Sigma-model solitons
on
noncommutative spaces

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Solitons of general topological charge over noncommutative tori
L. Dabrowski, M.S. Jakobsen, G.L. Franz Luef
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and earlier work with L. Dabrowski and T. Krajewski

and more recent work by J. Rosenberg and Mathai V.

Moyal plane:

$$[x_2, x_1] = \mathsf{i}\theta$$

IRA:

$$U_2U_1 = e^{2\pi\mathsf{i}\theta}U_1U_2$$

Moyal - IRA:

$$U_1 = e^{2\pi\mathsf{i}x_1} \qquad U_2 = e^{2\pi\mathsf{i}x_2}$$

Abstract

Sigma-model solitons over the Moyal plane and noncommutative tori
(also known as Irrational Rotation Algebra),

as source spaces, with a target space made of two points

A natural action functional leads to self-duality equations for projections in the source algebra

Solutions, having non-trivial topological content (instantons),
are constructed via suitable Morita duality bimodules
(obtained using the Schrödinger representation).

Inputs from time-frequency analysis and Gabor analysis

Solitons over NC tori via Gabor frames of time-frequency analysis.

The energy functional for projections in noncommutative tori becomes a functional for functions generating Gabor frames in $L^2(\mathbb{R} \times \mathbb{Z}_q)$

$q = 1$. Gabor frames for $L^2(\mathbb{R})$ of the form $\{E_{m\beta} T_{n\alpha} g\}_{m,n \in \mathbb{Z}}$, with g a Schwartz function; α and β are parameters in $\mathbb{R} \backslash \{0\}$ such that $|\alpha\beta| < 1$.

$(T_x) g(t) = g(t - x)$, $x, t \in \mathbb{R}$ the translation operator

$(E_\omega) g(t) = e^{2\pi i t \omega} g(t)$, $\omega, t \in \mathbb{R}$ the modulation operator.

The energy functional,

$$E(g) = \frac{\pi}{|\alpha\beta|} \sum_{n,m \in \mathbb{Z}} ((\alpha n)^2 + (\beta m)^2) \; |\langle g, E_{m\beta} T_{n\alpha} S_g^{-1} g \rangle|^2$$

is bounded from below by the constant $q = 1$ (a Chern number).

The Gaussian $g(x) = e^{-\pi x^2 - i\lambda x}$, $\lambda \in \mathbb{C}$ attains this minimum.

Non-linear σ-models:

field theories of maps X between the source space (Σ, g), and the target space (M, G). The action functional

$$S[X] = \frac{1}{2\pi} \int_\Sigma \sqrt{g}\; g^{\mu\nu}\, G_{ij}(X) \partial_\mu X^i\, \partial_\nu X^j\,,$$

The stationary points:

harmonic maps from Σ to M; describe minimal surfaces embedded in M.

Σ two dimensional, conformal invariance:

the action S is invariant for any rescaling of the metric $g \;\to\; e^\sigma g$.

Thus the action only depends on the conformal class of the metric and may be rewritten using a complex structure on Σ

$$S[X] = \frac{\mathrm{i}}{\pi} \int_\Sigma G_{ij}(X)\, \partial X^i \wedge \bar\partial X^j\,,$$

Here ∂ and $\bar{\partial}$, a complex structure and $\mathrm{d} = \partial + \overline{\partial}$.

In two dimensions

complex and conformal

are the same thing.

In two-dimensions, the conformal class of a general constant metric is parametrized by a complex number $\tau \in \mathbb{C}$, $\Im\tau > 0$.

Up to a conformal factor, the metric is

$$g = (g_{\mu\nu}) = \begin{pmatrix} 1 & \Re\tau \\ \Re\tau & |\tau|^2 \end{pmatrix} .$$

An algebraic generalization: by dualization and reformulation in terms of the $*$-algebras $\mathcal{A} = C^\infty(\Sigma, \mathbb{C})$ and $\mathcal{B} = C^\infty(M, \mathbb{C})$.

Embeddings X of Σ into M correspond to $*$-algebra morphisms $\pi_X : \mathcal{B} \to \mathcal{A}$, with correspondence

$$f \mapsto \pi_X(f) = f \circ X.$$

$*$-algebra morphisms make sense for general algebras $\mathcal{A}$ and $\mathcal{B}$.

The configuration space is all $*$-algebra morphisms from $\mathcal{B}$ to $\mathcal{A}$

The definition of the action functional involves generalizations of the conformal and Riemannian geometries.

Connes:
conformal is understood in the framework of positive Hochschild cohomology.

The tri-linear map $\phi : \mathcal{A}^{\otimes 3} \to \mathbb{R}$,

$$\phi(f_0, f_1, f_2) = \frac{\mathrm{i}}{\pi} \int_\Sigma f_0 \partial f_1 \wedge \bar{\partial} f_2$$

is an extremal of positive Hochschild cocycles belonging to the Hochschild cohomology class of the cyclic cocycle ψ defined by

$$\psi(f_0, f_1, f_2) = \frac{\mathrm{i}}{2\pi} \int_\Sigma f_0 \, \mathrm{d} f_1 \wedge \mathrm{d} f_2.$$

On the one hand ψ, the fundamental class, allows one to integrate 2-forms in dimension 2, so it is a metric independent object

On the other hand, ϕ defines a suitable positive scalar product

$$\langle a_0 da_1, b_0 db_1 \rangle = \phi(b_0^* a_0, a_1, b_1^*)$$

on 1-forms and depends on the conformal class of the metric.

Expressions like ϕ and ψ make sense for a general algebra $\mathcal{A}$.

Compose the cocycle ϕ with a morphism $\pi : \mathcal{B} \to \mathcal{A}$ to obtain a positive cocycle on $\mathcal{B}$

$$\phi_\pi = \phi \circ (\pi \otimes \pi \otimes \pi)$$

Evaluate the cocycle ϕ_π on a suitably element of $\mathcal{B}^{\otimes 3}$ which provides the noncommutative analogue of the metric on the target;

Easiest choice for this metric: a positive element $G = \sum_i b_0^i \delta b_1^i \delta b_2^i$ of the space of universal 2-forms $\Omega^2(\mathcal{B})$.

Thus, a well defined and positive quantity

$$S[\pi] = \phi_\pi(G) \tag{1}$$

a noncommutative analogue of the action functional of the non linear σ-model.

Here:

π is the dynamical variable (the embedding)

whereas ϕ (the conformal structure on the source) and
G (the metric on the target) are background structures that have been fixed.

The critical points of the σ-model for the action functional (1): generalizations of harmonic maps; "minimally embedded surfaces" in the (noncommutative) space associated with $\mathcal{B}$.

The role of the other cocycle ψ is to give a topological 'charge'.

More on this late on.

Two points as a target space $M = \{1, 2\}$

Any continuous map from a connected surface Σ to a discrete space is constant, a commutative theory would be trivial.

Not the case for a "noncommutative" source space: in general, not trivial such maps, 'dually', as $*$-algebra morphisms from the algebra of functions over $M = \{1, 2\}$, that is $\mathbb{C}^2$, to the algebra $\mathcal{A}$ of the noncommutative source.

As a vector space $\mathbb{C}^2$ is generated by the projection function e defined by $e(1) = 1$ and $e(2) = 0$;

$\Rightarrow$ any $*$-algebra morphism $\pi : \mathbb{C}^2 \to \mathcal{A}$ is the same as a projection $p = \pi(e) \in \mathcal{A}$.

The configuration space of a two point target space sigma-model is the collection of all projections $P(\mathcal{A})$ in the algebra $\mathcal{A}$.

Choosing the metric $G = \delta e \delta e$ on the space $M = \{1, 2\}$, and a Hochschild cocycle ϕ for the conformal structure, the action functional is simply

$$S[p] = \phi(1, p, p),$$

Positivity in Hochschild cohom implies it is bounded by a topological term

Noncommutative torus and Moyal plane as source space:

$$(\mathcal{A}, \mathrm{tr}, \partial_1, \partial_2)$$

the action functional is

$$S[p] = \frac{1}{4\pi}\mathrm{tr}(\partial p \overline{\partial} p).$$

with the natural complex structure on $\mathcal{A}$ given by

$$\partial = \partial_1 - \mathrm{i}\,\partial_2, \qquad \overline{\partial} = \partial_1 + \mathrm{i}\,\partial_2,$$

and derivations ∂_1 and ∂_2 infinitesimal generators of a $\mathbb{T}^2$-action

and tr an invariant trace.

All of above can be extended to more general metrics.

In two dimensions: Up to a conformal factor the general constant metric is parametrized by a complex number $\tau \in \mathbb{C}$, $\Im\tau > 0$.

The corresponding 'complex torus' $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z} + \tau\mathbb{Z}$ would act infinitesimally on $\mathcal{A}$ with two complex derivations

$$\partial = \partial_1 + \bar{\tau}\partial_2\,, \qquad \overline{\partial} = \partial_1 + \tau\partial_2.$$

As usual, the critical points of the action functional are obtained by equating to zero its first variation, that is the linear term in an infinitesimal variation

$$\delta S[p] = S[p + \delta p] - S[p], \qquad \text{for} \quad \delta p \in T_p(P(\mathcal{A}))\,.$$

One gets

$$p\ \Delta(p)\ (1-p) = 0 \qquad \text{and} \qquad (1-p)\ \Delta(p)\ p = 0\,,$$

or, equivalently the non-linear equations of the second order

$$p\ \Delta(p) - \Delta(p)\ p = 0\,. \tag{2}$$

with the Laplacian of the metric $\Delta = \frac{1}{2}(\partial\overline{\partial} + \overline{\partial}\partial)$

The cyclic 2-cocycle giving the fundamental class is

$$\psi(a_0, a_1, a_2) = \frac{1}{2\pi \mathrm{i}} \operatorname{tr}\big(a_0(\partial_1 a_1 \partial_2 a_2 - \partial_2 a_1 \partial_1 a_2)\big),$$

For any projection $p \in P(\mathcal{A})$, the quantity

$$c_1(p) := \psi(p, p, p)$$

is an integer: the index of a Fredholm operator.

For any $p \in P(\mathcal{A})$ it holds that

$$S[p] \geq |c_1(p)| \,.$$

The equality for projection p satisfying self-duality or anti-self duality eqns

$$p(\partial_1 \pm \mathrm{i}\, \partial_2)(p) = 0 \qquad (3)$$

These equation imply the EOM (2).

Solutions are σ-model solitons

$$p = |\xi><\eta|$$

$|\xi>, |\eta>$ in a suitable Hilbert-module

'field-theory' Hilbert space

Projection from Morita equivalence (Rieffel)

A Morita equivalence between (pre C^*-algebras) $\mathcal{A}$ and $\mathcal{B}$:

a $\mathcal{A} - \mathcal{B}$-bimodule $\mathcal{E}$

with a left-linear $\mathcal{A}$-valued hp ${}_\bullet\langle\cdot,\cdot\rangle$ and a right-linear $\mathcal{B}$-valued hp $\langle\cdot,\cdot\rangle_\bullet$.

There is an associativity condition:

$$ {}_\bullet\langle\xi,\eta\rangle\,\zeta = \xi\,\langle\eta,\zeta\rangle_\bullet $$

It follows an identification $\mathcal{B} \simeq \mathcal{K}_\mathcal{A}(\mathcal{E})$ (compact endomorphisms).

In particular, there exist elements $\{\eta_1, ..., \eta_n\}$ in $\mathcal{E}$ such that

$$ \sum_j \langle\eta_j,\eta_j\rangle_\bullet = 1_\mathcal{B}\,. $$

Then, the associativity condition gives that the matrix $p = (p_{jk})$

$$ p_{jk} = {}_\bullet\langle\eta_j,\eta_k\rangle $$

is a projection in the matrix algebra $M_n(\mathcal{A})$.

Both algebras $\mathcal{A}$ and $\mathcal{B}$ are in the joint smooth domain of two commuting derivations ∂_1 and ∂_2 ; and have faithful invariant tracial states, which are compatible in the sense that:

$$\mathrm{tr}\ {}_\bullet\langle \xi, \eta\rangle = \mathrm{tr}\ \langle \eta, \xi\rangle_\bullet$$

Derivations are lifted to $\mathcal{E}$ as (left and right) covariant derivatives:

$$\nabla_j : \mathcal{E} \to \mathcal{E}\,, \quad j = 1, 2,$$

$$\nabla_j(a\,\xi) = (\partial_j a)\,\xi\ + a\,(\nabla_j \xi) \quad \text{and} \quad \nabla_j(\xi\,b) = (\nabla_j \xi)\,b + \xi(\partial_j b)$$

compatible with both the
$\mathcal{A}$-valued hermitian structure ${}_\bullet\langle \cdot, \cdot\rangle$; the $\mathcal{B}$-valued hermitian structure $\langle \cdot, \cdot\rangle_\bullet$

$$\partial_j({}_\bullet\langle \xi, \eta\rangle) = {}_\bullet\langle \nabla_j \xi, \eta\rangle + {}_\bullet\langle \xi, \nabla_j \eta\rangle$$

and

$$\partial_j(\langle \xi, \eta\rangle_\bullet) = \langle \nabla_j \xi, \eta\rangle_\bullet + \langle \xi, \nabla_j \eta\rangle_\bullet\,.$$

Lifting self-duality equations: solitons

The holomorphic/anti-holomorphic, connection on $\mathcal{E}$,

$$\nabla = \nabla_1 - \mathrm{i}\,\nabla_2, \qquad \overline{\nabla} = \nabla_1 + \mathrm{i}\,\nabla_2$$

lift to $\mathcal{E}$ the complex derivations $\partial = \partial_1 - \mathrm{i}\,\partial_2$ or $\overline{\partial} = \partial_1 + \mathrm{i}\,\partial_2$.

The 'rank' one case:

Seek solutions of the s-d eqs (3) of the form

$$p_\psi := {}_\bullet\langle \psi, \psi \rangle \in \mathcal{A} \quad \text{with} \quad \psi \in \mathcal{E} \quad \text{such that} \quad \langle \psi, \psi \rangle_\bullet = 1_{\mathcal{B}}.$$

The projection p_ψ is a solution of the s-d eqs:

$$p_\psi \partial(p_\psi) = 0\,,$$

if and only if the vector ψ is a generalized eigenvector of $\overline{\nabla}$

i.e. there exists $\lambda \in \mathcal{B}$ such that

$$\overline{\nabla}\psi = \psi\lambda\,.$$

How to compute the topological charge:

The curvature of the covariant derivatives is defined as

$$F_{12} := \nabla_1 \nabla_2 \ - \nabla_2 \nabla_1$$

Let $\psi \in \mathcal{E}$ be such that $\langle \psi, \psi \rangle_{\bullet} = 1_{\mathcal{B}}$ and $p_\psi := {}_{\bullet}\langle \psi, \psi \rangle \in \mathcal{A}$ the corresponding projection. Then, its topological charge is:

$$c_1(p_\psi) = -\frac{1}{2\pi \mathrm{i}} \, \mathrm{tr} \, \langle \psi, F_{12} \psi \rangle_{\bullet} \, .$$

Constant curvature: $F_{12} = -2\pi \mathrm{i} \, q \, \mathrm{id}_{\mathcal{E}}$

the projection $p_\psi = {}_{\bullet}\langle \psi, \psi \rangle$ has then topological charge

$$c_1(p) = q \, \mathrm{tr}(1_B) \in \mathbb{Z}$$

note that neither q nor $\mathrm{tr}(1_B)$ need be an integer

Moyal plane from Schrödinger representation

The projective representation of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ defined for $\xi \in L^2(\mathbb{R})$ by

$$(\pi(z)\,\xi)(t) = e^{2\pi \mathrm{i} t\omega}\xi(t-x), \qquad \text{for} \quad z = (x,\omega). \tag{4}$$

$$\Rightarrow \qquad \pi(z)\,\pi(z') = e^{-2\pi \mathrm{i} x\omega'}\pi(z+z').$$

The map $c : \mathbb{R}\times\mathbb{R} \to \mathbb{T}$, $c(z,z') = e^{-2\pi\mathrm{i}(x\omega')}$ is a 2-cocycle.

Its matrix-coefficients are defined for $\xi,\eta \in L^2(\mathbb{R})$ by

$$V_\eta\xi(z) := \langle \xi, \pi(z)\eta\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \xi(t)\overline{\eta}(t-x)e^{-2\pi\mathrm{i}t\omega}\mathrm{d}t$$

In signal analysis $V_\eta\xi$ is known as the short time Fourier transform

Moyal's identity: $\langle V_\eta \xi, V_\psi \varphi \rangle_{L^2(\mathbb{R}^2)} = \langle \xi, \varphi \rangle_{L^2(\mathbb{R})} \overline{\langle \eta, \psi \rangle}_{L^2(\mathbb{R})}$

An additional important consequence of this identity:

is a reconstruction formula for $\xi \in L^2(\mathbb{R})$ in terms of $\{\pi(z)\eta : z \in \mathbb{R}^2\}$.

Let η and ψ be in $L^2(\mathbb{R})$ such that $\langle \psi, \eta \rangle \neq 0$. Then for any $\xi \in L^2(\mathbb{R})$,

$$\xi = \langle \psi, \eta \rangle^{-1} \iint_{\mathbb{R}^2} \langle \xi, \pi(z)\eta \rangle \pi(z)\psi \, \mathrm{d}z = \langle \psi, \eta \rangle^{-1} \iint_{\mathbb{R}^2} V_\eta \xi(z) \pi(z)\psi \, \mathrm{d}z.$$

The twisted group algebra $L^1(\mathbb{R}^2, c)$ of $\mathbb{R}^2$ associated to the cocycle c.

For k and l in $L^1(\mathbb{R}^2)$, the twisted convolution $(k \natural l)$:

$$(k \natural l)(z) = \iint k(z')l(z - z')c(z', z - z') \, \mathrm{d}z'$$

and twisted involution of $k \in L^1(\mathbb{R}^2)$:

$$k^\star(z) = c(z, z)\overline{k(-z)} = e^{-2\pi \mathrm{i} x\omega}\overline{k(-z)}$$

The integrated representation

$$K = \pi(k) = \iint_{\mathbb{R}}^{2} k(z)\pi(z)\,\mathrm{d}z$$

for $k \in L^1(\mathbb{R}^2)$, is a non-degenerate bounded representation of the twisted convolution algebra $L^1(\mathbb{R}, c)$ on $L^2(\mathbb{R}^2)$.

The adjoint of $K = \pi(k)$ is given by $K^* = \pi(k^\star)$ and the composition of $K = \pi(k)$ and $L = \pi(l)$ corresponds to $(k \natural l)$:

$$KL = \iint_{\mathbb{R}}^{2} (k \natural l)(z)\pi(z)\,\mathrm{d}z\,.$$

Denote by $\mathcal{A}$ the class of all operators $K = \pi(k)$ for $k \in \mathcal{S}(\mathbb{R}^2)$; they are all trace-class. Its norm closure is all compact operators.

$\mathcal{A}$ is a model of the Moyal plane: the Fourier transforms of the symbols defining elements of $\mathcal{A}$ yield the Moyal product:

$$k \star l = \mathcal{F}^{-1}\big(\mathcal{F}(k) \natural \mathcal{F}(l)\big) \qquad \text{for} \quad k, l \in \mathcal{S}(\mathbb{R}^2).$$

Rieffel:

The space $\mathcal{E} = \mathcal{S}(\mathbb{R})$ is an equivalence bimodule between $\mathcal{A}$ and $\mathbb{C}$ with respect to the actions:

$$K \cdot \xi = \iint k(z)\pi(z)\xi \, \mathrm{d}z,$$

$$\xi \cdot \lambda = \xi \, \overline{\lambda}$$

and $\mathcal{A}$ and $\mathbb{C}$-valued hermitian products:

$$_{\bullet}\langle \xi, \eta \rangle = \iint \langle \xi, \pi(z)\eta \rangle_{L^2(\mathbb{R})} \pi(z) \mathrm{d}z = \iint V_\eta \xi(z) \pi(z) \mathrm{d}z = \pi(V_\eta \xi)$$

$$\langle \xi, \eta \rangle_{\bullet} = \langle \eta, \xi \rangle_{L^2(\mathbb{R})} \, .$$

A two dimensional spectral geometry

Commuting derivations (an infinitesimal action of $\mathbb{T}^2$) ∂_1, ∂_2:

$$\partial_1 K = 2\pi\mathrm{i} \iint_{\mathbb{R}^2} x k(x,\omega)\pi(x,\omega)\,\mathrm{d}x\mathrm{d}\omega,$$

$$\partial_2 K = 2\pi\mathrm{i} \iint_{\mathbb{R}^2} \omega k(x,\omega)\pi(x,\omega)\,\mathrm{d}x\mathrm{d}\omega.$$

They lift to covariant derivatives on the equivalence bimodule $\mathcal{E}$:

$$(\nabla_1\xi)(t) = 2\pi\mathrm{i}\, t\,\xi(t) \qquad \text{and} \qquad (\nabla_2\xi)(t) = \xi'(t)$$

they are compatible with both left and right hermitian structures.

The connection has constant curvature:

$$F_{1,2} := [\nabla_1, \nabla_2] = -2\pi\mathrm{i}\ \mathrm{id}_{\mathcal{E}}$$

For $\psi \in \mathcal{S}(\mathbb{R})$ normalized as $\langle \psi, \psi \rangle_\bullet = \|\psi\|_2 = 1$,

$\Rightarrow$ a non-trivial projection $p_\psi = {}_\bullet\langle \psi, \psi \rangle$ in $\mathcal{A}$.

The projection p_ψ is a solution of the self-duality equations,

$$p_\psi(\partial p_\psi) = 0$$

if and only if, for some $\lambda \in \mathbb{C}$, the element ψ satisfies

$$\overline{\nabla}\psi = \psi\lambda\,.$$

Eigenfunction equations for $\overline{\nabla}$; solutions are generalized Gaussians:

$$\psi_\lambda(t) = c\, e^{-\pi t^2 - \mathrm{i}\lambda t}.$$

Explicitly,

$$p_\psi = {}_\bullet\langle \psi, \psi \rangle = \iint_{\mathbb{R}^2} V_\psi \psi(z) \pi(z)\, \mathrm{d}z$$

$$V_\psi \psi(x, \omega) = e^{-\frac{\pi}{2}(x^2+\omega^2)} e^{-\pi \mathrm{i} x\omega - \frac{\mathrm{i}}{2}(\bar{\lambda}+\lambda)x + \frac{1}{2}(\bar{\lambda}-\lambda)\omega}\,.$$

For its topological charge:

$$c_1(p_\psi) = \mathrm{tr}(p_\psi) = V_\psi\psi(0) = 1\,.$$

The constant curvature is none other than the Heisenberg commutation relations (in the Schrödinger representation).

The anti-holomorphic connection $\overline{\nabla} = \nabla_1 + \mathrm{i}\nabla_2$ is the annihilation operator;

the holomorphic $\nabla = \nabla_1 - \mathrm{i}\nabla_2$ is the creation operator.

The self-duality equation for the projections

is the equation for the minimizers of the Heisenberg uncertainty relation,

which explains why they are Gaussian ψ_λ.

The irrational rotation algebra (aka the noncommutative torus).

For $\theta \in \mathbb{R}$, the C^*-algebra A_θ of the noncommutative torus

is the norm closure of the span of $\{\pi(\theta k, l) : k, l \in \mathbb{Z}\}$: the restriction of the Schrödinger rep (4) of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ to $\theta\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$.

Denoting $\pi(0,1) = M_1$ and $\pi(\theta, 0) = T_\theta$ they satisfy:

$$M_1 T_\theta = e^{2\pi i \theta} T_\theta M_1 \,.$$

The smooth torus: subalgebra $\mathcal{A}_\theta$ of A_θ consisting of operators

$$\pi(\mathbf{a}) = \sum_{k,l \in \mathbb{Z}} a_{kl} \pi(\theta k, l), \qquad \text{for} \quad \mathbf{a} = (a_{kl}) \in \mathscr{S}(\mathbb{Z}^2).$$

With $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{S}(\mathbb{R})$ we have for their product

$$\pi(\mathbf{a})\pi(\mathbf{b}) = \pi(\mathbf{a} \natural \mathbf{b})$$

where $\mathbf{a} \natural \mathbf{b}$ is the twisted convolution

$$(\mathbf{a} \natural \mathbf{b})(k,l) = \sum_{m,n \in \mathbb{Z}} a_{mn} b_{k-m,n-l} e^{-2\pi i \theta n(k-m)}$$

while $\pi(\mathbf{a})^* = \pi(\mathbf{a}^*)$, where $\mathbf{a}^*$ is the twisted involution of $\mathbf{a}$:

$$(a_{kl})^* = e^{-2\pi i\theta kl}\overline{a_{-k,-l}}\,.$$

Operators commuting with $\pi(\theta k, l)$ are associated with the lattice $\mathbb{Z} \times \theta^{-1}\mathbb{Z}$.

They make up the algebra $\mathcal{A}_{1/\theta}$ of elements

$$b = \sum_{k,l\in\mathbb{Z}} b_{kl}\pi(k, \theta^{-1}l), \qquad \text{for} \quad \mathbf{b} = (b_{kl}) \in \mathscr{S}(\mathbb{Z}^2)$$

Take $\mathcal{A} = \mathcal{A}_\theta$ and $\mathcal{B} = (\mathcal{A}_{1/\theta})^{\mathsf{op}} \simeq \mathcal{A}_{-1/\theta}$ (this acts from the right)

The space $\mathcal{E} = \mathcal{S}(\mathbb{R})$ is an equivalence bimodule between the noncommutative tori $\mathcal{A}$ and $\mathcal{B}$ with respect to the actions:

$$a \cdot \xi = \sum_{k,l \in \mathbb{Z}} a_{kl} \pi(\theta k, l) \xi,$$

and $$\xi \cdot b = \sum_{k,l \in \mathbb{Z}} b_{kl} \pi(k, \theta^{-1} l)^* \xi,$$

and with hermitian products

$${}_\bullet\langle \xi, \eta \rangle = \theta \sum_{k,l \in \mathbb{Z}} V_\eta \xi(\theta k, l) \pi(\theta k, l),$$

and $$\langle \xi, \eta \rangle_\bullet = \sum_{k,l \in \mathbb{Z}} V_\xi \eta(k, l\theta^{-1}) \pi(k, \theta^{-1} l)\,.$$

A two dimensional spectral geometry

The infinitesimal action of an ordinary torus $\mathbb{T}^2$ on both algebras $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$, are derivations. On $\mathcal{A}_\theta$ they are

$$\partial_1(a) = 2\pi\mathrm{i} \sum_{k,l} k a_{k,l} \pi(\theta k, l)$$

$$\text{and} \qquad \partial_2(a) = 2\pi\mathrm{i} \sum_{k,l} l a_{k,l} \pi(\theta k, l)\,,$$

and the dual ones on $\mathcal{A}_{-1/\theta}$ are then

$$\partial_1(b) = -2\pi\mathrm{i}\theta^{-1} \sum_{k,l} k\, b_{k,l} \pi(k, \theta^{-1} l)^*$$

$$\text{and} \qquad \partial_2(b) = -2\pi\mathrm{i}\theta^{-1} \sum_{k,l} l\, b_{k,l} \pi(k, \theta^{-1} l)^*\,.$$

Lift to covariant derivatives ∇_1, ∇_2 on the bimodules $\mathcal{E} = \mathcal{S}(\mathbb{R})$:

$$(\nabla_1 \xi)(t) = 2\pi\mathrm{i}\, \theta^{-1}\, t\, \xi(t) \quad \text{and} \quad (\nabla_2 \xi)(t) = \frac{\mathrm{d}\xi(t)}{\mathrm{d}t} =: \xi'(t)\,.$$

The curvature is constant:

$$F_{1,2} := [\nabla_1, \nabla_2] = -2\pi\mathrm{i}\, \theta^{-1}\, \mathrm{id}_{\mathcal{E}}\,.$$

Frames

As a module over $\mathcal{A}_\theta$, the space $\mathcal{E} = \mathcal{S}(\mathbb{R})$ is of finite rank and projective and it admits a standard module Parseval frame $\{\eta_1, ..., \eta_n\}$ for $\mathcal{S}(\mathbb{R})$, that is each $\xi \in \mathcal{S}(\mathbb{R})$ has an expansion,

$$\xi = {}_\bullet\langle \xi, \eta_1 \rangle \, \eta_1 + \cdots + {}_\bullet\langle \xi, \eta_n \rangle \, \eta_n.$$

For $0 < \theta < 1$, the module $\mathcal{S}(\mathbb{R})$, is given by a projection in $\mathcal{A}_\theta$ itself: one can use a one-element Parseval frame η

From a standard module frame η one gets a Parseval frame $\widetilde{\eta}$ by taking the element $\widetilde{\eta} := \eta(\langle \eta, \eta \rangle_\bullet)^{-1/2}$

Then $\langle \widetilde{\eta}, \widetilde{\eta} \rangle_\bullet = 1$ and ${}_\bullet\langle \widetilde{\eta}, \widetilde{\eta} \rangle$ is a projection in $\mathcal{A}_\theta$.

Frames and projections:

- The Hermite function

$$\eta = \psi_k(t) = c_k e^{\pi t^2} \frac{d^k}{dt^k} e^{-2\pi t^2}$$

gives a projection $p_k = {}_\bullet\langle \widetilde{\eta}, \widetilde{\eta} \rangle \in \mathcal{A}_\theta$, if $0 < \theta < (k+1)^{-1}$.

- Let $\eta \in \mathcal{S}(\mathbb{R})$ be a totally positive function of finite type greater than 2. Then, $p_{\widetilde{\eta}} = {}_\bullet\langle \widetilde{\eta}, \widetilde{\eta} \rangle$ is a projection in $\mathcal{A}_\theta$ for $0 < \theta < 1$.

All of these projections have topological charge equal to 1. From

$$c_1(p) = q \, \mathrm{tr}(1_B)$$

with now $q = \theta^{-1}$ (the curvature) and $\mathrm{tr}(1_B) = \mathrm{tr}(\mathcal{A}_{-1/\theta}) = \theta$.

Duality and Gabor frames

For a Parseval frame, the *duality principle* (*Wexler-Raz identity*), reads as an expansion of each ξ in $\mathcal{S}(\mathbb{R})$ both over $\mathcal{A}$ and $\mathcal{B}$,

$$\xi = {}_\bullet\langle \xi, \widetilde{\eta} \rangle \, \widetilde{\eta} = \widetilde{\eta} \, \langle \widetilde{\eta}, \xi \rangle_\bullet \, ,$$

with ${}_\bullet\langle \xi, \widetilde{\eta} \rangle \in \mathcal{A}$ and $\langle \widetilde{\eta}, \xi \rangle_\bullet \in \mathcal{B}$ which are uniquely determined.

This helps for the soliton equation.

As before, the s-d eqs for the projection p_ψ obeys $p_\psi \partial(p_\psi) = 0$ translate to a generalized eigenvector equation

$$\overline{\nabla}\psi = \psi\lambda\,,$$

with now $\lambda = \big\langle \psi, \overline{\nabla}\psi \big\rangle_{\bullet} \in \mathcal{A}_{-1/\theta}$.

Using the duality principle we have that

$$\text{with } \psi := \eta(\langle \eta, \eta \rangle_{\bullet})^{-1/2}$$

$$\text{the projection } p_\psi = {}_{\bullet}\langle \psi, \psi \rangle \in \mathcal{A}_\theta$$

satisfies the s-d eqs:

- For $0 < \theta < (k+1)^{-1}$, if η is the k-th Hermite functions ψ_k.
- For $0 < \theta < 1$, if η is a tot pos fun in $\mathcal{S}(\mathbb{R})$ of finite type greater than 2.

In particular, the Gaussian function

$$\psi_\lambda(t) = c\, e^{-\frac{\pi}{\theta}\, t^2 - \mathrm{i}\lambda t}\,, \qquad \text{for} \;\; \lambda \in \mathbb{C}\,,$$

obeys the equation $\overline{\nabla}\psi_\lambda = \psi_\lambda \lambda$.

The right hermitian product $\langle \psi_\lambda, \psi_\lambda \rangle_{\bullet}$ is invertible in $\mathcal{A}_{-1/\theta}$ for all $0 < \theta < 1$,

so that the projections $p_\lambda = {}_{\bullet}\Big\langle \widetilde{\psi}_\lambda, \widetilde{\psi}_\lambda \Big\rangle$, with $\widetilde{\psi_\lambda} := \psi_\lambda (\langle \psi_\lambda, \psi_\lambda \rangle_{\bullet})^{-1/2}$

are solutions of the self-duality equations

The moduli space of such Gaussian solutions, is parametrised by possible λ's modulo gauge transformations

(implemented by invertible elements in $\mathcal{A}_{-1/\theta}$)

is a copy of the complex torus.

More examples and appications

Sigma-model solitons over the Moyal plane and noncommutative tori, as source spaces, with a target space made of two points

A natural action functional leads to self-duality equations for projections in the source algebra

Solutions, having non-trivial topological content, are constructed via suitable Morita duality bimodules,

Inputs from time-frequency analysis and Gabor analysis

More interesting cases

Uses in time-frequency analysis and Gabor analysis

Thank you