Nonlocality and Multiqubit States

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Introduction

- Quantum correlations in a system can lead to counter-intuitive consequences.
- One of this is nonlocality which is one of the most mysterious features of the quantum mechanics formalism.
- This feature allows us to carry out many tasks which would not have been otherwise possible.
- Apart from various tasks, the quantum correlations can also be explored using a set of inequalities (or equalities).
- One set of such inequalities are that of Bell-type.
- In the case of two-qubit pure states, the explorations of nonlocality using Bell-type inequalities is uneventful. There is even a relation between the violation of the inequality and entanglement.
- Beyond this, e.g, two-qudit pure states, mult-qubit/qudit pure states, or mixed states present their own challenges.

Introduction

- Our focus will be on three-qubit states, with some discussion on the generalization to multi-qubit states. So the title is perhaps too general.
- In 1964, Bell obtained an inequality and demonstrated the incompatibility between local-realism and quantum mechanics. It was done for a singlet state.
- After more than 25 years, in 1991, Gisin showed that any pure entangled state of a bipartite system violates a Bell's inequality, more accurately Clauser-Horne-Shimony-Holt (CHSH) inequality.
- Inequalities can be written in terms of correlations of observables. Maximum violation of CHSH inequality in quantum mechanics can be $2\sqrt{2}$ (Tsirelson's bound, 1980).
- One can establish a relationship between entanglement and nonlocality in the case of pure bipartite states.

Introduction

- The situation about mixed state is not clear. There are entangled states which don't violate standard CHSH inequality. Prototype example is Werner state. There are local hidden variable models for such entangled states.
- There appears to exist the phenomenon of hidden nonlocality. In literature, there are attempts to show that all entangled states are nonlocal. For example, Gisin (1996), Buscemi (2012), and Masanes et al (2008).
- Case of a multipartite state is more complex because we don't know how to characterize its entanglement. There can be multiple ways to characterize its nonlocality. But one may be able to discuss some categories of such states.
- One way to obtain inequalities is find facets of a polytope in the joint probability space.

Introduction

- CHSH inequality is a facet inequality. It is violated maximally by a the maximally entangled two-qubit states, the Bell states.
- It was then the naive expectation that the same may happen for other systems.
- It was thus a surprise when CGLMP inequality (D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, 2002), which is a facet inequality for a two-qudit sytem was shown to be violated more by a partially entangled state. It was termed as “anomaly of nonlocality”. This inequality is based on $(2, 2, d)$ scenario. There are, however, non-facet inequalities, like SLK (Son-Lee-Kim) inequalities, that are violated maximally by a maximally entangled two-qudit state.

Introduction

- We will consider three-qubit systems, with specific measurement scenario. We will obtain facet inequalities and also obtain a set of other inequalities. We will see that same phenomenon happens. A non-facet inequality is violated maximally by a maximally-entangled state, while a facet inequality is not.
- As a illustration, we will first discuss CHSH inequality as a facet inequality.
- We will first discuss CHSH inequality for two-qubit pure states and see its relation with a measure of entanglement, namely concurrence.
- We will then obtain a set of facet inequalities for a three-qubit system for a specific measurement scenario and show that they are not maximally violated by maximally entangled states.

Introduction

- Next we will introduce a set of Bell inequalities for three-qubit states. There can be a relation between the entanglement and nonlocality of a class of states – generalized GHZ states.
- These inequalities are not facet inequalities, but are maximally violated by maximally entangled three-qubit states.
- We will argue that this set of inequalities can separate states with genuine tripartite entanglement, biseparable states, and product states.
- We will also generalize these inequalities to n-qubit case. Because of the nature of construction, these inequalities will maximally violate n-qubit GHZ states. Also the entanglement, and nonlocality, as characterized by violation of these Bell-type inequalities will be linked.

Outline

Bell Polytopes

- In a typical Bell experiment, there can be two or more spatially separated parties - Alice, Bob, Charlie, ... These parties possess subsystems of a physical system. They can measure observables on their subsystems. They can collect data on joint probability distributions ($p(a, b, c, ...|x, y, z, ...)$). Here x, y, z are observables measured by Alice, Bob, and Charlie; a, b, c are measurement outcomes. These probability distributions have to satisfy normalization and no-signalling constraints. This reduces the number of these joint probabilities.
- A convex-hull of these probability distributions defines a Bell polytope. The probability distributions inside the polytope are classical distributiion. Alternately, one can characterize this polytope by facets. Each facet divides the probability space in two half, and is characterized by an inequality.
- A large fraction, as we will see in concrete examples, of these facets inequalities are just positivity conditions. Rest of the nontrivial inequalities can correspond to Bell-type inequalities.

Bell-CHSH Polytope

- Let us now look at the familiar case of CHSH inequality. The scenario is $(2, 2, 2)$. There are two parties, two measurements by each party, and two outcomes for each measurements.
- For this case there are 16 joint probability distributions $P(a, b|x, y)$. Taking into account conservation of probability and no-signalling conditions reduce the number to 8.
- So we have to consider polytope in 8 dimensions that have 16 vertices. As we will see, this consideration will give rise to 24 facets. The sixteen out of 24 are positivity conditions. Out of the remaining 8 four gives the upper bound and 4 the lower bound. There is only one independent inequality. Other follows from permutations.

Bell-CHSH Polytope

- The whole scenario is characterized by 16 joint probabilities. They satisfies normalization

$$\sum_{a,b} p(ab|xy) = 1 \qquad \forall \quad x, y = 0, 1.$$

- No signalling conditions are

$$p(a|x) \equiv \sum_{b} p(ab|xy) = 1 \quad \forall \quad a \text{ and } x, y = 0, 1,$$
$$p(b|y) \equiv \sum_{a} p(ab|xy) = 1 \quad \forall \quad b \text{ and } x, y = 0, 1.$$

- There are 4 normalization constraint and 12 no-signaling constraint. But these constraints are not all independent. Using normalization constraint we can reduce the no-signaling condition by 4.Therefore, there will be total 8 independent constraints. These 8 constraints will reduce the joint probabilities space to 8.

Bell-CHSH facets

- These probability space can be represented as

  $$p = [p(a_1), p(a_2), p(b_1), p(b_2), p(a_1b_1), p(a_1b_2), p(a_2b_1), p(a_2b_2)],$$

  where $p(a_x) = p(1|x)$, $p(b_y) = p(1|y)$ and $p(a_xb_y) = p(11|xy)$.
- As a_1, a_2, b_1 and b_2 can take two different diachomatic values, there will be total 16 different points in the above said probability space. So, the CHSH-polytope is 8 dimensional and described by 16 vertices. The polytope described in terms of vertices known as V-representation. One can find the faces of the polytope from this description by using a standard algorithm. There will be 24 facets as follows,

$$\begin{aligned}
&p(a_ib_j) \geqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2 \\
&-p(a_i) + p(a_ib_j) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2 \\
&-p(b_j) + p(a_ib_j) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2 \\
&p(a_i) + p(b_j) - p(a_ib_j) \leqslant 1, \quad i = 1,2 \quad \text{and} \quad j = 1,2
\end{aligned}$$

Bell-CHSH facets

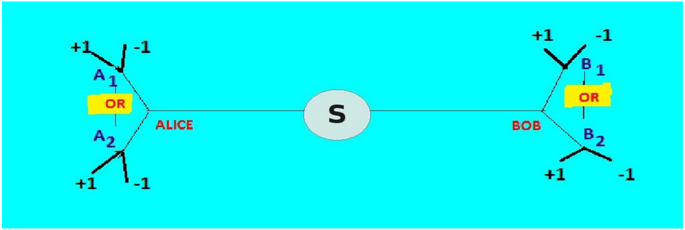
- The nontrivial facets are

$$
\begin{aligned}
p(a_1)+p(b_2)-p(a_1b_1)-p(a_1b_2)+p(a_2b_1)-p(a_2b_2) &\leqslant 1\\
p(a_1)+p(b_1)-p(a_1b_1)-p(a_1b_2)-p(a_2b_1)+p(a_2b_2) &\leqslant 1\\
p(a_2)+p(b_2)+p(a_1b_1)-p(a_1b_2)-p(a_2b_1)-p(a_2b_2) &\leqslant 1\\
p(a_2)+p(b_1)-p(a_1b_1)+p(a_1b_2)-p(a_2b_1)-p(a_2b_2) &\leqslant 1\\
-p(a_1)-p(b_2)+p(a_1b_1)+p(a_1b_2)-p(a_2b_1)+p(a_2b_2) &\leqslant 0\\
-p(a_1)-p(b_1)+p(a_1b_1)+p(a_1b_2)+p(a_2b_1)-p(a_2b_2) &\leqslant 0\\
-p(a_2)-p(b_2)-p(a_1b_1)+p(a_1b_2)+p(a_2b_1)+p(a_2b_2) &\leqslant 0\\
-p(a_2)-p(b_1)+p(a_1b_1)-p(a_1b_2)+p(a_2b_1)+p(a_2b_2) &\leqslant 0.
\end{aligned}
$$

- The last eight inequalities are the famous CH inequalities and are equivalent to CHSH inequalities. However, there is only one independent inequality. Here four give the lower, and four the upper bound. Out of the four, three can be obtained by permutations.

Outline

CHSH Inequality



The CHSH inequality (John Clauser, Michael Horne, Abner Shimony, and Richard Holt, 1969) is given in terms of the following combination of the observables,

$$I_{CHSH} = A_1\ B_1 + A_1\ B_2 + A_2\ B_1 - A_2\ B_2$$

In a local-realistic theory,

$$\langle I_{CHSH} \rangle \leq 2$$

CHSH Inequality

- An arbitrary two-qubit state after Schmidt decomposition can always be written as

$$|\psi_n\rangle = c_0|\hat{n}_+\hat{n}_+\rangle + c_1|\hat{n}_-\hat{n}_-\rangle.$$

- We choose the measurement settings in the following way

$$A_1 = \hat{m}_1 \cdot \vec{\sigma}, \qquad A_2 = \hat{m}_2 \cdot \vec{\sigma},$$
$$B_1 = \frac{1}{\sqrt{2}}(\hat{m}_1 \cdot \vec{\sigma} + \hat{m}_2 \cdot \vec{\sigma}), \quad B_2 = \frac{1}{\sqrt{2}}(\hat{m}_1 \cdot \vec{\sigma} - \hat{m}_2 \cdot \vec{\sigma}).$$

Here $\hat{n}$, $\hat{m}_1$ and $\hat{m}_2$ are the unit vectors perpendicular to each other.

- Now we have to obtain the expectation value of the CHSH operator in the state $|\psi_n\rangle$. We get

$$\langle\psi_n|I_S|\psi_n\rangle = 2\sqrt{2}\mathbb{C}.$$

CHSH Inequality

- Since concurrence is a measure of entanglement, we find that there is relation between an entanglement measure and the value of CHSH operator for any pure two-qubit state. Of course, these measurement settings have a flaw. Some of the entangled state don't violate CHSH inequality.
- Advantage of these settings is that the value of operator is zero for product states, and non-zero for entangled states.
- So, in this setting, CHSH operator can act as a witness to the entanglement as well as measure it.

CHSH Inequality

- Let us again consider a general two-qubit state,

$$|\psi_n\rangle = c_0|\hat{n}_+\hat{n}_+\rangle + c_1|\hat{n}_-\hat{n}_-\rangle.$$

- We choose the measurement settings in the following way

$$A_1 = \hat{n}\cdot\vec{\sigma}, \qquad A_2 = \hat{m}\cdot\vec{\sigma},$$
$$B_1 = \tfrac{1}{\sqrt{2}}(\hat{n}\cdot\vec{\sigma} + \hat{m}\cdot\vec{\sigma}), \quad B_2 = \frac{1}{\sqrt{2}}(\hat{n}\cdot\vec{\sigma} - \hat{m}\cdot\vec{\sigma}).$$

- We again find,

$$\langle\psi_0|I_{CHSH}|\psi_0\rangle = \sqrt{2}(1+\mathbb{C}).$$

  Here $\mathbb{C}$ is the concurrence of the state.
- We see that we have higher values and some entangled states do not violate CHSH inequality. But still there is a relation which can be used to measure entanglement.

CHSH Inequality

- Again we consider the general two-qubit state.

$$|\psi_n\rangle = c_0|\hat{n}_+\hat{n}_+\rangle + c_1|\hat{n}_-\hat{n}_-\rangle$$

- We choose third measurement settings in the following way

$$A_1 = \hat{n}\cdot\vec{\sigma}, \qquad A_2 = \hat{m}\cdot\vec{\sigma},$$
$$B_1 = \hat{n}\cdot\vec{\sigma}\ cos(\eta) + \hat{m}\cdot\vec{\sigma}\ sin(\eta), \quad B_2 = \hat{n}\cdot\vec{\sigma}\ cos(\eta) - \hat{m}\cdot\vec{\sigma}\ sin(\eta).$$

  Here $\hat{n}$ and $\hat{m}$ are the unit vectors perpendicular to each other. Also $cos(\eta) = \frac{1}{\sqrt{1+4c_0^2c_1^2}}$.
- We again get

$$\langle\psi_n|I_{CHSH}|\psi_n\rangle = 2\sqrt{1+\mathbb{C}^2}.$$

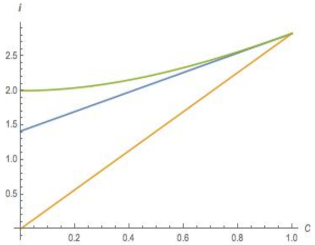
- Though these setting give the optimized value and largest violation, but there is a flaw. You have to know the state in advance for these settings.
- So if we wish to find how entangled an unknown state is, we should be using earlier settings.

CHSH Inequality

- The question may be asked what about the most general state-independent settings ? One can show that there is a relation,

$$\langle\psi_n|I_{CHSH}|\psi_n\rangle = A + B\,\mathbb{C}.$$

  Here A and B would depend on measurement setting angles.
- Here is a plot to show the value of CHSH operator for the three different settings:



Outline

Maximally Entangled States

- In the case of bipartite pure entangled states, the notion of maximally entangled state is unambiguous. For a two-qubit system, these are well known Bell states. By using LOCC, any other two-qubit state can be obtained from these states.
- As we go beyond bipartite case, the situation is not clear.
- For multipartite case, multiple notions exist. Two of these are concepts of Absolutely Maximally Entangled State (AMES) and Task-Oriented Maximally Entangled States (TMES).
- If a system is in AMES, then all its subsystems will have maximally allowed entropy. For the case of three qubits, GHZ state is AMES. For a four-qubit system, there are no AMES. There exist AMES for five and six-qubit systems. For eight-qubit systems and beyond there are no AMES. For a seven-qubit also there appears to be no AMES. In all such cases, one may look for states close to AMES.

Maximally Entangled States

- From another perspective, i. e., of TMES, in the case of three-qubit case, the GHZ state is TMES for a number of tasks, like teleportation, superdense coding, secret sharing, etc.
- For a three-qubit system, a modified-W state that is suitable for perfect teleportation and superdense coding can also be a TMES. But for a given modified-W state, it is possible in only one specific partition, while one can use GHZ state in any partition.
- If we consider GHZ state for a three-qubit system to be maximally entangled, then one may expect it to be maximally nonlocal also. For a quantum-mechanical system, entanglement is the source of nonlocality.
- If we use violation of Bell inequality as a measure of nonlocality, then as we will see, the situation is not straight forward.

Outline

Multipartite States

- In the case of multipartite states comparing the entanglement of even two states is not straightforward.
- The nature of entanglement is not very well understood.
- There is a long history of finding inequalities for multipartite states. We will particularly focus on three-qubit states.
- There are various issues with a number of popular inequalities. For example, Mermin's inequality is also violated by states that don't have genuine tripartite entanglement.
- Svetlichny inequality has problems with some states that have genuine tripartite entanglement.
- Bacal et al. introduced a weaker notion of nonlocality, and obtained a set of 185 inequalities as facets of no-signalling polytopes. They conjectured that all entangled staes will violate at least one of these inequalities.

Multipartite States

- Zukowski et al. have shown that generalized GHZ states for odd number of qubits do not violate any correlation inequalities.
- However Yu et al. have considered Hardy type nonlocality arguments, and shown that all entangled states violate a single Bell inequality.
- Without considering Hardy type arguments, for three-qubit states, we will give a set of inequalities that seem to be violated by all entangled states. These inequalities can also distinguish three different classes of states.
- These Bell inequalites have correlations based on measurements on three subsystems. But one makes only one messurement on one of the subsystems.

Introduction

- So in our scenario, there are three parties. Two parties, Alice and Bob make two measurements of diachomatic observables, while Charlie makes only one measurement of a diachomatic observable.
- In this scenario, there are 32 joint probability distributions. So there are 32 vertices. After taking into account the normalization and no-signalling, the probability space is 17 dimensional.
- There are 48 facets. 32 are just positivity conditions. There are 16 nontrivial facets. Let us recall, in the case of CHSH polytope, there were 8 nontrivial facets.
- Out of 16 nontrivial facets, 8 facets give upper bounds to inequalities, while the other 8 gives lower bounds.
- Out of 8 nontrivial inequalities, there are only 2 independent. Like CHSH case, there are two sets of 4 inequalities. Each set corresponds to only one inequality. The other three are just permutations.

Facets

- Here is the facet list:

$$p(a_ib_jc1) \geqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$
$$-p(a_ic1) + p(a_ib_jc1) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$
$$-p(b_jc1) + p(a_ib_jc1) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$
$$-p(a_ib_j) + p(a_ib_jc1) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$

$$-p(a_i) + p(a_ib_j) - p(a_ib_jc1) + p(a_ic1) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$
$$-p(b_j) + p(a_ib_j) - p(a_ib_jc1) + p(b_jc1) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$
$$-p(c1) + p(a_ic1) - p(a_ib_jc1) + p(b_jc1) \leqslant 0, \quad i = 1,2 \quad \text{and} \quad j = 1,2$$

$$p(c1) + p(a_i) + p(b_j) - p(a_ic1) + p(a_ib_jc1) - p(a_ib_j) - p(b_jc1) \leqslant 0,$$
$$i = 1,2 \quad \text{and} \quad j = 1,2$$

- These are 32 positivity conditions. Remaining 16 facets are non trivial.

Facet Inequalities

- Here are two independent and nontrivial inequalities.

$$-6 \leq (-A_2B_2 + A_2B_1 + A_1B_2 + A_1B_1) + (-A_2B_2 + A_2B_1 + A_1B_2 + A_1B_1)C_1 - 2C_1 \leq 2$$

$$-6 \leq (-A_2B_2 + A_2B_1 + A_1B_2 + A_1B_1) + (A_2B_2 - A_2B_1 - A_1B_2 - A_1B_1)C_1 + 2C_1 \leq 2$$

In terms of the well-known CHSH inequality this set can be written as,

$$\begin{aligned} -6 \leq \quad & I_{CHSH} + I_{CHSH}C_1 - 2C_1 \quad \leq 2, \\ -6 \leq \quad & I_{CHSH} - I_{CHSH}C_1 + 2C_1 \quad \leq 2. \end{aligned}$$

Facet Inequalities

- We can now check how these inequalities are violated by various states. We will maximize over all possible measurement settings.
- Let us define

$$I_3^{facet} = I_{CHSH} + I_{CHSH} C_1 - 2C_1$$

- Maximum value of $\langle I_3^{facet} \rangle$ for the GHZ state $= 2\sqrt{2} \approx 2.83$.
- Maximum value of $\langle I_3^{facet} \rangle$ for the W state ≈ 3.10.
- As we have discussed, for three qubits case, the GHZ state can be considered as maximally entangled state. But as we see, facet inequalities are violated more by W state.
- Actually, there are states that give even larger value.

Facet Inequalities

- If we consider generalized GHZ states

$$|GGHZ\rangle = \alpha|000\rangle + \beta|111\rangle, \quad (1)$$

  then violation of the facet inequalities is always smaller than that by the GHZ state. Actually, in this case, $\langle I_3^{facet} \rangle$ tracks the entanglement.
- Let us now consider another state in GHZ class:

$$|\psi_1\rangle = \sqrt{22/50}|000\rangle + \sqrt{3/50}|100\rangle + \sqrt{2/50}|101\rangle + \sqrt{21/50}|110\rangle + \sqrt{2/50}|111\rangle$$

- Maximum value of $\langle I_3^{facet} \rangle$ for the $|\psi_1\rangle \approx 3.38$.
- Let us now consider some other states in the W-class:

$$|\psi_2\rangle = \sqrt{1/6}|001\rangle + \sqrt{3/6}|010\rangle + \sqrt{2/6}|001\rangle$$

Facet Inequalities

- Another W class state

$$|\psi_3\rangle = \sqrt{1/10}|001\rangle + \sqrt{4/10}|010\rangle + \sqrt{5/10}|001\rangle$$

- Maximum value of $\langle I_3^{facet} \rangle$ for the $|\psi_2\rangle \approx 3.33$.
- Maximum value of $\langle I_3^{facet} \rangle$ for the $|\psi_3\rangle \approx 3.48$.
- These states seem to be less entangled than the W state, but give more violation.
- There does not seem to be any specific pattern for the violation.
- These inequalities are also violated by biseparable states like $\sqrt{1/2}(|00\rangle + |11\rangle)|0\rangle$. Here violation is more than that of GHZ state. The value is about 3.66.

Multipartite States

- For tripartite states, we have three qubits, A, B, and C. We will choose to make two measurements on two qubits, say A and B, and one measurement on the third qubit.
- This is motivated by the structure of Bell operator that gives maximal violation for the Bell state $|\varphi_+\rangle$, and GHZ state.
- Here these states are -

$$|\varphi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

- For the Bell state $|\varphi_+\rangle$, the operator is proportional to $(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)$. The state $|\varphi_+\rangle$ is also eigenstate of this operator.
- For the GHZ state, the operator is proportional to $(\sigma_x \otimes \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z \otimes I)$. The GHZ state is also eigenstate of this operator.

Outline

Multipartite States

- For a symmetric state like GHZ-state, only one of the following inequalities would be enough.
- The set of inequalities that we will discuss are:

$$\begin{aligned} A_1B_1(C_1+C_2)+B_2(C_1-C_2) &\leq 2, \\ A_2B_1(C_1+C_2)+A_1(C_1-C_2) &\leq 2, \end{aligned}$$

$$\begin{aligned} (B_1+B_2)C_2+A_1(B_1-B_2)C_1 &\leq 2, \\ A_1(B_1+B_2)+A_2(B_1-B_2)C_1 &\leq 2, \end{aligned}$$

$$\begin{aligned} (A_1+A_2)B_2+(A_1-A_2)B_1C_1 &\leq 2, \\ (A_1+A_2)C_1+(A_1-A_2)B_1C_2 &\leq 2. \end{aligned}$$

Multipartite States

- Like facet inequalities, there are two independent inequalities. In inequalities first and third, Alice makes only one measurement; while Bob and Charlie make two measurements each.
- Other 4 inequalities can be obtained by permutation. In the case of second and sixth inequalities Bob makes only one measurement. Charlie is making only one measurement in fourth and fifth inequalities.
- These inequalities have similarities with conventional CHSH inequality, except that there is only one measurement on one of the qubits.
- These inequalities can be generalized to arbitrary number of qubits. One will have to make distinction between states of odd number and even number of qubits. In the case of odd number, one will make only one measurement, at least on one qubit.

Quantum Bound

- We will obtain the bound for the first inequality and the analysis will be similar for others. Let us call the corresponding Bell operator for the first inequality as,

$$B_3 = A_1 B_1 (C_1 + C_2) + B_2 (C_1 - C_2)$$

- If we take the square of this expression we get,

$$B_3^2 = 4I + A_1 [C_1, C_2][B_1, B_2].$$

- Here, we have used ${A_1}^2 = {B_1}^2 = {B_2}^2 = {C_1}^2 = {C_2}^2 = I$. Now, we know that, for two bounded operators X and Y,

$$\parallel [X, Y] \parallel \leq 2 \parallel X \parallel \parallel Y \parallel$$

- Using this relation, we notice that the maximum value will be obtained when B_3^2 is $8I$ and hence $\parallel B_3 \parallel \leq 2\sqrt{2}$.

Generalized GHZ States

- Generalized GHZ states,

$$|GGHZ\rangle = \alpha|000\rangle + \beta|111\rangle, \qquad (2)$$

  have been a problem for a number of different type of inequalities.
- All pure states in generalized GHZ class violate our Bell inequalities. This is not surprising. Our Bell inequalities were designed for GHZ states.
- We will now show that the violation of the inequalities depends on the entanglement of these states. For this we will take α and β to be real.
- This GHZ state is symmetric under the permutation of qubits. So we can choose any of the inequalities. We choose the inequality to be,

$$A_1(B_1 + B_2) + A_2(B_1 - B_2)C_1 \leq 2 \qquad (3)$$

Generalized GHZ States

- Let us choose following set of measurements -

$$A_1 = \sigma_z, \qquad A_2 = \sigma_x,$$
$$B_1 = \cos\theta\sigma_x + \sin\theta\sigma_z, \qquad B_2 = -\cos\theta\sigma_x + \sin\theta\sigma_z,$$
$$C_1 = \sigma_x.$$

- These measurement settings are inspired by that of two-qubit Bell state case. On qubit 'A', there are two measurements along orthogonal directions. On qubit 'C', there is only one measurement. We will choose the angle θ such that the value of the Bell operator is maximum.
- We can now compute the expectation value of the Bell operator for the generalized GHZ state.

$$\langle GGHZ|(A_1(B_1 + B_2) + A_2(B_1 - B_2)C_1)|GGHZ\rangle \qquad (4)$$

- The result is

$$2[2\alpha\beta\cos\theta + (\alpha^2 + \beta^2)\sin\theta] = 2[2\alpha\beta\cos\theta + \sin\theta]$$

Generalized GHZ States

- We can now use,

$$a \sin\phi + b\cos\phi \leq \sqrt{a^2 + b^2}$$

.

- This gives

$$\langle GGHZ|(A_1(B_1+B_2)+A_2(B_1-B_2)C_1)|GGHZ\rangle \leq 2\sqrt{1+4\alpha^2\beta^2}$$

.

- We see that the expectation value of the operator is always greater than 2 for nonzero values of α and β, i.e., as long as the state is entangled.
- The maximum value is $2\sqrt{2}$ for the conventional GHZ state. RHS can be written in a suggestive way as $2\sqrt{1+\mathcal{C}^2}$, where $\mathcal{C} = 2\alpha\beta$.
- We see that for generalized GHZ state, the violation depends on the amount of entanglement in the state.
- This result can be generalized for n-qubit $|GGHZ\rangle$ state.

General three-qubit state

- What we have considered is a special class of genuinely entangled three-qubit states. One would like to show that any genuinely tripartite entangled state violates one of our inequalities.
- There is a parametrization of genuinely tripartite entangled three-qubit states, due to Acin et al. It involves 6 parameters, including one phase.

$$|\psi\rangle = \lambda_0|0\rangle|0\rangle|0\rangle + \lambda_1 e^{i\phi}|1\rangle|0\rangle|0\rangle + \lambda_2|1\rangle|0\rangle|1\rangle + \lambda_3|1\rangle|1\rangle|0\rangle + \lambda_4|1\rangle|1\rangle|1\rangle \quad (5)$$

- For genuine tripartite entanglement, these parameters have to satisfy some conditions. These conditions are: $\lambda_i \geq 0$, $\sum_i \lambda_i^2 = 1$, $\lambda_0 \neq 0$, $\lambda_2 + \lambda_4 \neq 0$, $\lambda_3 + \lambda_4 \neq 0$ and $\phi \in [0, \pi]$.
- We can use this parametrization to test our Bell inequalities.

General three-qubit state

- We will generate a large number of these states by choosing the parameter values at random. For each state, we will check if a Bell inequality is violated.
- Any of these random choice, will almost always have non-zero values of parameters. To strengthen our tests, we divided these states in number of classes by choosing some of the parameters as zero.
- The classes we test for are only $\lambda_1 = 0$, only $\lambda_2 = 0$, only $\lambda_3 = 0$, only $\lambda_4 = 0$; Only λ_1 and $\lambda_2 = 0$, only λ_1 and $\lambda_3 = 0$, only λ_1 and $\lambda_4 = 0$, only λ_2 and $\lambda_3 = 0$; only λ_1, λ_2, and $\lambda_3 = 0$; all λ's are non-zero. For each class ϕ is arbitrary. We have taken 5000 random values of each parameter for first 9 classes and found violations within the set of 12 inequalities in each case. For all non-zero parameters, we have tested for 25000 states (A total of 70000 states). The results are displayed in the plots below.

Three-qubit states

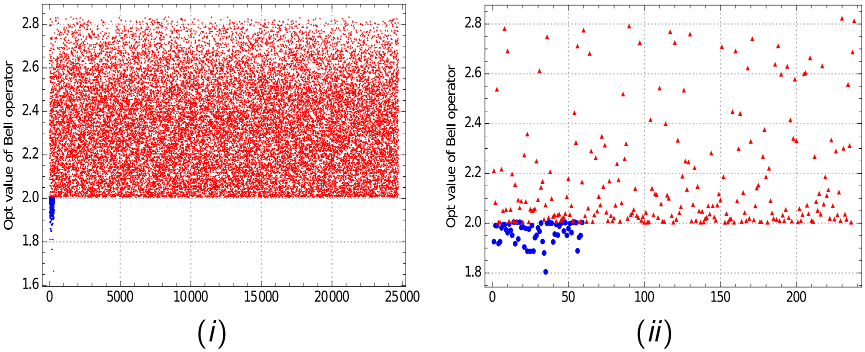


Figure: Maximum value of the Bell operator (i) Inequality 1 (ii) Inequality 2.

- First inequality is not violated by 297 states. Of these 59 states do not violate second inequality.

Three-qubit states

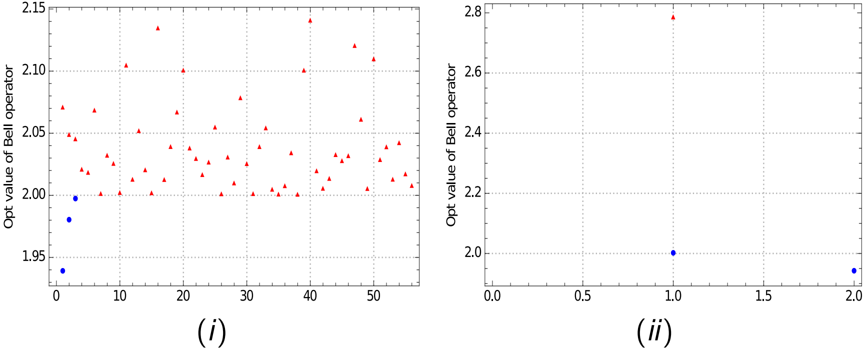
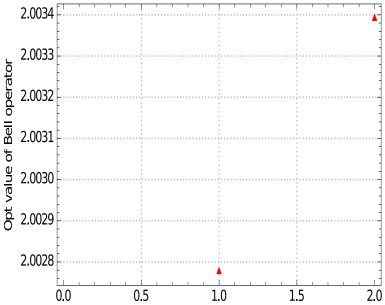


Figure: Maximum value of the Bell operator (i) Inequality 3 (ii) Inequality 4.

- Out of 59 states, 3 states do not violate 3rd inequality. Of these 2 states do not violate 4th inequality.

Three-qubit states

- We have seen that number of states that are remaining now is 2 only. Using 5th inequality, we see that now all states violate one of the inequalities.



- So we see that all the 25,000 randomly generated genuinely entangled states violate one of the inequalities.

Three-qubit states

- As we discussed earlier, none of these states have any vanishing parameter. One may suspect that such cases might be special and may or may not violate our inequalities.
- To take of this situation, we have generated 5000 states for each of the special 9 classes. In each, we find that one of the inequality is at least violated.
- As our inequalities are similar to CHSH inequality, so they are not violated by product states. One can show this explicitly analytically as well as numerically.
- Similarly, we can consider two-way entangled, i.e., biseparable, tripartite states. For such cases, each state violates only four of the twelve inequalities. Furthermore, the amount of the violation is same in each case.
- So we see that our inequalities can distinguish the three classes of states, as well as seem to be violated by all entangled states.

n-qubit states

- As we discussed earlier, we can generalize above Bell inequalities to n-qubit case. One has to distinguish between odd and even number of qubits case.
- For odd number of qubits, there will be a total of $2n(n-1)$ inequalities. The first two inequalities can be written as:

$$A_1A_2A_3A_4A_5..(A_n + A'_n) + A'_2A'_3A'_4A'_5..(A_n - A'_n) \leq 2, \quad (6)$$

and

$$A_2A_3A_4A_5..(A_n + A'_n) + A_1A'_2A'_3A'_4A'_5..(A_n - A'_n) \leq 2. \quad (7)$$

- Here, A_i and A'_i are two dichotomic observable for i^{th} party. In these inequalities, one measurement has been made on first qubit. Similarly one can make single measurement on $(n-2)$ other qubits. This will lead to $2(n-1)$ inequalities.

n-qubit states

- We can write n such $2(n-1)$ inequalities with $(A_i \pm A_i')$ for i^{th} qubit. This will give a total of $2n(n-1)$ inequalities.
- For even number of qubits, there will be a total of n inequalities.

$$(A_1 + A_1')A_2A_3A_4A_5..A_n + (A_1 - A_1')A_2'A_3'A_4'A_5'..A_n' \leq 2. \quad (8)$$

- Here, A_i and A_i' are two dichotomic observable for i^{th} party. Similarly, n such inequalities with $(A_i \pm A_i')$ can be written.
- These are simplest possible generalizations. More will be possible.
- In the case of generalized n-qubit GHZ state, again, like earlier, we will need only one inequality. Irrespective of number of qubits, the maximal violation would be $2\sqrt{2}$, for n-qubit GHZ state.

Outline

Conclusions

- We have seen that CHSH operator can be used to measure the entanglement of a pure qubit state with several different settings. A state-dependent setting gives the largest violation.
- For the two-qubit systems, CHSH is the facet inequality, and is violated maximally by the maximally entangled states.
- However, this fact is far from the norm. It is more of a norm that a facet inequality is violated more by a non-maximally entangled state.
- For tripartite states, we have introduced a new set of inequalities. For a special class of states, the generalized GHZ states, there is a direct relation between the entanglement and the violation. For more general states, the nature of entanglement is poorly understood, so its relationship with amount of nonlocality is not clear.

Conclusions

- Our inequalities also separate the three classes of the states of genuine tripartite entangled states, biseparable states, and product states.
- There was a generalization to n-qubit states. There again seems to be relation between entanglement and nonlocality of generalized n-qubit states.

The effect of these above operators on the state is

$$\hat{m}_1 \cdot \vec{\sigma} |\hat{n}_+\rangle = -|\hat{n}_-\rangle, \quad \hat{m}_1 \cdot \vec{\sigma} |\hat{n}_-\rangle = -|\hat{n}_+\rangle,$$
$$\hat{m}_2 \cdot \vec{\sigma} |\hat{n}_+\rangle = -i|\hat{n}_-\rangle, \quad \hat{m}_2 \cdot \vec{\sigma} |\hat{n}_-\rangle = +i|\hat{n}_+\rangle.$$

CHSH Inequality

- We had written above the CHSH inequality in terms of correlation functions. It is possible to rewrite this inequality in terms of joint probabilities:

$$I_{CHSH} = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1).$$

- In this expression $P(A = B + k)$, more generally, stands for

$$P(A = B + k) = \sum_{j=0}^{d-1} P(A = j + k \mod d, B = j).$$

For qubits $d = 2$. P(A=j, B=k) is a joint probability of obtaining $A = j$ and $B = k$ on measuring A and B.
- This form of CHSH inequality was generalized to qudits and is known as CGLMP inequality. (D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, 2002)

Introduction

- MABK (Mermin, Ardehali, Belinskii and Klyshko) inequality

$$A_1(B_1C_2 + B_2C_1) + A_2(B_1C_1 - B_2C_2) \leq 2 \qquad (9)$$

Bell-type Inequalities and Qudits

- It is natural to examine inequalities, where the observables can take d different values - like $0, 1, 2, ..., d-1$. One can also measure more than 2 observables on each qudit.
- Our focus will be on inequalities with two observables with d values on each side.
- One early development in this direction was the introduction of CGLMP inequality. (D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, 2002). This inequality has its own advantages and disadvantages.
- Subsequently, many more inequalities for two or more qudit systems have been proposed. We will pick one such inequality, which was one of many that were proposed by W. Son, J. Lee and M. S. Kim (2006). It is called SLK inequality.
- We will obtain a relation between an entanglement measure and the expectation value of SLK operator in a particular set of observation settings. For these settings no such relation exists for CGLMP operator.

CGLMP Inequality

- The CGLMP inequality is a generalization of CHSH inequality. It is a specific generalization in terms of joint probability distributions:

$$\begin{aligned} I_d \;=\; & \sum_{k=0}^{[\frac{d}{2}]-1} (1 - \frac{2k}{(d-1)})[(P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\ & + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)) \\ & - (P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\ & + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1))] \end{aligned} \tag{10}$$

- This generalization was obtained by first trying to find an optimum expression for $d = 3$ and $d = 4$.
- There are other ways to write it, as we will see later.
- The maximum local-realistic value for this is 2 and maximum possible value is 4.

CGLMP Inequality

- This has been more popular qudit inequality. It has been tested experimentally also. (Dada et al, 2011)
- However this inequality has one drawback. A nonmaximally entangled state violates it more than a maximally entangled state. Following table from Acin, Durt, Gisin, and Latorre (2002) illustrates this.

Dimension	Violation for $\|\Psi\rangle$	Maximal violation (for $\|\Psi_{mv}\rangle$)	Difference (%)
3	2.8729	2.9149	1.4591
4	2.8962	2.9727	2.6398
5	2.9105	3.0157	3.6133
6	2.9202	3.0497	4.4345
7	2.9272	3.0776	5.1411
8	2.9324	3.1013	5.7588

- Given this, it would appear unlikely that a relation where a relation like that for CHSH inequality may exist for CGLMP inequality.