A pullback structure of trimmable graph C^*-algebras

(joint work with: F. Arici, P.M. Hajac, M. Tobolski)

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27/11/2018

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Kolkata, India, 27.11.2018 – 30.11.2018

Pushout diagrams

$U(1)$-equivariant pushout diagram:

$$\begin{tikzcd}
 & S^{2n+1} & \\
S^{2n-1} \arrow[ur, hook] & & B^{2n} \times S^1 \arrow[ul, hook'] \\
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A commutative diagram (of sets):

$$\begin{tikzcd}
P & Y \arrow[l] \\
X \arrow[u] & Z \arrow[l, "g"] \arrow[u, "f"']
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is a pushout diagram if

$$P \simeq \frac{X \sqcup Y}{f(z) \sim g(z)}$$

We get P by "gluing" X and Y along Z.

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For $n = 1$:

$$\begin{tikzcd}
 & S^2 & \\
\{p\} \atop \text{(north pole)} \arrow[ur] & & B^2 \atop \text{(closed disk)} \arrow[ul] \\
 & S^1 \atop \text{(disk's boundary)} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Pullbacks of function algebras

Dualizing:

$$\begin{tikzcd}[column sep=tiny] & C(S^{2n+1}) \arrow[dl] \arrow[dr] & \\ C(S^{2n-1}) \arrow[dr] & & C(B^{2n}) \otimes C(S^1) \arrow[dl] \\ & C(S^{2n-1}) \otimes C(S^1) & \end{tikzcd}$$

with $U(1)$-invariant subalgebras:

$$\begin{tikzcd}[column sep=tiny] & C(\mathbb{CP}^n) \arrow[dl] \arrow[dr] & \\ C(\mathbb{CP}^{n-1}) \arrow[dr] & & C(B^{2n}) \arrow[dl] \\ & C(S^{2n-1}) & \end{tikzcd}$$

A commutative diagram of C^*-algebras:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a pullback diagram if

$$A \simeq \left\{(b, c) \in B \times C : f(b) = g(c)\right\}$$

$$\mathrel{\stackrel{!}{\|}}$$

$$B \times_D C$$

Applications? Get K-theory recursively $\Rightarrow$ Milnor's connecting homomorphism.

Interlude: Mayer-Vietoris and pushouts

If $\{X, Y\}$ is an open cover of a smooth n-manifold P, one has the pushout diagram:

$$\begin{tikzcd} P & Y \arrow[l] \\ X \arrow[u] & Z := X \cap Y \arrow[l] \arrow[u, hook] \end{tikzcd}$$

a short exact sequence of k-forms, and a long exact sequence in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0_{dR}(P) \arrow[r] & H^0_{dR}(X) \oplus H^0_{dR}(Y) \arrow[r] & H^0_{dR}(Z) \arrow[dll, rounded corners] \\ & H^1_{dR}(P) \arrow[r] & H^1_{dR}(X) \oplus H^1_{dR}(Y) \arrow[r] & H^1_{dR}(Z) \arrow[dll, rounded corners] \\ & H^2_{dR}(P) \arrow[r] & \cdots\cdots\cdots \arrow[r] & H^n_{dR}(Z) \arrow[r] & 0 \end{tikzcd}$$

This holds for more general (co)homology theories (e.g. singular) and one-injective pushout diagrams (e.g. of CW complexes).

Interlude: Mayer-Vietoris in K-theory

From a one-surjective pullback diagram of C^*-algebras

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, two heads, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

we get a six-term exact sequence

$$\begin{tikzcd} K_0(A) \arrow[r, "(i_*, j_*)"] & K_0(B \oplus C) \arrow[r, "f_* - g_*"] & K_0(D) \arrow[d, dashed, "d_{01}"] \\ K_1(D) \arrow[u, dashed, "d_{10}"] & K_1(B \oplus C) \arrow[l, "f_* - g_*"'] & K_1(A) \arrow[l, "(i_*, j_*)"'] \end{tikzcd}$$

with d_{10}, d_{01} the “connecting homomorphisms”.

Quantum spaces

There is a $U(1)$-equivariant commutative diagram:

$$
\begin{tikzcd}
 & C(S_q^{2n+1}) \arrow[dl, two heads, "\pi"] \arrow[dr] & \\
C(S_q^{2n-1}) \arrow[dr, hook, "\delta"] & & C(B_q^{2n}) \otimes C(S^1) \arrow[dl, two heads, "\partial\otimes\mathrm{id}"'] \\
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$\delta = C(S^1)$-coaction

$\partial =$ evaluation at the boundary

$\pi =$ killing one complex "coordinate"

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The $U(1)$-invariant part is automatically a (one-surjective) pullback diagram:

$$\begin{tikzcd}
 & C(\mathbb{CP}_q^n) \arrow[dl] \arrow[dr] & \\
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 & C(S_q^{2n-1}) &
\end{tikzcd}$$

K-theory of q-projective spaces

We know that:

	$\mathbb{CP}^n_q$	B^{2n}_q	S^{2n-1}_q
K_0	$\mathbb{Z}^{n+1}$	$\mathbb{Z}$	$\mathbb{Z}$
K_1	0	0	$\mathbb{Z}$

The six-term exact sequence:

$$\begin{tikzcd} K_0(C(\mathbb{CP}^n_q)) \arrow[r] & K_0(C(\mathbb{CP}^{n-1}_q)) \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[u, "d_{10}"] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

gives:

$$K_0(C(\mathbb{CP}^n_q)) \simeq K_0(C(\mathbb{CP}^{n-1}_q)) \oplus d_{10}\big(K_1(C(S^{2n-1}_q))\big)$$

The extra ($n+1$-th) generator of $K_0(C(\mathbb{CP}^n_q))$ comes from the generator of $K_1(C(S^{2n-1}_q))$.

Graph C^*-algebras

A graph $G = (G^0, G^1, s, t)$ consists of

- a countable set G^0 of vertices;
- a countable set G^1 of edges;
- source and target maps $s, t : G^1 \to G^0$.

G is row-finite

$$\Updownarrow$$

$s^{-1}(v)$ is finite $\forall\, v \in G^0$.

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$C^*(G)$:= universal C^*-algebra generated by mutually orthogonal projections $\{P_v : v \in G^0\}$ and partial isometries $\{S_e : e \in G^1\}$ such that:

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$$\sum_{e \in s^{-1}(v)} S_e S_e^* = P_v \quad \forall\, v \in G^0 : s^{-1}(v) \neq \varnothing$$

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Examples:

- all Cuntz algebras
- all finite-dim. C^*-algebras (G finite, no cycles)
- $C(S^1)$, $\mathcal{K}$, $\mathcal{T}$, $M_n(C(S^1))$, certain q-algebras

Up to Morita equivalence, graph C^*-algebras include:

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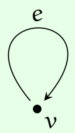

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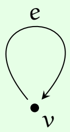
$C(S^1)$:



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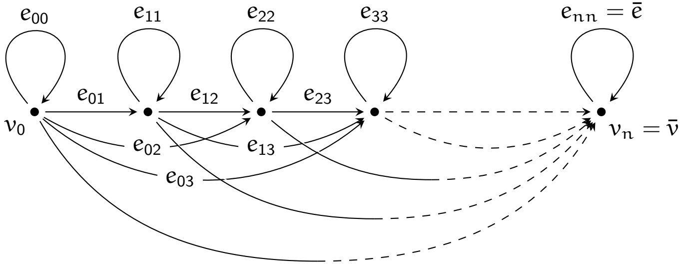
q-spheres

G has $n+1$ vertices and an edge $e_{ij} : v_i \to v_j$ for all $i \leqslant j$.



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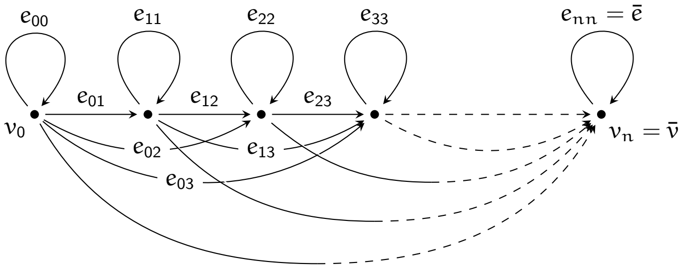

$C(S_q^{2n+1})$ generated by $\{z_i, z_i^*\}_{i=0}^n$ with commutation relations: $\boxed{z_1 z_0 = q z_0 z_1}$, etc.

and sphere condition: $\boxed{z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* = 1}$

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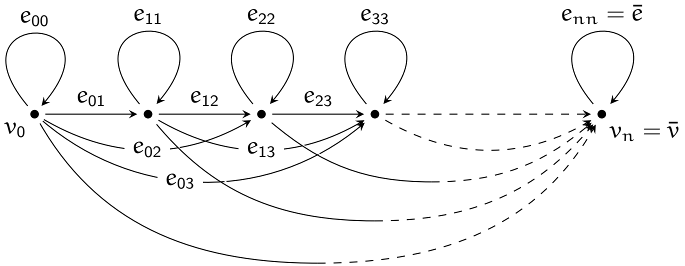

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$C(S_q^{2n+1}) \simeq C(S_{q=0}^{2n+1})$ $\forall$ $0 < q < 1$, and $C(S_{q=0}^{2n+1}) \simeq C^*(G)$ via: $\boxed{z_i \mapsto \sum_{j=i}^{n} S_{e_{ij}}}$

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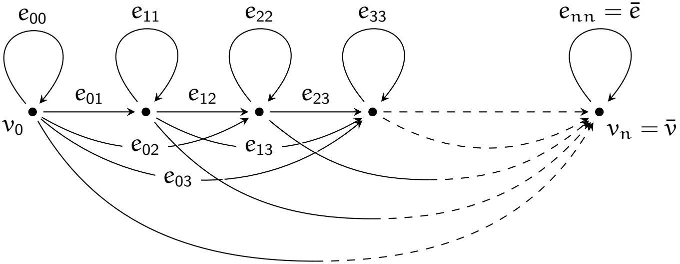

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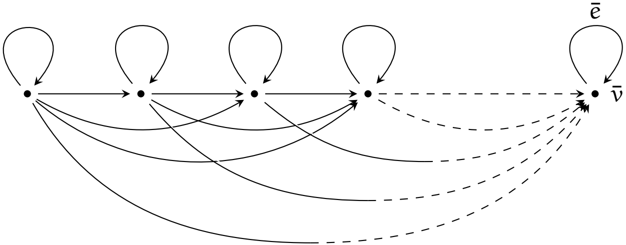
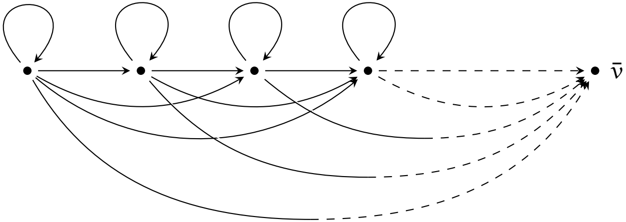
K_1 generated by $\boxed{U := S_{\bar{e}} + (1 - P_{\bar{v}})}$

Morphisms



S_q^{2n+1}



B_q^{2n}

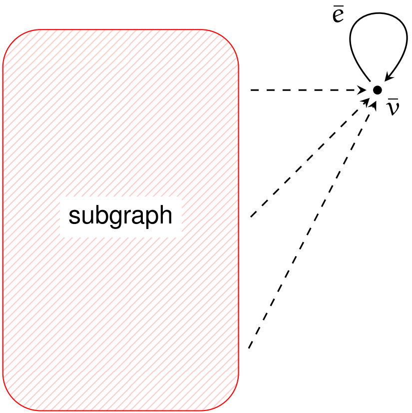
$$\begin{tikzcd} C(S_q^{2n+1}) \arrow[r, "r"] \arrow[rr, bend right, "\pi"'] & C(B_q^{2n}) \arrow[r, "\partial"] & C(S_q^{2n-1}) \end{tikzcd}$$

$r : S_{\bar{e}} \mapsto P_{\bar{v}}$

$\partial : t^{-1}(\bar{v}) \mapsto 0,\ P_{\bar{v}} \mapsto 0$

$\pi : z_n \mapsto 0$

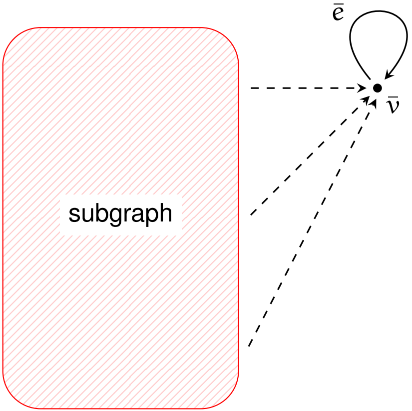
Trimmable graphs



A graph with a distinguished vertex $\bar{v}$ is called $\bar{v}$-trimmable if:

1. $\bar{v}$ emits one loop $\bar{e}$ and no other edges;
2. $\bar{v}$ is target of other edges, besides $\bar{e}$;
3. every vertex of the subgraph emitting an arrow ending in $\bar{v}$, also emits (at least) another arrow not ending in $\bar{v}$.

Trimmable graphs



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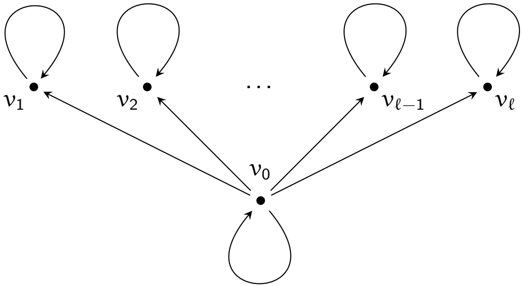
Examples:

- Vaksman-Soibelman quantum spheres S_q^{2n+1},
- quantum lens spaces $L_q^3(\ell; 1, \ell)$ (with $\,.\,/\mathrm{U}(1) = \mathbb{WP}_q^1(1, \ell)$ quantum teardrops),
- one loop extensions, ...

Example: quantum lens spaces

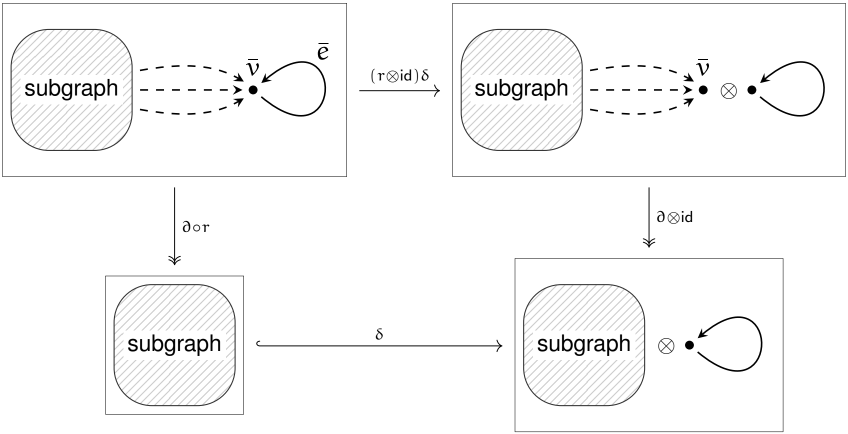
$C(L^3_q(\ell; 1, \ell))$ is the graph C^*-algebra of the graph:



Every vertex is trimmable except v_0.

Pullback structure of trimmable graph C^*-algebras

A $U(1)$-equivariant (cf. gauge action) commutative diagram:



Want to prove: it is a pullback diagram.

Sketch of the proof

Consider morphisms of associative algebras (g injective):

$$B \xrightarrow{f} D \xhookleftarrow{g} C$$

Fact 1. $B \times_D C$ isomorphic to the subalgebra P of B given by: $\quad P := f^{-1}\big(g(C)\big)$

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Fact 1. $B \times_D C$ isomorphic to the subalgebra P of B given by: $\quad P := f^{-1}\big(g(C)\big)$

Fact 2. Given a one-injective one-surjective diagram:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, two heads] & B \arrow[d, "f"] \\ C \arrow[r, hook, "g"'] & D \end{tikzcd}$$

φ always maps A into P: we only have to show that it is a bijection with P.

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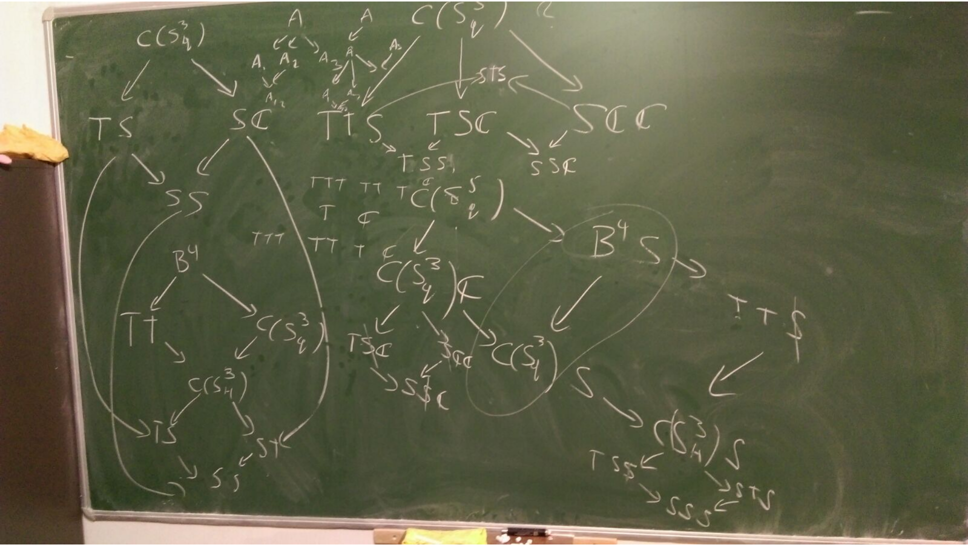
In our case, injectivity comes from the "gauge invariant uniqueness theorem":

$$\varphi\ U(1)\text{-equivariant} \quad \& \quad \varphi(P_v) \neq 0\ \forall\, v \quad \Longrightarrow \quad \varphi \text{ is injective}$$

For surjectivity one uses properties of ideals associated to "saturated hereditary" subsets.

Forthcoming…



Thank you for your attention.