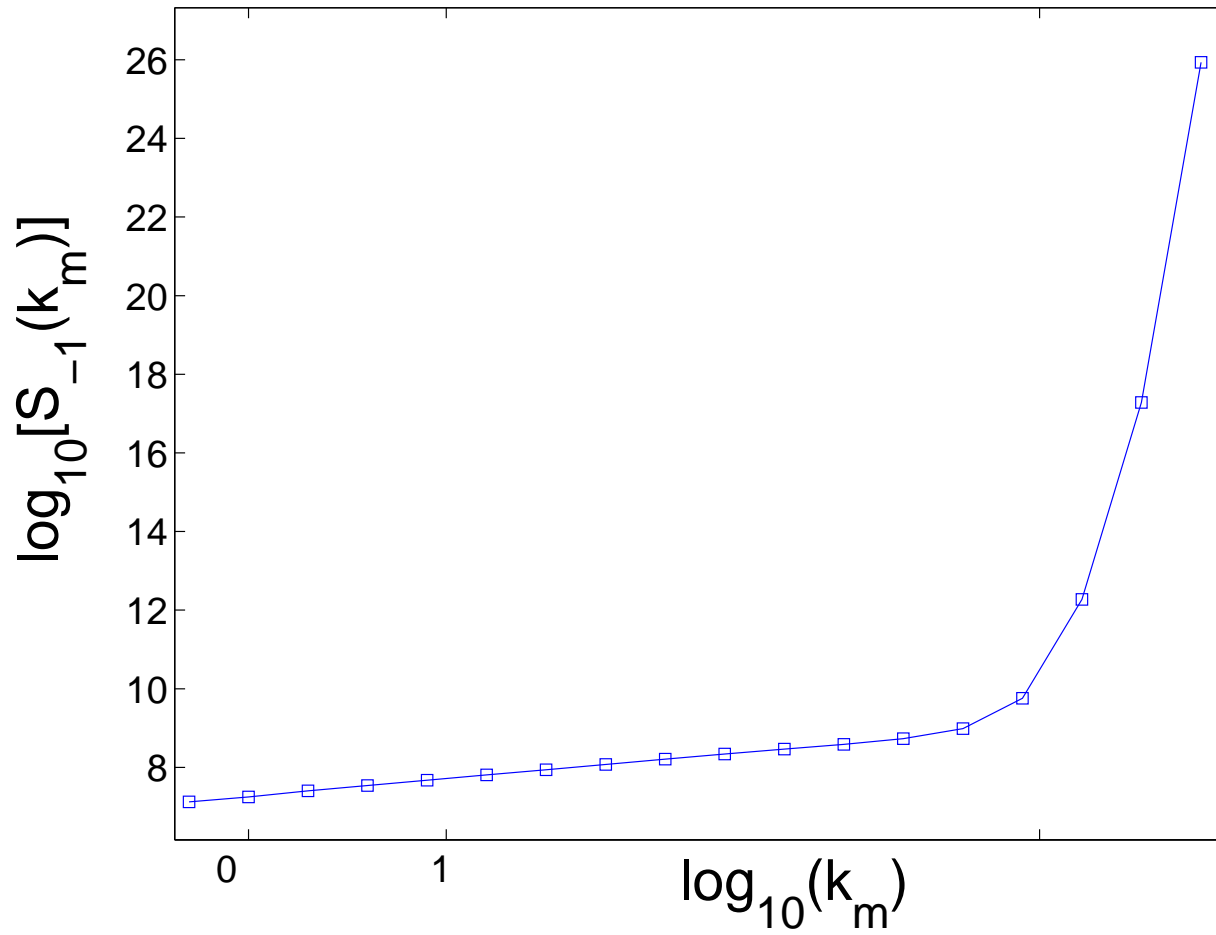
# Cumulative pdf for $u_m$



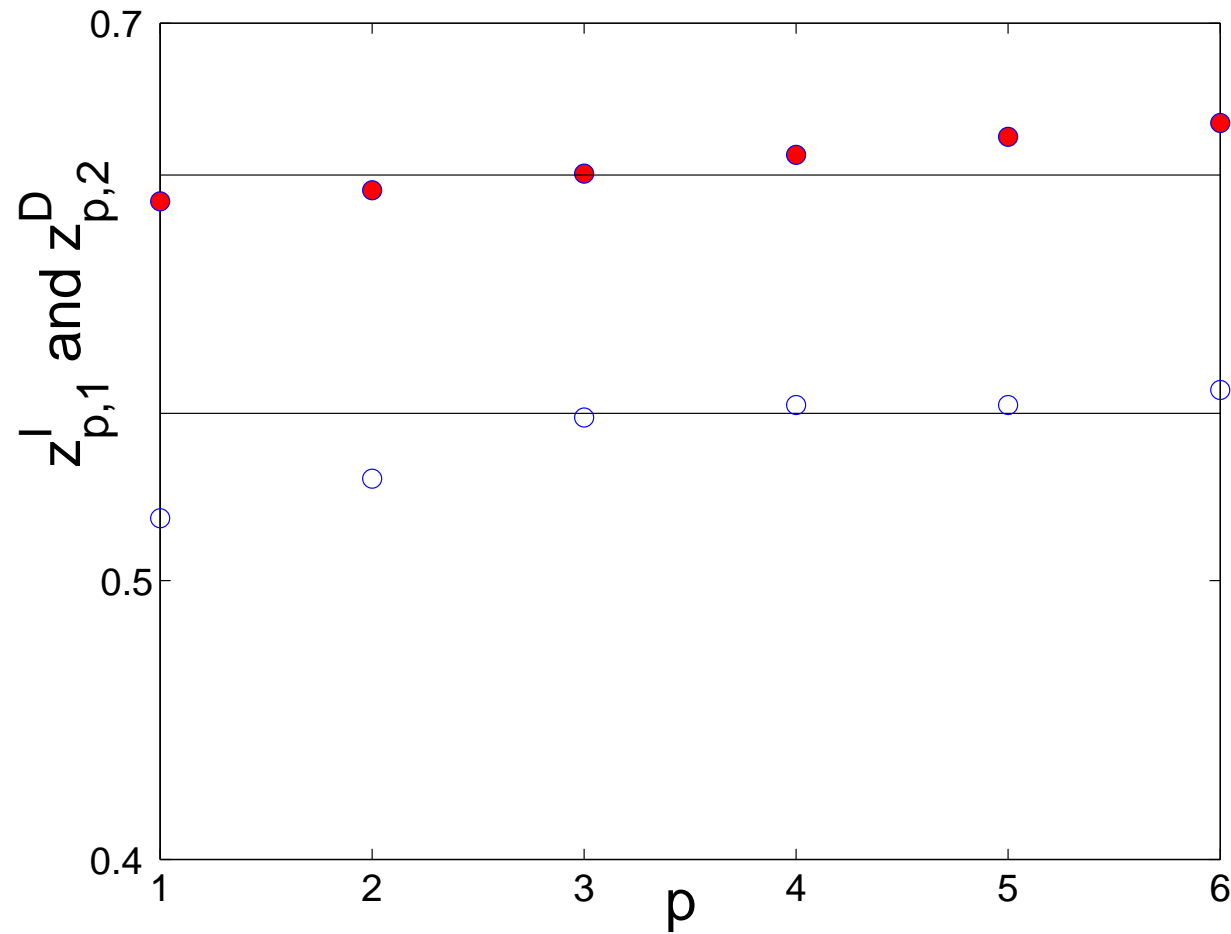
# Negative exponents



- For small  $|u_m|$ ,  $P^{cum}[|u_m|] \sim |u_m|^{1.8}$ .
- $P[|u_m|] \sim |u_m|^{0.8}$
- $S_{-1}(m) \equiv \int P[x] \frac{1}{x} dx \sim \int x^{-0.2} dx$ ; **exists.**
- But  $S_p(m)$  for  $p$ , for  $p \approx -1.8$  doesnot.
- $T_{p,M}^I$  for  $M > 2$  doesnot exists.
- Measurement of a static quantity ( $P(x)$ ) gives us information about existence of a dynamic quantity  $T_{p,M}^I$ .



# Dynamic multiscaling



- Analytically tractable model exists, (Kraichnan model) but shows simple dynamic scaling.
- Multifractal model gives correct result.
- If the advecting field is multiscaling, dynamic multiscaling is obtained.
- $z_p$  need not necessary be a non-linear function of  $p$  for dynamic multiscaling.
- 3-d passive-vector should have same dynamic scaling like passive scalar.

# References



1. Dynamic Multiscaling in fluid turbulence : An Overview. **D. Mitra** and **R. Pandit**, *Physica A*, **318**, 179 (2003).
2. Varieties of Dynamic Multiscaling in Fluid Turbulence. **D. Mitra** and **R. Pandit**. *Phys. Rev. Lett.* **93**, 024501 (2004) (nlin.CD/0309037)
3. Dynamic scaling and multiscaling in passive scalar turbulence. **D. Mitra** and **R. Pandit**. Manuscript in preparation.