On the causal evolution of N-particle systems

Tomasz Miller

Work in progress with **Ryszard Horodecki**, **Paweł Horodecki** and **Michał Eckstein** (KCIK, UG, Poland)

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A **Lorentzian spectral triple** can be (loosely) defined as $(\mathcal{A}, \mathcal{K}, \mathcal{D})$, where:

- $\mathcal{A}$ is a (dense $*$-subalgebra of a) C^*-algebra with a preferred unitisation $\widetilde{\mathcal{A}}$,
- $\mathcal{K}$ is a Krein space endowed with an (indefinite) inner product $(.\,,.)$ together with a faithful representation of $\mathcal{A}$,
- $\mathcal{D}$ is an unbounded operator on $\mathcal{K}$ such that $(\phi, \mathcal{D}\psi) = -(\mathcal{D}\phi, \psi)$ for all $\phi, \psi \in \mathrm{dom}\mathcal{D}$ and such that $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{K}$ for any $a \in \widetilde{\mathcal{A}}$.

Example constructed upon $\mathcal{M}$ – glob. hyp. spacetime

$$\mathcal{A} := C_c^\infty(\mathcal{M}), \quad \mathcal{K} := L^2(\mathcal{M}, \mathcal{S}), \quad \mathcal{D} := -i\gamma^\mu \nabla_\mu^{\mathcal{S}}$$

Motivations and goals

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Definition (N. Franco, M. Eckstein, 2013)

Let $\mathcal{C}$ be the cone of all Hermitian $a \in \widetilde{\mathcal{A}}$ such that $(\phi, [\mathcal{D}, a]\phi) \leq 0$ for all $\phi \in \mathcal{K}$. For any two states ξ, η on $\mathcal{A}$ we define the **causal relation**

$$\xi \preceq \eta \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \forall\, a \in \mathcal{C} \quad \xi(a) \leq \eta(a).$$

In the case of $\mathcal{A} = C_c^\infty(\mathcal{M})$

- $\mathcal{C}$ is the set of smooth bounded causal functions.
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Causal precedence relation $\preceq$ (J^+) between **events**

$p \preceq q$ if $\exists$ a piecewise smooth fut-dir causal curve from p to q (or $p = q$).

On the other hand: $\delta_p \preceq \delta_q \iff \forall f \in \mathcal{C} \quad f(p) \leq f(q)$

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Here the measures can be spread also in the timelike direction.

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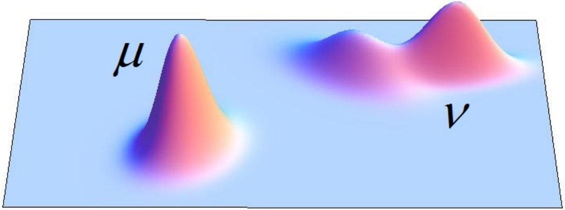

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Causality for probability measures

What does it mean that $\mu \preceq \nu$? [M. Eckstein, TM '17]

Let $\mathcal{M}$ be a spacetime. Then for any $\mu, \nu \in \mathscr{P}(\mathcal{M})$

$\mu \preceq \nu \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists \omega \in \mathscr{P}(\mathcal{M}^2)$ such that:

- $\forall_{B \text{ – Borel}} \quad \omega(B \times \mathcal{M}) = \mu(B), \quad \omega(\mathcal{M} \times B) = \nu(B)$,
- $\omega(J^+) = 1$,

where $J^+ := \{(p,q) \in \mathcal{M}^2 \mid p \preceq q\}$.

- ω can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p$, $\nu = \delta_q$, the only coupling is $\omega = \delta_{(p,q)}$ and so $\delta_p \preceq \delta_q$ iff $p \preceq q$.
- $\preceq$ is reflexive and transitive. It is antisymmetric for $\mathcal{M}$ past/future distinguishing.

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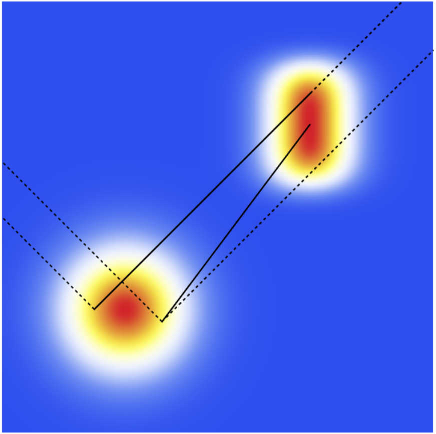
Each infinitesimal part of the probability measure should travel along a future-directed causal curve.



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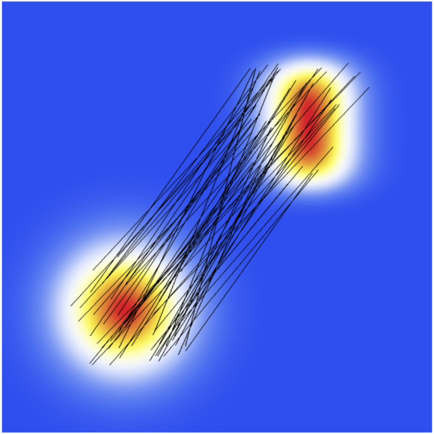
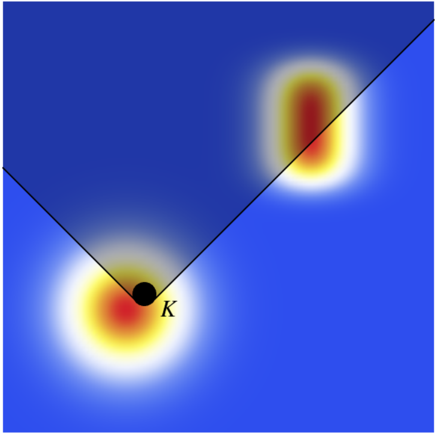


Causality for probability measures

For $\mathcal{M}$ causally simple ($\preceq$ antisymmetric + topologically closed):

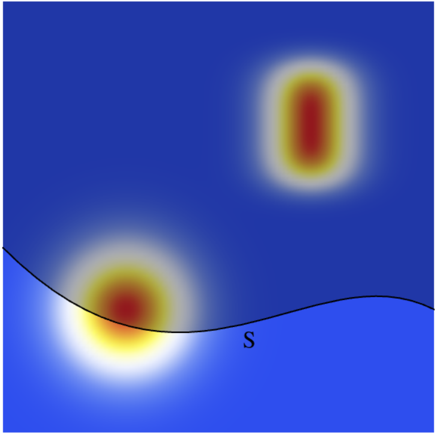
$$\mu \preceq \nu \quad \Longleftrightarrow \quad \text{for any compact } \mathcal{K} \subset \operatorname{supp} \mu \quad \mu(\mathcal{K}) \leq \nu(J^+(\mathcal{K}))$$



Causality for probability measures

For $\mathcal{M}$ globally hyperbolic:

$\mu \preceq \nu \quad \Longleftrightarrow \quad$ for any Cauchy hypersurface $\Sigma \quad \mu(J^+(\Sigma)) \leq \nu(J^+(\Sigma))$



Causality for probability measures

Definition (M. Eckstein, TM 2017)

Let $s \in (0,1]$. For any $\mu, \nu \in \mathscr{P}(\mathcal{M})$ their s^{th} Lorentz–Wasserstein distance is defined via

$$LW_s(\mu,\nu) := \begin{cases} \sup\limits_{\omega \text{ – c.c. of } \mu \text{ and } \nu} \left[\int_{\mathcal{M}^2} d(p,q)^s d\omega(p,q) \right]^{1/s} & \text{if } \mu \preceq \nu \\ 0 & \text{if } \mu \npreceq \nu \end{cases}$$

This notion has already been picked by:

- R. McCann in his study of the relation between the Hawking–Penrose strong energy condition and the geodesic concavity of the Boltzmann–Shannon entropy.
- A. Mondino and S. Suhr in their optimal-transport-theoretic formulation of the Einstein equations.

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Causal time-evolution of measures ($\mathcal{M}$ – Minkowski)

Causal time-evolution of a pointlike particle

A curve $\gamma : I \to \mathcal{M}$ with $\gamma(t) = (t, x(t))$ is a worldline of a physical particle if

$$\forall s,t \in I \quad s \leq t \;\Rightarrow\; \gamma(s) \preceq \gamma(t).$$

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A map $\mu : I \to \mathscr{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\operatorname{supp} \mu_t \subset \{t\} \times \mathbb{R}^3$ for all $t \in I$ is a **causal evolution of a measure** if

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- Fix a Cauchy temporal function $\mathcal{T}$.

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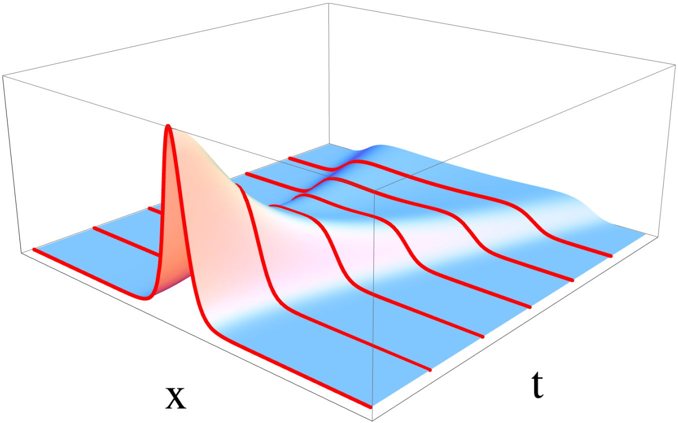

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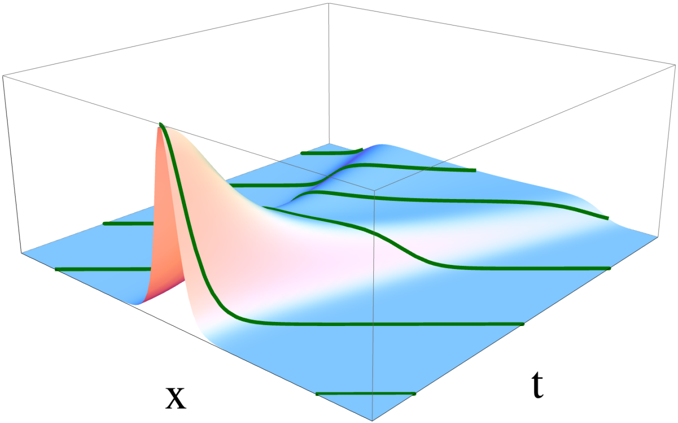

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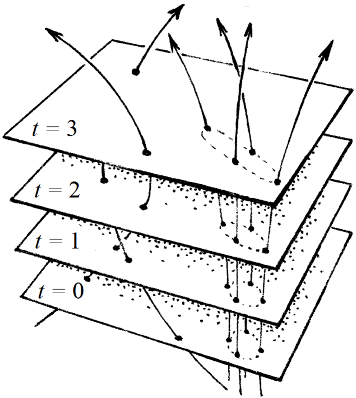
Theorem [TM '17]

Fix a Cauchy temporal function $\mathcal{T}$.
Consider a map $t \mapsto \mu_t \in \mathscr{P}(\mathcal{M})$ satisfying $\operatorname{supp} \mu_t \subset \mathcal{T}^{-1}(t)$ for all $t \in I$.
TFAE:

- The map $t \mapsto \mu_t$ is *causal*, i.e.
  $$\forall s, t \in I \quad s \leq t \;\Rightarrow\; \mu_s \preceq \mu_t.$$
- There exists a **probability measure on the space of worldlines**, from which one can recover μ_t for all $t \in I$.

The "space of worldlines" is suitably topologized so as to ensure **Polishness**.



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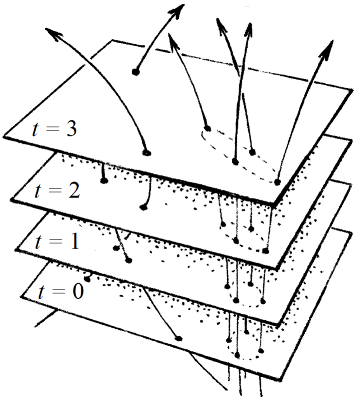

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- Choose a Cauchy temporal function $\mathcal{T}$.
- This fixes a Geroch–Bernal–Sànchez splitting $\mathcal{M} \cong \mathbb{R} \times \mathcal{S}$.
- Call $\mathbb{R} \times \mathcal{S}^N$ an "N-particle glob. hyp. spacetime".
- Causal relation:
  $(s, x_1, \ldots, x_N) \preceq (t, y_1, \ldots, y_N) \quad \Leftrightarrow \quad \forall i \; (s, x_i) \preceq (t, y_i)$.
- Extend $\preceq$ onto $\mathscr{P}(\mathbb{R} \times \mathcal{S}^N)$ employing the notion of a coupling.

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- This fixes a Geroch–Bernal–Sànchez splitting $\mathcal{M} \cong \mathbb{R} \times \mathcal{S}$.
- Call $\mathbb{R} \times \mathcal{S}^N$ an "N-particle glob. hyp. spacetime".
- Causal relation:
  $(s, x_1, \ldots, x_N) \preceq (t, y_1, \ldots, y_N) \quad \Leftrightarrow \quad \forall i \ (s, x_i) \preceq (t, y_i)$.
- Extend $\preceq$ onto $\mathscr{P}(\mathbb{R} \times \mathcal{S}^N)$ employing the notion of a coupling.

Causal time-evolution of an N-particle probability measure

A map $\mu : I \to \mathscr{P}(\mathbb{R} \times \mathcal{S}^N)$, $t \mapsto \mu_t$ such that $\operatorname{supp} \mu_t \subseteq \{t\} \times \mathcal{S}^N$ for all $t \in I$ is a **causal evolution of an N-particle measure** if

$$\forall s, t \in I \quad s \leq t \ \Rightarrow \ \mu_s \preceq \mu_t.$$

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Causality for N-particle systems (work in progress)

We retain most of the characterisations of $\preceq$!

Fix $s, t \in I$. TFAE:

- $\mu_s \preceq \mu_t$
- For any compact $\mathcal{K} \subseteq \operatorname{supp} \mu_s \quad \mu_s(\mathcal{K}) \leq \mu_t(J^+(\mathcal{K}))$
- For any Cauchy hypersurface $\Sigma \subset \mathbb{R} \times \mathcal{S}^N \quad \mu_s(J^+(\Sigma)) \leq \mu_t(J^+(\Sigma))$
- For any time function f, $\int_{\mathbb{R}\times\mathcal{S}^N} f d\mu_s \leq \int_{\mathbb{R}\times\mathcal{S}^N} f d\mu_t$

where f being time means that $f \in C(\mathbb{R} \times \mathcal{S}^N)$ and the condition that $\forall i \ (s, x_i) \preceq (t, y_i)$ implies that $f(s, x_1, \ldots, x_N) < f(t, y_1, \ldots, y_N)$.

We also retain the equivalence between the evolution being causal and the existence of a single probability measure on the (space of worldlines)^N.

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Conclusions and take-home messages

- The causal relation J^+ can be **naturally** extended onto $\mathscr{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.
- One can use thus extended relations to describe the **causal evolution of probability measures** in glob. hyperbolic spacetimes.
  - Time-evolution of a **pointlike** particle $\leftrightsquigarrow$ **single** worldline.
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- One can generalize the above formalism to an N-particle setting, constructing the "N-particle glob. hyp. spacetime" $\mathbb{R} \times \mathcal{S}^N$ upon the Geroch–Bernal–Sànchez splitting $\mathcal{M} \cong \mathbb{R} \times \mathcal{S}$.
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Thank you for your attention!

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A1: Relationship with the continuity equation ($\mathcal{M}$ – Minkowski)

Theorem [M. Eckstein, TM '17]

Suppose $\rho(t,x)$ satisfies the **continuity equation** $\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0$ with a velocity field such that $\|\mathbf{v}(t,x)\| \leq 1$. Then μ_t defined via

$$d\mu_t = \delta_t \otimes \rho(t,x)\, d^3x$$

evolves **causally**.

More generally, suppose μ_t satisfies:

$$\forall\, \Phi \in C_c^\infty(I \times \mathbb{R}^n) \quad \int_I \int_{\mathcal{M}} (\partial_t + \mathbf{v} \cdot \nabla)\, \Phi\, d\mu_t dt = 0$$

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Conjecture

Fix a Cauchy temporal function $\mathcal{T}$. Suppose μ_t (such that $\mathrm{supp}\ \mu_t \subseteq \mathcal{T}^{-1}(t)$) satisfies:

$$\forall\, \Phi \in C_c^\infty(\mathcal{T}^{-1}(I)) \quad \int_I \int_{\mathcal{M}} X\Phi \, d\mu_t dt = 0 \tag{$\star$}$$

with a certain **causal** vector field X. Then μ_t evolves causally.

Converse result (preliminary!)

Fix a Cauchy temporal function $\mathcal{T}$. Suppose μ_t evolves causally. Then there exists a **causal** vector field X such that $(\star)$ holds.

X is generally rather low-regular. Namely, $L^2(\mathcal{T}^{-1}(I), \int_I \mu_t dt)$-regular.

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A2: Polish spaces of causal curves

- **Q**: How to topologize sets of (fut-dir) causal curves?
  A (naïve): Induce topology from $C(I, \mathcal{M})$ (the compact-open top.)
- Too large a space! Various parameterizations of an unparameterized curve treated as distinct elements!
- Two ways out:
  - Take a quotient modulo (continuous strictly increasing) reparameterizations $\Leftrightarrow$ focus on unparameterized curves, and use the C^0**-topology**.
  - Choose the "canonical" parameterization of each curve — e.g. the arc-length parameterization — and use the **compact-open topology**.

A2: Polish spaces of causal curves

Spaces of causal curves parameterized "in accordance with $\mathcal{T}$"

$\mathcal{M}$ – stably causal spacetime, $\mathcal{T}$ – time function, I – interval.
$C^I_{\mathcal{T}} :=$ the space of all fut-dir causal curves $\gamma \in C(I, \mathcal{M})$ such that

$$\exists\, c_\gamma > 0 \;\; \forall s, t \in I \qquad \mathcal{T}(\gamma(t)) - \mathcal{T}(\gamma(s)) = c_\gamma (t - s),$$

endowed with the compact-open topology induced from $C(I, \mathcal{M})$.

- $C^I_{\mathcal{T}}$ is separable and completely metrizable (i.e. Polish).
- $\mathscr{C} :=$ the space of all *compact* unparameterized causal curves with the C^0-topology. Theorem: $\quad C^{[a,b]}_{\mathcal{T}} \cong \mathscr{C} \quad$ and hence:
  - $\mathscr{C}$ is Polish!
  - $C^{[a,b]}_{\mathcal{T}_1} \cong C^{[c,d]}_{\mathcal{T}_2}$.
- $\mathcal{M}$ – glob. hyperbolic, $\mathcal{T}_1, \mathcal{T}_2$ – Cauchy temporal functions.
  Theorem: $C^{\mathbb{R}}_{\mathcal{T}_1} \cong C^{\mathbb{R}}_{\mathcal{T}_2}$.

Theorem [TM '17]

Fix a Cauchy temporal function $\mathcal{T}$.
Consider a map $t \mapsto \mu_t \in \mathscr{P}(\mathcal{M})$ satisfying $\mathrm{supp}\ \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$.
TFAE:

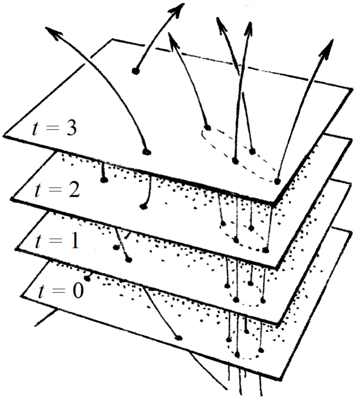
- The map $t \mapsto \mu_t$ is *causal*, i.e.

$$\forall s, t \in I \quad s \leq t \ \Rightarrow \ \mu_s \preceq \mu_t.$$

- $\exists\, \sigma \in \mathscr{P}(C^I_{\mathcal{T}})$ such that

$$(\mathrm{ev}_t)_\# \sigma = \mu_t,$$

where $\mathrm{ev}_t : C^I_{\mathcal{T}} \to \mathcal{M}$, $\gamma \mapsto \gamma(t)$.



Adapted from R. Penrose's "Road to Reality"