PREPARATION UNCERTAINTY RELATIONS BEYOND HEISENBERG'S

Arun Kumar Pati

Quantum Information and Computation Group
Harish-Chandra Research Institute, Allahabad, India

S. N. Bose Center, Jan 29-Feb 2, 2018

Plan

- Introduction.

Plan

- Introduction.
- Stronger Uncertainty relations

Plan

- Introduction.
- Stronger Uncertainty relations
- Tighter uncertainty relations

Plan

- Introduction.
- Stronger Uncertainty relations
- Tighter uncertainty relations
- Reverse uncertainty relations

Plan

- Introduction.
- Stronger Uncertainty relations
- Tighter uncertainty relations
- Reverse uncertainty relations
- Summary and conclusions

History

- Quantum mechanics is the finest theory of nature that explains behavior of atoms and subatomic particles.
- It has been enormously successful in giving correct results in practically every situation.
- In spite of the overwhelming success of quantum mechanics, the foundations of the subject contain many paradoxes.

History

- (Heisenberg) Matrix mechanics [Zeitschrift für Physik, **33**, 879-893 (1925)].
- (Schrödinger) wave mechanics. [E. Schrödinger, Ann. der Physik, **384**, 361-376 (1926)].
- The wave mechanics appealed to many physicists. It is equivalent to matrix mechanics.
- Matrix mechanics was difficult to visualize.
- Intense debate between the alternative versions of quantum mechanics formed the background for the development of uncertainty relation and the Copenhagen Interpretation.

History

- I knew of [Heisenberg's] theory, of course, but I felt discouraged, not to say repelled, by the methods of transcendental algebra, which appeared difficult to me, and by the lack of visualizability.
  -Schrödinger in 1926

- The more I think about the physical portion of Schödinger's theory, the more repulsive I find it...What Schrödinger writes about the visualizability of his theory 'is probably not quite right,' in other words it's crap.
  –Heisenberg, writing to Pauli (1926)

History

- It is meaningless to ascribe any properties or even existence to anything that has not been measured.

  Bohr: “Nothing is real unless it is observed”.
- Einstein: It is the theory which decides what we can observe.
- I believe that the existence of the classical “path” can be formulated as follows: The ”path” comes into existence only when we observe it.
  –Heisenberg, in uncertainty principle paper (1927)

Uncertainty Principle

- (1927) Heisenberg's uncertainty principle is a fundamental result in quantum physics and gives profound insights to quantum world.
- Minimum amount of unavoidable momentum disturbance and inaccuracy in position measurement $\Delta x \Delta p \sim h$
- However, he did not give a precise definition for Δx and Δp.

  One can never know with perfect accuracy both of those two important factors which determine the movement of one of the smallest particle–its position and its velocity.

Uncertainty Relation

- Robertson Uncertainty Relation (1927): Bounds the product of the variances for two observables through the expectation value of the commutator

$$\Delta A^2 \Delta B^2 \geq |\frac{1}{2}\langle\psi|[A, B]|\psi\rangle|^2 \tag{1}$$

- There is a limit to the precision with which a pair of non-commuting observables can be measured.

- Robertson and Kennard (1927) relation $\Delta x \Delta p \geq \frac{\hbar}{2}$.

  More precisely the position of a particle is determined, the less precisely its momentum can be known.

Uncertainty Principle vs Relation

- Uncertainty principle is different than uncertainty relation. Both appear in similar formulations that even many practicing physicists tend to confuse.
- Heisenberg version is about uncertainty principle: in observing a quantum particle we inevitably disturb it.
- When we measure the position of an electron with an error Δx, we disturb the momentum of the electron by the amount Δp.

- This (UR) inequality shows that the fluctuation exists regardless whether we measur or not–preparation uncertainty.
- This inequality does not say anything about what happens when a measurement is performed.
- Robertson-Kennard formulation is therefore totally different from Heisenberg's.
- But many physicists, probably including Heisenberg himself, have been under the impression that both formulations describe same phenomenon.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.
- Existence of incompatible observables lead to uncertainty relations.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.
- Existence of incompatible observables lead to uncertainty relations.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.
- Existence of incompatible observables lead to uncertainty relations.
- Owing to the seminal works by Heisenberg, Robertson and Schrödinger, lower bounds were shown to exist for the product of variances of two non-commuting observables.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.
- Existence of incompatible observables lead to uncertainty relations.
- Owing to the seminal works by Heisenberg, Robertson and Schrödinger, lower bounds were shown to exist for the product of variances of two non-commuting observables.

Introduction

- Quantum mechanics has many distinguishing features from classical mechanics in the microscopic world.
- Existence of incompatible observables lead to uncertainty relations.
- Owing to the seminal works by Heisenberg, Robertson and Schrödinger, lower bounds were shown to exist for the product of variances of two non-commuting observables.
- Entropic uncertainty relations also capture the essence of quantum uncertainty and the incompatibility between two observables, but in a state-independent way.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.
- Further, it has been used in entanglement detection, security analysis of quantum key distribution in quantum cryptography, quantum metrology and quantum speed limit.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.
- Further, it has been used in entanglement detection, security analysis of quantum key distribution in quantum cryptography, quantum metrology and quantum speed limit.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.
- Further, it has been used in entanglement detection, security analysis of quantum key distribution in quantum cryptography, quantum metrology and quantum speed limit.
- In most of these areas, particularly, in quantum entanglement detection and quantum metrology or quantum speed limit, where a small fluctuation in an unknown parameter of the state of the system is needed to detect, state-dependent relations may be useful.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.
- Further, it has been used in entanglement detection, security analysis of quantum key distribution in quantum cryptography, quantum metrology and quantum speed limit.
- In most of these areas, particularly, in quantum entanglement detection and quantum metrology or quantum speed limit, where a small fluctuation in an unknown parameter of the state of the system is needed to detect, state-dependent relations may be useful.

Uncertainty relations

- With the advent of quantum information theory, uncertainty relations in particular, have been established as important tools for a wide range of applications.
- Further, it has been used in entanglement detection, security analysis of quantum key distribution in quantum cryptography, quantum metrology and quantum speed limit.
- In most of these areas, particularly, in quantum entanglement detection and quantum metrology or quantum speed limit, where a small fluctuation in an unknown parameter of the state of the system is needed to detect, state-dependent relations may be useful.
- Thus, a focus on the study of the state dependent, tighter uncertainty and the reverse uncertainty relations based on the variance is important.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.
- In this sense, uncertainty relations in terms of the sum of variances capture the concept of incompatibility more accurately.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.
- In this sense, uncertainty relations in terms of the sum of variances capture the concept of incompatibility more accurately.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.
- In this sense, uncertainty relations in terms of the sum of variances capture the concept of incompatibility more accurately.
- It may be noted that earlier uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables, entropic uncertainty relations, sum uncertainty relation for angular momentum observables, sum uncertainty relations for N-incompatible observables, uncertainty relations for noise and disturbance and also uncertainty relations for non-Hermitian operators.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.
- In this sense, uncertainty relations in terms of the sum of variances capture the concept of incompatibility more accurately.
- It may be noted that earlier uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables, entropic uncertainty relations, sum uncertainty relation for angular momentum observables, sum uncertainty relations for N-incompatible observables, uncertainty relations for noise and disturbance and also uncertainty relations for non-Hermitian operators.

Uncertainty relations

- Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms— product form and sum form.
- Although, both of these kinds of uncertainty relations express limitations in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute.
- In this sense, uncertainty relations in terms of the sum of variances capture the concept of incompatibility more accurately.
- It may be noted that earlier uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables, entropic uncertainty relations, sum uncertainty relation for angular momentum observables, sum uncertainty relations for N-incompatible observables, uncertainty relations for noise and disturbance and also uncertainty relations for non-Hermitian operators.
- Recently, experiments have also been performed to test various uncertainty relations.

Stronger Uncertainty Relations

Stronger Uncertainty Relations

- Stronger Uncertainty Relation 1

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle\psi|[A,B]|\psi\rangle + \left|\langle\psi|(A \pm iB)|\psi^{\perp}\rangle\right|^2 \tag{2}$$

which is valid for arbitrary states $|\psi^{\perp}\rangle$ orthogonal to the state of the system $|\psi\rangle$, where the sign should be chosen so that $\pm i\langle[A,B]\rangle$ (a real quantity) is positive.

Stronger Uncertainty Relations

- Stronger Uncertainty Relation 1

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle\psi|[A,B]|\psi\rangle + \left|\langle\psi|(A \pm iB)|\psi^{\perp}\rangle\right|^2 \tag{2}$$

  which is valid for arbitrary states $|\psi^{\perp}\rangle$ orthogonal to the state of the system $|\psi\rangle$, where the sign should be chosen so that $\pm i\langle[A,B]\rangle$ (a real quantity) is positive.

- The lower bound in UR1 is nonzero for almost any choice of $|\psi^{\perp}\rangle$ if $|\psi\rangle$ is not a common eigenstate of A and B. We can choose $|\psi^{\perp}\rangle$ that is orthogonal to $|\psi\rangle$ but not orthogonal to the state $(A \pm iB)|\psi\rangle$. Such a choice is always possible unless $|\psi\rangle$ is a joint eigenstate of A and B.

Stronger Uncertainty Relations

- Stronger Uncertainty Relation 1

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle\psi|[A,B]|\psi\rangle + \left|\langle\psi|(A \pm iB)|\psi^{\perp}\rangle\right|^2 \tag{2}$$

  which is valid for arbitrary states $|\psi^{\perp}\rangle$ orthogonal to the state of the system $|\psi\rangle$, where the sign should be chosen so that $\pm i\langle[A,B]\rangle$ (a real quantity) is positive.

- The lower bound in UR1 is nonzero for almost any choice of $|\psi^{\perp}\rangle$ if $|\psi\rangle$ is not a common eigenstate of A and B. We can choose $|\psi^{\perp}\rangle$ that is orthogonal to $|\psi\rangle$ but not orthogonal to the state $(A \pm iB)|\psi\rangle$. Such a choice is always possible unless $|\psi\rangle$ is a joint eigenstate of A and B.

L. Maccone and A. K. Pati, Phys. Rev. Lett. **113**, 2604401 (2014)

- How to choose $|\psi^{\perp}\rangle$: if $|\psi\rangle$ is an eigenstate of A one can choose $|\psi^{\perp}\rangle = (B - \langle B\rangle)|\psi\rangle/\Delta B \equiv |\psi_B^{\perp}\rangle$, or $|\psi^{\perp}\rangle = (A - \langle A\rangle)|\psi\rangle/\Delta A \equiv |\psi_A^{\perp}\rangle$ if $|\psi\rangle$ is an eigenstate of B.

- How to choose $|\psi^{\perp}\rangle$: if $|\psi\rangle$ is an eigenstate of A one can choose $|\psi^{\perp}\rangle = (B - \langle B\rangle)|\psi\rangle/\Delta B \equiv |\psi_B^{\perp}\rangle$, or $|\psi^{\perp}\rangle = (A - \langle A\rangle)|\psi\rangle/\Delta A \equiv |\psi_A^{\perp}\rangle$ if $|\psi\rangle$ is an eigenstate of B.

- If $|\psi\rangle$ is not an eigenstate of either and $|\psi_A^{\perp}\rangle \neq |\psi_B^{\perp}\rangle$, one can choose $|\psi^{\perp}\rangle \propto (I - |\psi_B^{\perp}\rangle\langle\psi_B^{\perp}|)|\psi_A^{\perp}\rangle$, or $|\psi^{\perp}\rangle = |\psi_A^{\perp}\rangle$ if $|\psi_A^{\perp}\rangle = |\psi_B^{\perp}\rangle$.

- How to choose $|\psi^{\perp}\rangle$: if $|\psi\rangle$ is an eigenstate of A one can choose $|\psi^{\perp}\rangle = (B - \langle B\rangle)|\psi\rangle/\Delta B \equiv |\psi_B^{\perp}\rangle$, or $|\psi^{\perp}\rangle = (A - \langle A\rangle)|\psi\rangle/\Delta A \equiv |\psi_A^{\perp}\rangle$ if $|\psi\rangle$ is an eigenstate of B.

- If $|\psi\rangle$ is not an eigenstate of either and $|\psi_A^{\perp}\rangle \neq |\psi_B^{\perp}\rangle$, one can choose $|\psi^{\perp}\rangle \propto (I - |\psi_B^{\perp}\rangle\langle\psi_B^{\perp}|)|\psi_A^{\perp}\rangle$, or $|\psi^{\perp}\rangle = |\psi_A^{\perp}\rangle$ if $|\psi_A^{\perp}\rangle = |\psi_B^{\perp}\rangle$.

  An optimization of $|\psi^{\perp}\rangle$ (namely, the choice that maximizes the lower bound), will saturate the inequality: it becomes an equality.

Stronger Uncertainty Relation 2

Stronger Uncertainty Relation 2

- A second inequality with nontrivial bound even if $|\psi\rangle$ is an eigenstate either of A or of B is

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}|\langle\psi^{\perp}_{A+B}|(A+B)|\psi\rangle|^2 \tag{3}$$

where $|\psi^{\perp}_{A+B}\rangle \propto (A + B - \langle A + B\rangle)|\psi\rangle$ is a state orthogonal to $|\psi\rangle$ (where $\langle O\rangle$ denotes the expectation value of the observable O).

Stronger Uncertainty Relation 2

- A second inequality with nontrivial bound even if $|\psi\rangle$ is an eigenstate either of A or of B is

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}|\langle\psi^{\perp}_{A+B}|(A+B)|\psi\rangle|^2 \tag{3}$$

where $|\psi^{\perp}_{A+B}\rangle \propto (A + B - \langle A+B\rangle)|\psi\rangle$ is a state orthogonal to $|\psi\rangle$ (where $\langle O\rangle$ denotes the expectation value of the observable O).

- The form of $|\psi^{\perp}_{A+B}\rangle$ implies that the right-hand-side is nonzero unless $|\psi\rangle$ is an eigenstate of $(A+B)$.

- Both inequalities can be combined in a single uncertainty relation for the sum of variances:

$$\Delta A^2 + \Delta B^2 \geq \max(\mathcal{L}_{(1)}, \mathcal{L}_{(2)}) \tag{4}$$

with $\mathcal{L}_{(1),(2)}$ the right-hand-side of UR1 and UR2, respectively.

- Both inequalities can be combined in a single uncertainty relation for the sum of variances:

$$\Delta A^2 + \Delta B^2 \geq \max(\mathcal{L}_{(1)}, \mathcal{L}_{(2)}) \tag{4}$$

  with $\mathcal{L}_{(1),(2)}$ the right-hand-side of UR1 and UR2, respectively.

- Remarks: (i) UR1 and UR2 involve the sum of variances, so one must introduce some dimensional constants in the case in which A and B are measured with different units; (ii) removing the last term in UR1, we have $\Delta A^2 + \Delta B^2 \geq |\langle\psi|[A, B]|\psi\rangle|$ which is also implied by the Heisenberg-Robertson relation

- To prove UR1 use the parallelogram law in Hilbert space

$$2\Delta A^2 + 2\Delta B^2 = \|C + \alpha D|\psi\rangle\|^2 + \|C - \alpha D|\psi\rangle\|^2, \tag{5}$$

for $C = A - \langle A\rangle$, $D = B - \langle B\rangle$, and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

- Take $\alpha = \pm i$ and note that

$$\|(C \pm iD)|\psi\rangle\|^2 = \Delta A^2 + \Delta B^2 \pm i\langle[A, B]\rangle \tag{6}$$

- Second term can be lower bounded through the Schwarz inequality as

$$\begin{aligned} |\langle\psi|(A \pm iB)|\psi^\perp\rangle|^2 &= |\langle\psi|(A \pm iB - \langle A \pm iB\rangle|\psi^\perp\rangle|^2 \\ &= |\langle\psi|(C \pm iD)|\psi^\perp\rangle|^2 \leq \|(C \mp iD)|\psi\rangle\|^2 \end{aligned} \tag{7}$$

valid for all $|\psi^\perp\rangle$ orthogonal to $|\psi\rangle$. Hence the proof.
The equality condition for follows from the equality condition of the Schwarz inequality, namely iff $|\psi^\perp\rangle \propto (A \mp iB - \langle(A \mp iB)\rangle)|\psi\rangle$.

- To prove UR1 use the parallelogram law in Hilbert space

$$2\Delta A^2 + 2\Delta B^2 = \|C + \alpha D|\psi\rangle\|^2 + \|C - \alpha D|\psi\rangle\|^2, \tag{5}$$

  for $C = A - \langle A\rangle$, $D = B - \langle B\rangle$, and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.
- Take $\alpha = \pm i$ and note that

$$\|(C \pm iD)|\psi\rangle\|^2 = \Delta A^2 + \Delta B^2 \pm i\langle[A, B]\rangle \tag{6}$$

- Second term can be lower bounded through the Schwarz inequality as

$$\begin{aligned} |\langle\psi|(A \pm iB)|\psi^\perp\rangle|^2 &= |\langle\psi|(A \pm iB - \langle A \pm iB\rangle|\psi^\perp\rangle|^2 \\ &= |\langle\psi|(C \pm iD)|\psi^\perp\rangle|^2 \leq \|(C \mp iD)|\psi\rangle\|^2 \end{aligned} \tag{7}$$

  valid for all $|\psi^\perp\rangle$ orthogonal to $|\psi\rangle$. Hence the proof.
  The equality condition for follows from the equality condition of the Schwarz inequality, namely iff $|\psi^\perp\rangle \propto (A \mp iB - \langle(A \mp iB)\rangle)|\psi\rangle$.

- To prove UR2 we use the parallelogram law in Hilbert space to obtain

$$2\Delta A^2 + 2\Delta B^2 = \|C + \alpha D|\psi\rangle\|^2 + \|C - \alpha D|\psi\rangle\|^2, \tag{8}$$

for $C = A - \langle A\rangle$, $D = B - \langle B\rangle$, and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Since $\Delta(A+B) = \|(C+D)|\psi\rangle\|$, $\Delta(A-B) = \|(C-D)|\psi\rangle\|$, for $\alpha = 1$ is equal to

$$\begin{aligned} \Delta A^2 + \Delta B^2 &= \frac{1}{2}[\Delta(A+B)^2 + \Delta(A-B)^2] \\ &\geq \frac{1}{2}\Delta(A+B)^2, \end{aligned} \tag{9}$$

which is equivalent to UR2 since $\Delta(A+B)^2 = |\langle\psi^{\perp}_{A+B}|(A+B)|\psi\rangle|^2$.

- For any observable O

$$O|\psi\rangle = \langle O\rangle|\psi\rangle + \Delta O|\psi^{\perp}_{O}\rangle. \tag{10}$$

- Hence, we have

$$\Delta O^2 = |\langle\psi^{\perp}_{O}|O|\psi\rangle|^2 \tag{11}$$

- The equality condition for UR2: $|\psi\rangle$ must be an eigenstate of $A - B$. Also, note that the lower bound in UR2 is nonzero unless $|\psi\rangle$ is an eigenstate of $(A + B)$.

- The equality condition for UR2: $|\psi\rangle$ must be an eigenstate of $A - B$. Also, note that the lower bound in UR2 is nonzero unless $|\psi\rangle$ is an eigenstate of $(A + B)$.

- Clearly $|\psi\rangle$ can be an eigenstate of $A + B$ without being an eigenstate of either A or B, but in the interesting case when $|\psi\rangle$ is an eigenstate of one of the two (which trivializes both Heisenberg's and Schrödinger's uncertainty relations), the lower bound must be nonzero unless $|\psi\rangle$ is an eigenstate of both.

- The equality condition for UR2: $|\psi\rangle$ must be an eigenstate of $A - B$. Also, note that the lower bound in UR2 is nonzero unless $|\psi\rangle$ is an eigenstate of $(A + B)$.

- Clearly $|\psi\rangle$ can be an eigenstate of $A + B$ without being an eigenstate of either A or B, but in the interesting case when $|\psi\rangle$ is an eigenstate of one of the two (which trivializes both Heisenberg's and Schrödinger's uncertainty relations), the lower bound must be nonzero unless $|\psi\rangle$ is an eigenstate of both.

- It is also easy to modify the inequality UR2 so that it has always a nontrivial lower bound except when $|\psi\rangle$ is a joint eigenstate of A and B, namely

$$\begin{aligned}\Delta A^2 + \Delta B^2 \quad &\geq \quad \max\Big(\frac{1}{2}\langle\psi^{\perp}_{A+B}|(A+B)|\psi\rangle|^2, |\langle\psi^{\perp}_{A}|A|\psi\rangle|^2, \\ &|\langle\psi^{\perp}_{B}|B|\psi\rangle|^2\Big) \qquad (12)\end{aligned}$$

Example

Example

- Stronger uncertainty relations suggest that the quantum world is more uncertain than what the Heisenberg-Robertson uncertainty relation makes us believe.

Example

- Stronger uncertainty relations suggest that the quantum world is more uncertain than what the Heisenberg-Robertson uncertainty relation makes us believe.

- Harmonic Oscillator Hamiltonian $H = \frac{1}{2}(P^2 + X^2)$ $(\hbar = 1, m = \omega = 1)$.
- For any state $|\psi\rangle$, the stronger uncertainty relation for the position and momentum

$$\Delta X^2 + \Delta P^2 \geq 1 + \left| \sqrt{2} \langle \psi | a^\dagger | \psi^\perp \rangle \right|^2, \tag{13}$$

where $a^\dagger = \frac{1}{\sqrt{2}}(X - iP)$ is the annihilation operator.

Example

- Stronger uncertainty relations suggest that the quantum world is more uncertain than what the Heisenberg-Robertson uncertainty relation makes us believe.

- Harmonic Oscillator Hamiltonian $H = \frac{1}{2}(P^2 + X^2)$ ($\hbar = 1, m = \omega = 1$).
- For any state $|\psi\rangle$, the stronger uncertainty relation for the position and momentum

$$\Delta X^2 + \Delta P^2 \geq 1 + \left|\sqrt{2}\langle\psi|a^\dagger|\psi^\perp\rangle\right|^2, \tag{13}$$

where $a^\dagger = \frac{1}{\sqrt{2}}(X - iP)$ is the annihilation operator.

- From the Heisenberg-Robertson UR:

$$\Delta X^2 + \Delta P^2 \geq 1 \tag{14}$$

- If $|\psi\rangle = |n\rangle$ is the nth eigenstate of the HO Hamiltonian, then we can choose $|\psi^{\perp}\rangle$ as $|n-1\rangle$. UR1 implies

$$\Delta X^2 + \Delta P^2 \geq (1 + 2n) \tag{15}$$

- This is stronger than what is implied by the Heisenberg-Robertson uncertainty relation.
- For $|\psi\rangle = |n\rangle$, $\Delta X^2 = (n + \frac{1}{2})$ and $\Delta P^2 = (n + \frac{1}{2})$. This shows that *all* eigenstates of HO Hamiltonian saturate the stronger UR.
- Ground state of HO Hamiltonian is a minimum uncertainty state with respect to the Heisenberg-Robertson UR. However, all the eigenstates of HO Hamiltonian are minimum uncertainty states with respect to the stronger UR1.

Measurement Uncertainty and Preparation Uncertainty

Measurement Uncertainty and Preparation Uncertainty

- Heisenberg's original principle is about measurement uncertainty.
- Robertson-Schrödinger relation is about preparation uncertainty.
- Recently, there has been an interesting and lively debate on how to interpret the uncertainty principle [Ozawa, Busch-Lahti-Werner].
- To elucidate the relation between these results and ours, we introduce Peres' nomenclature that distinguishes between uncertainty *relation* and uncertainty *principle*.
- The former refers solely to the preparation of the system which induces a spread in the measurement outcomes, and does not refer to the disturbance induced by the measurement or to joint measurements. [Peres]

  "The only correct interpretation of [the uncertainty relations for x and p] is the following: If the same preparation procedure is repeated many times, and is followed either by a measurement of x, or by a measurement of p, the various results obtained for x and for p have standard deviations, Δx and Δp, whose product cannot be less than $\hbar/2$.

- There never is any question here that a measurement of x 'disturbs' the value of p and vice-versa, as sometimes claimed. These measurements are indeed incompatible, but they are performed on *different* particles (all of which were identically prepared) and therefore these measurements cannot disturb each other in any way.

- There never is any question here that a measurement of x 'disturbs' the value of p and vice-versa, as sometimes claimed. These measurements are indeed incompatible, but they are performed on *different* particles (all of which were identically prepared) and therefore these measurements cannot disturb each other in any way.

- The uncertainty relation reflects the intrinsic randomness of the outcomes of quantum tests.
- We emphasize that the uncertainty *relation* must not be confused with the uncertainty *principle*.
- The latter entails also the measurement disturbance by the apparatus and the impossibility of joint measurements of incompatible observables.

Applications

Applications

- Peng Xue group have tested our stronger uncertainty relatons for 3-dimensional system and have confirmed both relations UR1 and UR2.

Applications

- Peng Xue group have tested our stronger uncertainty relatons for 3-dimensional system and have confirmed both relations UR1 and UR2.

  Experimental investigation of the stronger uncertainty relations for all incompatible observables
  Kunkun Wang, Xiang Zhan, Zhihao Bian, Jian Li, Yongsheng Zhang, and Peng Xue Phys. Rev. A **93**, 052108 (2016).
- Stronger Error Disturbance Relations for Incompatible Quantum Measurements C. Mukhopadhyay, N. Shukla, A. K. Pati, Europhysics Letters **113**, 50002 (2016)
- Tighter Einstein-Podolsky-Rosen steering inequality based on the sum uncertainty relation A. G. Maity, S. Datta, A. S. Majumdar, Phys. Rev. A **96**, 052326 (2017)

Tighter Uncertainty relations

- Here, we provide a tighter UR compared to the Robertson-Schrödinger uncertainty relation.

Tighter Uncertainty relations

- Here, we provide a tighter UR compared to the Robertson-Schrödinger uncertainty relation.

Tighter Uncertainty relations

- Here, we provide a tighter UR compared to the Robertson-Schrödinger uncertainty relation.

- Let us consider two observables A and B in their eigenbasis as $A = \sum_i a_i|a_i\rangle\langle a_i|$ and $B = \sum_i b_i|b_i\rangle\langle b_i|$. Let us define $(A - \langle A\rangle) = \overline{A} = \sum_i \tilde{a}_i|a_i\rangle\langle a_i|$ and $(B - \langle B\rangle) = \overline{B} = \sum_i \tilde{b}_i|b_i\rangle\langle b_i|$. We express $|f\rangle = \overline{A}|\Psi\rangle$ and $|g\rangle = \overline{B}|\Psi\rangle$ as $|f\rangle = \sum_n \alpha_n|\psi_n\rangle$ and $|g\rangle = \sum_n \beta_n|\psi_n\rangle$, where $\{|\psi_n\rangle\}$ is an arbitrary complete orthonormal basis.

Tighter Uncertainty relations

- Here, we provide a tighter UR compared to the Robertson-Schrödinger uncertainty relation.

- Let us consider two observables A and B in their eigenbasis as $A = \sum_i a_i |a_i\rangle\langle a_i|$ and $B = \sum_i b_i |b_i\rangle\langle b_i|$. Let us define $(A - \langle A\rangle) = \overline{A} = \sum_i \tilde{a}_i |a_i\rangle\langle a_i|$ and $(B - \langle B\rangle) = \overline{B} = \sum_i \tilde{b}_i |b_i\rangle\langle b_i|$. We express $|f\rangle = \overline{A}|\Psi\rangle$ and $|g\rangle = \overline{B}|\Psi\rangle$ as $|f\rangle = \sum_n \alpha_n |\psi_n\rangle$ and $|g\rangle = \sum_n \beta_n |\psi_n\rangle$, where $\{|\psi_n\rangle\}$ is an arbitrary complete orthonormal basis.

Tighter Uncertainty relations

- Here, we provide a tighter UR compared to the Robertson-Schrödinger uncertainty relation.

- Let us consider two observables A and B in their eigenbasis as $A = \sum_i a_i |a_i\rangle\langle a_i|$ and $B = \sum_i b_i |b_i\rangle\langle b_i|$. Let us define $(A - \langle A\rangle) = \overline{A} = \sum_i \tilde{a}_i |a_i\rangle\langle a_i|$ and $(B - \langle B\rangle) = \overline{B} = \sum_i \tilde{b}_i |b_i\rangle\langle b_i|$. We express $|f\rangle = \overline{A}|\Psi\rangle$ and $|g\rangle = \overline{B}|\Psi\rangle$ as $|f\rangle = \sum_n \alpha_n |\psi_n\rangle$ and $|g\rangle = \sum_n \beta_n |\psi_n\rangle$, where $\{|\psi_n\rangle\}$ is an arbitrary complete orthonormal basis.

- New uncertainty relation:

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}\Big(\sum_n \Big| \langle [\overline{A}, \overline{B}_n^{\psi}] \rangle_\Psi + \langle \{\overline{A}, \overline{B}_n^{\psi}\} \rangle_\Psi \Big| \Big)^2. \qquad (16)$$

Tighter uncertainty relations

- Using the Cauchy-Schwarz inequality for two real vectors $\vec{\alpha} = \Big(|\alpha_1|, |\alpha_2|, |\alpha_3|, ...\Big)$, $\vec{\beta} = \Big(|\beta_1|, |\beta_2|, |\beta_3|, ...\Big)$, we have

$$
\begin{aligned}
\Delta A^2 \Delta B^2 &= \langle f|f\rangle\langle g|g\rangle = \sum_{n,m} |\alpha_n|^2 |\beta_m|^2 \\
&\geq \Big(\sum_n |\alpha_n||\beta_n|\Big)^2 = \Big(\sum_n |\alpha_n^* \beta_n|\Big)^2 \\
&= \Big(\sum_n |\langle\Psi|\overline{A}|\psi_n\rangle\langle\psi_n|\overline{B}|\Psi\rangle|\Big)^2 \\
&= \Big(\sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\Big)^2, \qquad (17)
\end{aligned}
$$

where $\overline{B}_n^{\psi} = |\psi_n\rangle\langle\psi_n|\overline{B}$, $\alpha_n = \langle\psi_n|\overline{A}|\Psi\rangle$ and $\beta_n = \langle\psi_n|\overline{B}|\Psi\rangle$.

Tighter uncertainty relations

- Using the Cauchy-Schwarz inequality for two real vectors $\vec{\alpha} = \Big(|\alpha_1|, |\alpha_2|, |\alpha_3|, ...\Big)$, $\vec{\beta} = \Big(|\beta_1|, |\beta_2|, |\beta_3|, ...\Big)$, we have

$$\begin{aligned}
\Delta A^2 \Delta B^2 &= \langle f|f\rangle\langle g|g\rangle = \sum_{n,m} |\alpha_n|^2|\beta_m|^2 \\
&\geq \Big(\sum_n |\alpha_n||\beta_n|\Big)^2 = \Big(\sum_n |\alpha_n^* \beta_n|\Big)^2 \\
&= \Big(\sum_n |\langle\Psi|\overline{A}|\psi_n\rangle\langle\psi_n|\overline{B}|\Psi\rangle|\Big)^2 \\
&= \Big(\sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\Big)^2, \qquad (17)
\end{aligned}$$

where $\overline{B}_n^{\psi} = |\psi_n\rangle\langle\psi_n|\overline{B}$, $\alpha_n = \langle\psi_n|\overline{A}|\Psi\rangle$ and $\beta_n = \langle\psi_n|\overline{B}|\Psi\rangle$.

- On expressing $\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle = \frac{1}{2}\big(\langle[\overline{A}, \overline{B}_n^{\psi}]\rangle_\Psi + \langle\{\overline{A}, \overline{B}_n^{\psi}\}\rangle_\Psi\big)$, the new uncertainty relation can be written as

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}\Big(\sum_n \Big|\langle[\overline{A}, \overline{B}_n^{\psi}]\rangle_\Psi + \langle\{\overline{A}, \overline{B}_n^{\psi}\}\rangle_\Psi\Big|\Big)^2. \qquad (18)$$

Tighter uncertainty relations

- The new uncertainty relation is tighter than the Robertson-Schrödinger uncertainty relation. To prove this let us start with the right hand side of Eq. (17) and note that

$$\begin{aligned} \Big(\quad \sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\Big)^2 &\geq \Big|\sum_n \langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle\Big|^2 \\ = \quad \Big|\langle\Psi|\overline{A}\,\overline{B}|\Psi\rangle\Big|^2, \end{aligned} \tag{19}$$

where we have used the fact that $|\sum_i z_i|^2 \leq (\sum_i |z_i|)^2$, $z_i \in \mathbb{C}$ for all i. Here, the last line in Eq. (19) is nothing but the bound obtained in Eq. (**??**).

Tighter uncertainty relations

- The new uncertainty relation is tighter than the Robertson-Schrödinger uncertainty relation. To prove this let us start with the right hand side of Eq. (17) and note that

$$\begin{aligned} \left(\sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\right)^2 &\geq \left|\sum_n \langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle\right|^2 \\ &= \left|\langle\Psi|\overline{A}\,\overline{B}|\Psi\rangle\right|^2, \end{aligned} \tag{19}$$

  where we have used the fact that $|\sum_i z_i|^2 \leq (\sum_i |z_i|)^2$, $z_i \in \mathbb{C}$ for all i. Here, the last line in Eq. (19) is nothing but the bound obtained in Eq. (**??**).
- Thus, our bound is indeed tighter than the Robertson-Schrödinger uncertainty relation.

Tighter uncertainty relations

- The new uncertainty relation is tighter than the Robertson-Schrödinger uncertainty relation. To prove this let us start with the right hand side of Eq. (17) and note that

$$\begin{aligned} \Big(\quad \sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\Big)^2 &\geq \Big|\sum_n \langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle\Big|^2 \\ = \quad & \Big|\langle\Psi|\overline{A}\,\overline{B}|\Psi\rangle\Big|^2, \end{aligned} \tag{19}$$

  where we have used the fact that $|\sum_i z_i|^2 \leq (\sum_i |z_i|)^2$, $z_i \in \mathbb{C}$ for all i. Here, the last line in Eq. (19) is nothing but the bound obtained in Eq. (**??**).
- Thus, our bound is indeed tighter than the Robertson-Schrödinger uncertainty relation.
- This uncertainty relation in Eq. (18) can further be tightened by optimizing over the sets of complete orthonormal bases as

$$\Delta A^2 \Delta B^2 \geq \max_{\{|\psi_n\rangle\}} \frac{1}{4}\Big(\sum_n \Big|\langle[\overline{A},\overline{B}_n^{\psi}]\rangle_\Psi + \langle\{\overline{A},\overline{B}_n^{\psi}\}\rangle_\Psi\Big|\Big)^2. \tag{20}$$

Tighter uncertainty relations

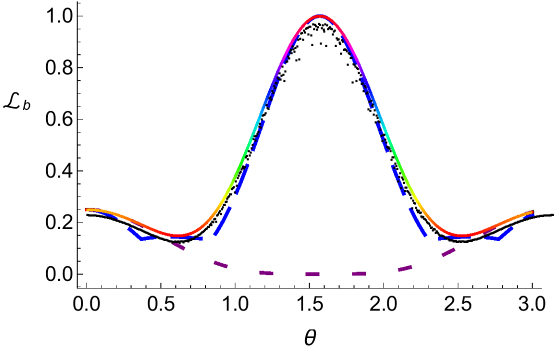


Figure : Here, we plot the lower bound of the product of variances of two incompatible observables, $A = L_x$ and $B = L_y$, two components of the angular momentum for spin 1 particle with a state $|\Psi\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle$, where the state $|1\rangle$ and $|0\rangle$ are the eigenstates of L_z corresponding to eigenvalues 1 and 0 respectively. The long dashed (blue colored) line shows the lower bound of the product of variances given by (21), the flattest (purple colored, tiny dashed) curve stands for the bound given by Schrödinger uncertainty relation given by Eq. (**??**) and the continuous line (hue colored) plot denotes the product of two variances. Scattered black points denote the optimized uncertainty bound achieved by Eq. (20).

As shown above, an optimization over different bases indeed gives tighter bound.

Tighter uncertainty relations

- Next, we derive an optimization-free uncertainty relation for two incompatible observables. For that we consider (say) $\overline{A}^2 = \sum_{i,j}(a_i - a_j F_\Psi^{a_j})^2 |a_i\rangle\langle a_i| = \sum_i (\tilde{a}_i)^2 |a_i\rangle\langle a_i|$ and $\overline{B}^2 = \sum_{i,j}(b_i - b_j F_\Psi^{b_j})^2 |b_i\rangle\langle b_i| = \sum_i (\tilde{b}_i)^2 |b_i\rangle\langle b_i|$, where F_Ψ^x is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ ($|x\rangle = |a_i\rangle, |b_i\rangle$), $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$.

Tighter uncertainty relations

- Next, we derive an optimization-free uncertainty relation for two incompatible observables. For that we consider (say) $\overline{A}^2 = \sum_{i,j}(a_i - a_j F_{\Psi}^{a_j})^2 |a_i\rangle\langle a_i| = \sum_i (\tilde{a}_i)^2 |a_i\rangle\langle a_i|$ and $\overline{B}^2 = \sum_{i,j}(b_i - b_j F_{\Psi}^{b_j})^2 |b_i\rangle\langle b_i| = \sum_i (\tilde{b}_i)^2 |b_i\rangle\langle b_i|$, where F_{Ψ}^{x} is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ $(|x\rangle = |a_i\rangle, |b_i\rangle)$, $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$.
- Using the Cauchy-Schwarz inequality, we obtain

$$\Delta A^2 \Delta B^2 \geq \left(\sum_i \sqrt{F_{\Psi}^{a_i}} \sqrt{F_{\Psi}^{b_i}} \tilde{a}_i \tilde{b}_i \right)^2, \tag{21}$$

Tighter uncertainty relations

- Next, we derive an optimization-free uncertainty relation for two incompatible observables. For that we consider (say)
$\overline{A}^2 = \sum_{i,j}(a_i - a_j F_{\Psi}^{a_j})^2 |a_i\rangle\langle a_i| = \sum_i (\tilde{a}_i)^2 |a_i\rangle\langle a_i|$ and
$\overline{B}^2 = \sum_{i,j}(b_i - b_j F_{\Psi}^{b_j})^2 |b_i\rangle\langle b_i| = \sum_i (\tilde{b}_i)^2 |b_i\rangle\langle b_i|$, where F_{Ψ}^{x} is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ $(|x\rangle = |a_i\rangle, |b_i\rangle)$, $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$.
- Using the Cauchy-Schwarz inequality, we obtain

$$\Delta A^2 \Delta B^2 \geq \left(\sum_i \sqrt{F_{\Psi}^{a_i}} \sqrt{F_{\Psi}^{b_i}} \tilde{a}_i \tilde{b}_i \right)^2, \tag{21}$$

- To prove this use the inequality for two real vectors $\vec{u}$ and $\vec{v}$ defined as
$\vec{u} = \left(\tilde{a}_1 \sqrt{F_{\Psi}^{a_1}}, \tilde{a}_2 \sqrt{F_{\Psi}^{a_2}}, \tilde{a}_3 \sqrt{F_{\Psi}^{a_3}}, \ldots\right)$, $\vec{v} = \left(\tilde{b}_1 \sqrt{F_{\Psi}^{b_1}}, \tilde{b}_2 \sqrt{F_{\Psi}^{b_2}}, \tilde{b}_3 \sqrt{F_{\Psi}^{b_3}}, \ldots\right)$ and the quantities $\sqrt{F_{\Psi}^{a_i}}\tilde{a}_i$, $\sqrt{F_{\Psi}^{b_i}}\tilde{b}_i$ are arranged such that $\sqrt{F_{\Psi}^{a_{i+1}}}\tilde{a}_{i+1} \geq \sqrt{F_{\Psi}^{a_i}}\tilde{a}_i$ and $\sqrt{F_{\Psi}^{b_{i+1}}}\tilde{b}_{i+1} \geq \sqrt{F_{\Psi}^{b_i}}\tilde{b}_i$.

Tighter uncertainty relations

- Next, we derive an optimization-free uncertainty relation for two incompatible observables. For that we consider (say) $\overline{A}^2 = \sum_{i,j}(a_i - a_j F_{\Psi}^{a_j})^2 |a_i\rangle\langle a_i| = \sum_i (\tilde{a}_i)^2 |a_i\rangle\langle a_i|$ and $\overline{B}^2 = \sum_{i,j}(b_i - b_j F_{\Psi}^{b_j})^2 |b_i\rangle\langle b_i| = \sum_i (\tilde{b}_i)^2 |b_i\rangle\langle b_i|$, where F_{Ψ}^{x} is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ $(|x\rangle = |a_i\rangle, |b_i\rangle)$, $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$.
- Using the Cauchy-Schwarz inequality, we obtain

$$\Delta A^2 \Delta B^2 \geq \left(\sum_i \sqrt{F_{\Psi}^{a_i}} \sqrt{F_{\Psi}^{b_i}} \tilde{a}_i \tilde{b}_i \right)^2, \tag{21}$$

- To prove this use the inequality for two real vectors $\vec{u}$ and $\vec{v}$ defined as $\vec{u} = \left(\tilde{a}_1\sqrt{F_{\Psi}^{a_1}}, \tilde{a}_2\sqrt{F_{\Psi}^{a_2}}, \tilde{a}_3\sqrt{F_{\Psi}^{a_3}}, \ldots\right)$, $\vec{v} = \left(\tilde{b}_1\sqrt{F_{\Psi}^{b_1}}, \tilde{b}_2\sqrt{F_{\Psi}^{b_2}}, \tilde{b}_3\sqrt{F_{\Psi}^{b_3}}, \ldots\right)$ and the quantities $\sqrt{F_{\Psi}^{a_i}}\tilde{a}_i$, $\sqrt{F_{\Psi}^{b_i}}\tilde{b}_i$ are arranged such that $\sqrt{F_{\Psi}^{a_{i+1}}}\tilde{a}_{i+1} \geq \sqrt{F_{\Psi}^{a_i}}\tilde{a}_i$ and $\sqrt{F_{\Psi}^{b_{i+1}}}\tilde{b}_{i+1} \geq \sqrt{F_{\Psi}^{b_i}}\tilde{b}_i$.
- This new uncertainty relation depends on the transition probability between the state of the system and the eigenbases of the observables. The incompatibility is captured here not by the non-commutativity, rather by the non-orthogonality of the state of the system $|\Psi\rangle$ and the eigenbases of the observables $|a_i\rangle$ and $|b_i\rangle$.

Tighter uncertainty relations

- We use the the parallelogram law for two real vectors to improve the bound on the sum of variances for two incompatible observables.

Tighter uncertainty relations

- We use the the parallelogram law for two real vectors to improve the bound on the sum of variances for two incompatible observables.
- Using the parallelogram law for two real vectors $\vec{u}$ and $\vec{v}$, one can derive a lower bound on the sum of variances of two observables as

$$\Delta A^2 + \Delta B^2 \quad \geq \quad \frac{1}{2} \sum_i \left(\tilde{a}_i \sqrt{F_{\Psi}^{a_i}} + \tilde{b}_i \sqrt{F_{\Psi}^{b_i}} \right)^2 . \tag{22}$$

This is one of the tightest optimization free bound.

Tighter uncertainty relations

- We use the the parallelogram law for two real vectors to improve the bound on the sum of variances for two incompatible observables.
- Using the parallelogram law for two real vectors $\vec{u}$ and $\vec{v}$, one can derive a lower bound on the sum of variances of two observables as

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \sum_i \left(\tilde{a}_i \sqrt{F_\Psi^{a_i}} + \tilde{b}_i \sqrt{F_\Psi^{b_i}} \right)^2. \tag{22}$$

  This is one of the tightest optimization free bound.
- If one allows the optimization over a set of states, then the procedure used to derive the uncertainty relation given in Eq. (20) can be used to derive another set of uncertainty relations using the parallelogram law for two real vectors $\vec{\alpha}$ and $\vec{\beta}$.

Reverse uncertainty relations

- Does quantum mechanics restrict upper limit to the product and sum of variances of two incompatible observables?

Reverse uncertainty relations

• Does quantum mechanics restrict upper limit to the product and sum of variances of two incompatible observables?

• Here, for the first time, we introduce the reverse bound, i.e., the upper bound to the product and the sum of variances of two incompatible observables.

Reverse uncertainty relations

• Does quantum mechanics restrict upper limit to the product and sum of variances of two incompatible observables?

• Here, for the first time, we introduce the reverse bound, i.e., the upper bound to the product and the sum of variances of two incompatible observables.

• To prove the reverse uncertainty relation for the product of variances of two observables, we use the reverse Cauchy-Schwarz inequality for positive real numbers.

Reverse uncertainty relations

• Does quantum mechanics restrict upper limit to the product and sum of variances of two incompatible observables?

• Here, for the first time, we introduce the reverse bound, i.e., the upper bound to the product and the sum of variances of two incompatible observables.

• To prove the reverse uncertainty relation for the product of variances of two observables, we use the reverse Cauchy-Schwarz inequality for positive real numbers.

• This states that *for two sets of positive real numbers* $c_1, ..., c_n$ *and* $d_1, ...d_n$*, if* $0 < c \leq c_i \leq C < \infty$*,* $0 < d \leq d_i \leq D < \infty$ *for some constants* c*,* d*,* C *and* D *for all* $i = 1, ...n$*, then*

$$\sum_{i,j} c_i^2 d_j^2 \leq \frac{(CD + cd)^2}{4cdCD} \Big(\sum_i c_i d_i \Big)^2. \tag{23}$$

Reverse uncertainty relations

- Using this inequality for $c_i = \sqrt{F_\Psi^{a_i}}|\tilde{a}_i|$ and $d_i = \sqrt{F_\Psi^{b_i}}|\tilde{b}_i|$, one can show that the product of variances of two observables satisfies the relation

$$\Delta A^2 \Delta B^2 \leq \Omega_{ab}^{\Psi} \left(\sum_i \sqrt{F_\Psi^{a_i}} \sqrt{F_\Psi^{b_i}} |\tilde{a}_i||\tilde{b}_i| \right)^2, \quad (24)$$

where $\Omega_{ab}^{\Psi} = \frac{\left(M_\Psi^a M_\Psi^b + m_\Psi^a m_\Psi^b\right)^2}{4 M_\Psi^a M_\Psi^b m_\Psi^a m_\Psi^b}$ with $M_\Psi^a = \max\{\sqrt{F_\Psi^{a_i}}|\tilde{a}_i|\}$, $m_\Psi^a = \min\{\sqrt{F_\Psi^{a_i}}|\tilde{a}_i|\}$, $M_\Psi^b = \max\{\sqrt{F_\Psi^{b_i}}|\tilde{b}_i|\}$ and $m_\Psi^b = \min\{\sqrt{F_\Psi^{b_i}}|\tilde{b}_i|\}$.
- If one uses the reverse Cauchy-Schwarz inequality for the two real positive vectors $\vec{\alpha}$ and $\vec{\beta}$, we have

$$\begin{aligned} \Delta A^2 \quad \Delta B^2 &\leq \Lambda_{\alpha\beta}^{\psi\Psi} \left(\sum_n |\alpha_n||\beta_n| \right)^2 \\ &= \frac{\Lambda_{\alpha\beta}^{\psi\Psi}}{4} \left(\sum_n \left| \langle [\overline{A}, \overline{B}_n^{\psi}] \rangle + \langle \{\overline{A}, \overline{B}_n^{\psi}\} \rangle \right| \right)^2, \end{aligned} \quad (25)$$

where $\Lambda_{\alpha\beta}^{\psi\Psi} = \frac{\left(M_{\psi\Psi}^{\alpha} M_{\psi\Psi}^{\beta} + m_{\psi\Psi}^{\alpha} m_{\psi\Psi}^{\beta}\right)^2}{4 M_{\psi\Psi}^{\alpha} M_{\psi\Psi}^{\beta} m_{\psi\Psi}^{\alpha} m_{\psi\Psi}^{\beta}}$ with $M_{\psi\Psi}^{\alpha} = \max\{|\alpha_n|\}$, $m_{\psi\Psi}^{\alpha} = \min\{|\alpha_n|\}$, $M_{\psi\Psi}^{\beta} = \max\{|\beta_n|\}$ and $m_{\psi\Psi}^{\beta} = \min\{|\beta_n|\}$.
- One can optimize the right hand side.

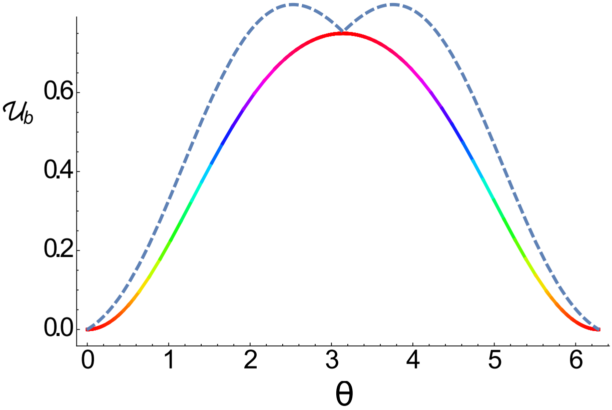


Figure : Here, we plot the upper bound of the product of variances for two incompatible observables, $A = \sigma_x$ and $B = \sigma_z$, two components of the angular momentum for spin $\frac{1}{2}$ particle with a state $\rho = \frac{1}{2}\left(I_2 + \cos\frac{\theta}{2}\sigma_x + \frac{\sqrt{3}}{2}\sin\frac{\theta}{2}\sigma_y + \frac{1}{2}\sin\frac{\theta}{2}\sigma_z\right)$. Blue dashed line curve is the upper bound of the product of the two variances given by (24) and the continuous line (hue colored) plot denotes the product of the two variances.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.
- Significance of the uncertainty and the reverse relations is that for a fixed amount of ‘spread’ of the distribution of measurement outcomes of one observable, the ‘spread’ for the other observable is bounded from both the sides.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.
- Significance of the uncertainty and the reverse relations is that for a fixed amount of ‘spread’ of the distribution of measurement outcomes of one observable, the ‘spread’ for the other observable is bounded from both the sides.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.
- Significance of the uncertainty and the reverse relations is that for a fixed amount of ‘spread’ of the distribution of measurement outcomes of one observable, the ‘spread’ for the other observable is bounded from both the sides.
- These URS may play important role in quantum metrology, quantum speed limits and many other fields of quantum information theory due to the fact that these relations are optimization free, state-dependent and tighter than the most of the existing bounds.

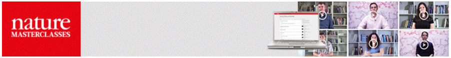
Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.
- Significance of the uncertainty and the reverse relations is that for a fixed amount of ‘spread’ of the distribution of measurement outcomes of one observable, the ‘spread’ for the other observable is bounded from both the sides.
- These URS may play important role in quantum metrology, quantum speed limits and many other fields of quantum information theory due to the fact that these relations are optimization free, state-dependent and tighter than the most of the existing bounds.

Summary and conclusions

- Arguably, the uncertainty relations are the most fundamental relations in quantum theory.
- Proved stronger URs, tighter, state-dependent uncertainty relations both in the sum as well as the product form.
- Proved state-dependent reverse uncertainty relations.
- Significance of the uncertainty and the reverse relations is that for a fixed amount of ‘spread’ of the distribution of measurement outcomes of one observable, the ‘spread’ for the other observable is bounded from both the sides.
- These URS may play important role in quantum metrology, quantum speed limits and many other fields of quantum information theory due to the fact that these relations are optimization free, state-dependent and tighter than the most of the existing bounds.
- Reverse uncertainty relations may give upper bound on error in measurement and the upper bound for the speed of quantum evolutions.

natureINDIA



RESEARCH HIGHLIGHTS

doi:10.1038/nindia.2017.98 Published online 1 August 2017

Quantum physics gets a reverse uncertainty relation

Theoretical physicists from the Harish-Chandra Research Institute (HRI) in Allahabad have derived a new kind of relation in quantum mechanics called the "Reverse Uncertainty Relation", that could find applications in various areas of quantum physics, quantum information and quantum technology[1].

Debasis Mondal, Shrobona Bagchi and Arun Kumar Pati from HRI show, for the first time, that there is an upper limit to how accurately one can simultaneously measure the position and momentum of a particle.

The original uncertainty principle introduced in 1927 by Werner Heisenberg is a rule in quantum mechanics which sets a "lower" limit on the product of the "variances" of two "incompatible observables" (such as position and momentum), but it was not known if there is any "upper" limit.

"We show that there is indeed an upper limit," Pati, one of the authors, told *Nature India*. "The reverse uncertainty relation shows that there is a "spread" or "range" for both the sum and product of variances of two non-commuting observables," he said. In addition to the reverse uncertainty relation, the authors have proved a new and tighter uncertainty relation from which the Heisenberg uncertainty relation directly follows.

The new relation may be useful in setting an upper limit in "quantum metrology," which exploits quantum systems to reach unprecedented levels of precision in measurements. "Thus, this is not only of fundamental interest but can have applications in diverse areas of quantum physics," Pati said adding "the reverse uncertainty relation should open up a whole new direction of explorations in quantum mechanics which we have not thought of."

References

1. Mondal, D. *et al*. Tighter uncertainty and reverse uncertainty relations. *Phys. Rev. A* **95**, 052117 (2017) doi: 10.1103/PhysRevA.95.052117

References

- W. Heisenberg, Zeitschrift fur Physik (in German) **43**, 172 (1927).
- H. P. Robertson, Phys. Rev. **34**, 163 (1929).
- E. Schrodinger, Proc. Cambridge Philos. Soc. **31**, 553 (1935).
- P. Busch, T. Heinonen, P. J. Lahti, Phys. Rep. **452**, 155 (2007).
- P. Busch, P. Lahti, R. F. Werner, Phys. Rev. Lett. **111**, 160405 (2013)
- L. Maccone and A. K. Pati, Phys. Rev. Lett. **113**, 260401 (2014).
- K. Wang, X. Zhan, Z. Bian, J. Li, Y. Zhang, and P. Xue, Phys. Rev. A **93**, 052108 (2016).
- S. Bagchi and A. K. Pati, Phys. Rev. A **94**, 042104 (2016).
- D. Mondal, S. Bagchi and A. K. Pati, Phys. Rev. A **95**, 052117 (2017).

THANK YOU