Renormalization in the DFR Quantum Spacetime

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Andrés F. Reyes Lega

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Motivation

Principle of Gravitational Stability against localization of events:

"The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localize an event in spacetime should not be so strong to hide the event itself to any distant observer-distant compared to the Planck scale"
(Doplicher, Fredenhagen, Roberts - 1995)

- Localization of events with extreme precision may cause gravitational collapse: spacetime loses operational meaning
- Fate of locality?

The DFR spacetime uncertainty relations (Doplicher, Fredenhagen, Roberts):

$$c\Delta t(\Delta x + \Delta y + \Delta z) \geq \lambda_P^2 \tag{1}$$
$$\Delta x \Delta y + \Delta x \Delta z + \Delta y \Delta z \geq \lambda_P^2 \tag{2}$$

STUR (Tomassini, Viaggiu)

$$(\Delta x)^2(\Delta y)^2(\Delta z)^2 \geq 12\lambda_P^4(\Delta x \Delta y + \Delta x \Delta z + \Delta y \Delta z) \tag{3}$$
$$(c\Delta t)^2(\Delta x \Delta y + \Delta x \Delta z + \Delta y \Delta z) \geq 12\lambda_P^4 \tag{4}$$

- “Small scale structure of Minkowski space”
- Original derivation: linearized gravity..
- Recent, stronger form (STUR) implies DFR uncertainty relations (L. Tomassini, S. Viaggiu, 2011)

Quantum Spacetime

The DFR spacetime

$$[q_\mu, q_\nu] = i\lambda_P^2 Q_{\mu\nu} \tag{5}$$

Covariance:

$$q_\mu \overset{(\Lambda,a)}{\longmapsto} q'_\mu = \Lambda_\mu^\nu q_\nu + a_\mu \tag{6}$$

This implies:

$$Q'_{\mu\nu} = Q_{\rho\tau}\Lambda_\mu^\rho \Lambda_\nu^\tau \quad \text{(rank-2 tensor)} \tag{7}$$

There are two invariants:

$$Q_{\mu\nu}Q^{\mu\nu} \quad \text{and} \quad Q_{\mu\nu}(*Q)^{\mu\nu}. \tag{8}$$

In the simplest model, the $Q_{\mu\nu}$ are central:

$$[Q_{\mu\nu}, q_\alpha] = 0 \quad \forall \mu, \nu, \alpha \in \{0, 1, 2, 3\}. \tag{9}$$

$\Rightarrow$ Joint spectrum of the $Q_{\mu\nu}$:

$$\Sigma = \Sigma_+ \cup \Sigma_- \sim SL(2, \mathbb{C})/D. \tag{10}$$

$$\sigma^{\mu\nu} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix}, \quad \sigma = \sigma(\vec{e}, \vec{m}). \tag{11}$$

"Quantum Conditions": $$Q_{\mu\nu}Q^{\mu\nu} = 0, \ \left(\frac{1}{4}Q_{\mu\nu}(*Q)^{\mu\nu}\right)^2 = 1. \tag{12}$$

In terms of the spectral values, the quantum conditions become:

$$\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \quad = \quad \|\vec{m}\|^2 - \|\vec{e}\|^2 \stackrel{!}{=} 0, \tag{13}$$

$$\frac{1}{4}\sigma_{\mu\nu}(*\sigma)^{\mu\nu} \quad = \quad \vec{e} \cdot \vec{m} \stackrel{!}{=} \pm 1. \tag{14}$$

Joint spectrum of the $Q_{\mu\nu}$:

$$\Sigma = \Sigma_+ \cup \Sigma_- = SL(2, \mathbb{C})/D, \tag{15}$$

with

$$\Sigma_\pm = \{\sigma(\vec{e}, \vec{m}) \,|\, \vec{e} \cdot \vec{m} = \pm 1, \ \|\vec{m}\| = \|\vec{e}\|\}. \tag{16}$$

Commutation relations in Weyl form:

$$e^{i\alpha_\mu q^\mu} e^{i\beta_\nu q^\nu} = e^{-\frac{i}{2}\lambda_P^2 \alpha_\mu Q^{\mu\nu}\beta_\nu} e^{i(\alpha+\beta)_\mu q^\mu} \tag{17}$$

A Lorentz covariant (regular) representation can be explicitly constructed as follows:
On $T^*\mathbb{R}^2$ consider the standard symplectic form

$$\sigma_0 = \begin{pmatrix} \mathbb{0}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{0}_2 \end{pmatrix}. \tag{18}$$

Associated to σ_0 we have generators satisfying the CCR:

$$[Q_i, P_j] = i\delta_{ij}, \quad [Q_i, Q_j] = 0, \quad [P_i, P_j] = 0 \ \ (i = 1, 2). \tag{19}$$

We regard the operators Q_i, P_i $(i = 1, 2)$ as acting on

$$\boxed{\mathcal{H} = L^2(\mathbb{R}^2, d^2 s)} \tag{20}$$

through the Schrödinger respresentation.

Rename these operators in the following way:

$$(X^0, X^1, X^2, X^3) \equiv (P_1, P_2, Q_1, Q_2). \tag{21}$$

It then follows that

$$[X^\mu, X^\nu] = i\sigma_0^{\mu\nu}. \tag{22}$$

Let $\mathscr{L}$ denote the Lorentz group, provided with the Haar measure $d\Lambda$. Define the Hilbert space

$$\mathscr{H}_q := L^2(\mathscr{L}, \mathcal{H}) \cong L^2(\mathscr{L} \times \mathbb{R}^2, d\Lambda\, d^2 s). \tag{23}$$

Elements Ψ of $\mathscr{H}_q$ are maps

$$\begin{aligned} \Psi : \mathscr{L} &\longrightarrow \mathcal{H} = L^2(\mathbb{R}^2, d^2 s) \\ \Lambda &\longmapsto \Psi_\Lambda. \end{aligned} \tag{24}$$

The inner product is defined as

$$\langle \Psi, \Phi \rangle_{\mathscr{H}_q} := \int_{\mathscr{L}} d\Lambda \langle \Psi_\Lambda, \Phi_\Lambda \rangle_{\mathcal{H}}. \tag{25}$$

Quantum coordinate operators

Let $\Psi \in \mathscr{H}_q = L^2(\mathscr{L}, \mathcal{H})$. Recall that $\Psi_\Lambda \in \mathcal{H} = L^2(\mathbb{R}^2, d^2 s)$.
Define

$$(q^\mu \Psi)_\Lambda := \Lambda^\mu_{\ \nu} X^\nu \Psi_\Lambda, \tag{26}$$

as well as

$$(U(\Lambda)\Psi)_{\widetilde{\Lambda}} := \Psi_{\Lambda^{-1}\widetilde{\Lambda}}. \tag{27}$$

$\Rightarrow$

Lorentz covariance of the quantum coordinate operators:

$$U(\Lambda^{-1})\, q^\mu\, U(\Lambda) = \Lambda^\mu_{\ \nu}\, q^\nu. \tag{28}$$

The above treatment can be easily generalized in order to obtain a set of *Poincaré* covariant quantum coordinate operators.

The Moyal product

$$\text{CCR} \longrightarrow \quad \boxed{[q,p] = i}$$

Weyl form of the CCR:

$$e^{i(\alpha_1 q + \alpha_2 p)} e^{i(\beta_1 q + \beta_2 p)} = e^{-\frac{i}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)} e^{i((\alpha_1+\beta_1)q + (\alpha_2+\beta_2)p)} \tag{29}$$

- Symplectic structure: $\sigma(\alpha,\beta) := \alpha_1\beta_2 - \alpha_2\beta_1, \qquad \alpha,\beta \in \mathbb{R}^2$
- Weyl generators: $W(\alpha) := e^{i(\alpha_1 q + \alpha_2 p)} \in U(L^2(\mathbb{R}))$
- Weyl algebra: $W(\alpha)W(\beta) = e^{-\frac{i}{2}\sigma(\alpha,\beta)} W(\alpha+\beta)$
- $\sigma(\alpha,\beta) = \alpha \cdot \Theta\beta, \;\; \Theta^{-1} = -\Theta = \Theta^T$
- Relation between (Weyl) CCR and Moyal product $f \star_\Theta g$?

$$\begin{aligned}\mathcal{F}(f)(k) &\equiv \hat{f}(k) = \int d^n x\, e^{-ik\cdot x} f(x),\\ \overline{\mathcal{F}}(g)(x) &\equiv \check{g}(x) = \frac{1}{(2\pi)^n}\int d^n k\, e^{ik\cdot x} g(x).\end{aligned}$$

The idea of (Moyal-Weyl) quantization is the following ($n = 2$):

- $f = \mathcal{F}^{-1}(\hat{f}\,) \longleftrightarrow f(x) = (2\pi)^{-2}\int d^2u\, e^{iu\cdot x}\hat{f}(u)$.
- Replace $\exp(i(u_1x_1 + u_2x_2))$ by $W(u) = \exp(i(u_1\mathbf{q} + u_2\mathbf{p}))$.
- As $W(u) \in U(L^2(\mathbb{R}))$, the resulting operator "$f(\mathbf{q}, \mathbf{p})$" is best understood as an operator $\pi(f)$, providing a representation of a noncommutative algebra of functions (Moyal algebra).
- Define, for f in a suitable class, $\pi(f) := \frac{1}{(2\pi)^2}\int d^2u\, W(u)\hat{f}(u)$.

Moyal product

We want to define a product $f \star_\Theta g$ through the relation

$$\pi(f)\pi(g) \stackrel{!}{=} \pi(f \star_\Theta g). \tag{30}$$

- In order to establish what is the star product, we need to compute the product $\pi(f)\pi(g)$.
- The relation $\pi(f \star_\Theta g) = \pi(f)\pi(g)$ then implies that π is an algebra homomorphism.

Result:

$$\begin{aligned}
(f \star_\Theta g)(x) &= \frac{1}{(2\pi)^2}\int d^2u \int d^2t\, f(x - \tfrac{1}{2}\Theta u)g(x+t)e^{-iu\cdot t} \\
&= \frac{1}{(2\pi)^2}\int d^2s \int d^2t\, f(x+s)g(x+t)e^{-is\cdot\Theta^{-1}t} \\
&= \frac{1}{(2\pi)^2}\int d^2a \int d^2b\, f(a)g(b)e^{2i\sigma(a-x,b-x)}. \qquad (31)
\end{aligned}$$

The DFR algebra

Scalar quantum field on quantum spacetime

$$\phi(q) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{2E_k} \left(e^{-iq_\mu k^\mu} \otimes a_{\vec{k}} + e^{iq_\mu k^\mu} \otimes a^\dagger_{\vec{k}} \right), \quad (32)$$

$$\mathfrak{H} = \mathscr{H}_q \otimes \mathcal{F}_\phi. \quad (33)$$

▷ Definition of interaction terms requires an algebra closed under the product of the $e^{iq_\mu k^\mu}$'s.

More generally, if we try to quantize functions (say $f, \hat{f} \in L^1(\mathbb{R}^4)$) using the Weyl prescription

$$f(q) = \int d^4k \hat{f}(k) e^{iq_\mu k^\mu}, \quad (34)$$

we realize that we need to enlarge the space of functions to be quantized.

- In the previous Moyal example, L^1-functions f on phase space are quantized to $\pi(f) \equiv f(\hat{q}, \hat{p})$.
- In the DFR case, exponentials $e^{iq_\mu k^\mu}$ do not form a closed algebra.
- So we enlarge the class of functions to be quantized from $L^1(\mathbb{R}^4)$, to $C_0(\Sigma, L^1(\mathbb{R}^4))$.
- The quantized functions will be denoted as $f(Q, q)$.
- Following the same idea as in the Moyal example, we can induce a star product on $C_0(\Sigma, L^1(\mathbb{R}^4))$.
- It is fixed by the following condition:

$$\pi_{\text{DFR}}(f)\pi_{\text{DFR}}(g) \stackrel{!}{=} \pi_{\text{DFR}}(f \star g). \tag{35}$$

More explicitly:

▷ For $f \in C_0(\Sigma, L^1(\mathbb{R}^4))$, consider the Fourier transform

$$\hat{f}(\sigma, k) = \int_{\mathbb{R}^4} d^4x f(\sigma, x) e^{-ik_\mu x^\mu}. \tag{36}$$

▷ Let $Q^{\mu\nu} = \int_\Sigma \sigma^{\mu\nu} dE(\sigma)$ denote the (joint) spectral resolution of the $Q^{\mu\nu}$'s.

▷ Then, for functions f such that $f, \hat{f} \in C_0(\Sigma, L^1(\mathbb{R}^4))$, define

$$\pi_{\text{DFR}}(f) := \int_\Sigma dE(\sigma) \int_{\mathbb{R}^4} d^4k \hat{f}(\sigma, k) e^{ik_\mu q^\mu}. \tag{37}$$

$$(f \star g)(\sigma, x) = \frac{1}{(2\pi)^4} \int d^4a \int d^4b f(\sigma, a) g(\sigma, b) e^{2i(a-x)_\mu \sigma^{\mu\nu} (b-x)_\nu}. \tag{38}$$

An equivalent expression in *momentum space* is given by the twisted convolution "$\times$":

$$(\hat{f} \times \hat{g})(\sigma, k) = \int \frac{d^4k'}{(2\pi)^4} \hat{f}(\sigma, k') \hat{g}(\sigma, k - k') e^{\frac{i}{2} k_\mu \sigma^{\mu\nu} k'_\nu}. \qquad (39)$$

Star product vs. twisted convolution

For $f, g \in C_0(\Sigma, L^1(\mathbb{R}^4))$ we have:

$$f \star g = (\check{f} \times \check{g})\hat{} = (2\pi)^{-4} (\hat{f} \times \hat{g})\check{}. \qquad (40)$$

Alternative point of view, of relevance for field theory

As in (34), to every function f with $f, \hat{f} \in L^1(\mathbb{R}^4)$ we can associate an operator $f(q) = \int d^4\alpha \check{f}(\alpha) e^{-i\alpha_\mu q^\mu}$. Then, we want to make sense of

$$f_1(q) \cdots f_n(q) = (\check{f}_1 \times \cdots \times \check{f}_n)\hat{}(q). \qquad (41)$$

Example ($n = 2$):

$$
\begin{aligned}
f_1(q)f_2(q) &= \int d^4\alpha \int d^4\beta \check{f}_1(\alpha)\check{f}_2(\beta) e^{-\frac{i}{2}\alpha_\mu Q^{\mu\nu}\beta_\nu} e^{-i(\alpha+\beta)_\mu q^\mu} \\
&= \int d^4\alpha \left[\int d^4\alpha' \check{f}_1(\alpha')\check{f}_2(\alpha-\alpha') e^{\frac{i}{2}\alpha_\mu Q^{\mu\nu}\alpha'_\nu} \right] e^{-i\alpha_\mu q^\mu} \\
&= \int d^4\alpha \underbrace{(\check{f}_1 \times \check{f}_2)(Q,\alpha)}_{\in Z(\mathscr{E})} e^{-i\alpha_\mu q^\mu}.
\end{aligned}
\tag{42}
$$

Repeating the process we can obtain a general expression for the twisted convolution:

$$
f_1(q)\cdots f_n(q) = \int d^4\alpha (\check{f}_1 \times \cdots \times \check{f}_n)(Q,\alpha) e^{-i\alpha_\mu q^\mu}. \tag{43}
$$

Integrals

$$\int_{q^0=t} d^3q \; F(Q,q) := \int d^3\vec{x} \; F(Q,(t,\vec{x})). \tag{44}$$

$$\int d^4q \; F(Q,q) := \int d^4x \; F(Q,x). \tag{45}$$

$$\int dt \int_{q^0=t} d^3q \; F(Q,q) = \int d^4q \; F(Q,q). \tag{46}$$

Using the twisted convolution, we can express both types of integration by means of integral kernels:

$$f_1(q)\cdots f_n(q) = \int d^4x_1 \cdots d^4x_n f_1(x_1)\cdots f_n(x_n) C_n(x_1 - x, \dots, x_n - x) \tag{47}$$

$$\mathscr{D}_n(x_1, x_2, \dots, x_n; t) := \int_{x^0=t} d^3\vec{x}\, C_n((x_1 - x, \dots, x_n - x)) \tag{48}$$

Non-locality from space integration

$$\int_{q^0=t} d^3q\, f_1(q)\cdots f_n(q) = \\ = \int d^4x_1 \cdots d^4x_n \mathscr{D}_n(x_1, \dots, x_n; t) f_1(x_1)\cdots f_n(x_n) \tag{49}$$

Renormalization

Renormalization

Causality and the origin of UV divergences

Dyson series

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \int dt_2 \cdots \int dt_n T(\tilde{V}(t_1) \cdots \tilde{V}(t_n))$$

- $T(A(t_1)A(t_2)) = \theta(t_1 - t_2)A(t_1)A(t_2) + \theta(t_2 - t_1)A(t_2)A(t_1)$
- Non-relativistic QM: ok
- QFT: $\tilde{V}(t) = -\int d^3x \mathcal{L}_{int}(\varphi(x), \partial^\mu \varphi(x))$
- Problem: multiplication of distributions

Multiplication of distributions

Example

Multiplication of distributions

Example

- $\hat{f}(k) = \int f(x)e^{-ikx}dx$

Multiplication of distributions

Example

- $\hat{f}(k) = \int f(x)e^{-ikx}dx$
- $f * g(x) = \int dy f(y)g(x-y)$

Multiplication of distributions

Example

- $\hat{f}(k) = \int f(x) e^{-ikx} dx$
- $f * g(x) = \int dy f(y) g(x-y)$
- $(fg)\hat{} = (2\pi)^{-1} \hat{f} * \hat{g}$

Multiplication of distributions

Example

- $\hat{f}(k) = \int f(x)e^{-ikx}dx$
- $f * g(x) = \int dy f(y)g(x-y)$
- $(fg)\hat{} = (2\pi)^{-1}\hat{f} * \hat{g}$
- $\hat{\delta}(k) = \int dx\, \delta(x)e^{-ikx} = 1$

Multiplication of distributions

Example

- $\hat{f}(k) = \int f(x) e^{-ikx} dx$
- $f * g(x) = \int dy f(y) g(x-y)$
- $(fg)\hat{} = (2\pi)^{-1} \hat{f} * \hat{g}$
- $\hat{\delta}(k) = \int dx\, \delta(x) e^{-ikx} = 1$
- $\hat{\theta}(k) = \lim_{\epsilon \to 0} \int \theta(x) e^{-ikx - \epsilon x} dx = -\frac{i}{k - i\varepsilon}$

Multiplication of distributions

Example

- $\hat{f}(k) = \int f(x) e^{-ikx} dx$
- $f * g(x) = \int dy f(y) g(x-y)$
- $(fg)\hat{} = (2\pi)^{-1} \hat{f} * \hat{g}$
- $\hat{\delta}(k) = \int dx\, \delta(x) e^{-ikx} = 1$
- $\hat{\theta}(k) = \lim_{\epsilon \to 0} \int \theta(x) e^{-ikx - \epsilon x} dx = -\frac{i}{k - i\varepsilon}$
- $(\text{“}\theta\delta\text{”})\hat{}(k) := (2\pi)^{-1} \hat{\delta} * \hat{\theta}(k) = \frac{-i}{2\pi} \int \frac{dk'}{k' - i\varepsilon}$

Introduce switching functions:

$$S(g) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \int dt_2 \cdots \int dt_n T(\tilde{V}(t_1) \cdots \tilde{V}(t_n)) g(t_1) \cdots g(t_n)$$

Causality condition

$$g_1 < g_2 \Rightarrow S(g_1 + g_2) = S(g_2) S(g_1)$$

Generalization: Epstein-Glaser

Scattering matrix as operator valued distribution

$$S(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \int dt_2 \cdots \int dt_n T_n(t_1, \ldots, t_n) g(t_1) \cdots g(t_n)$$

Basic principles: COVARIANCE, CAUSALITY, UNITARITY

Causality condition, perturbatively:

$\{t_1, \ldots, t_m\} > \{t_{m+1}, \ldots, t_n\} \Rightarrow$

$$T_n(t_1, \ldots, t_n) = T_m(t_1, \ldots, t_m) T_{n-m}(t_{m+1}, \ldots, t_n)$$

Example: QM

Define:

$$\begin{aligned}
A_2'(t_1,t_2) &= -T_1(t_1)T_1(t_2), \\
R_2'(t_1,t_2) &= -T_1(t_2)T_1(t_1), \\
A_2(t_1,t_2) &= A_2'(t_1,t_2) + T_2(t_1,t_2), \\
R_2(t_1,t_2) &= R_2'(t_1,t_2) + T_2(t_1,t_2).
\end{aligned}$$

Using the causality condition, we obtain:

$$\begin{aligned}
t_1 > t_2 &\quad\Rightarrow\quad A_2(t_1,t_2) = 0, \\
t_1 < t_2 &\quad\Rightarrow\quad R_2(t_1,t_2) = 0.
\end{aligned}$$

A_2 y R_2: unknown, but with known support. A_2' y R_2': known. Furthermore,

$$D_2 := R_2 - A_2 \equiv R_2' - A_2'.$$

Then, if $t_1 > t_2$, $A_2 = 0$, so that

$$\begin{aligned} R_2(t_1, t_2) &= R_2(t_1, t_2) - 0 = D_2(t_1, t_2) \\ &= R_2'(t_1, t_2) - A_2'(t_1, t_2) \equiv T_1(t_1)T_1(t_2) - T_1(t_2)T_1(t_1) \end{aligned}$$

On the other hand, if $t_1 < t_2$, then $R_2 = 0$. It follows that

$$R_2(t_1, t_2) = \theta(t_1 - t_2)\Big(T_1(t_1)T_1(t_2) - T_1(t_2)T_1(t_1)\Big).$$

From R_2 we can now obtain T_2, this leding to

$$\begin{aligned} T_2(t_1, t_2) &= R_2(t_1, t_2) - R_2'(t_1, t_2) = R_2(t_1, t_2) + T_1(t_2)T_1(t_1) \\ &= \theta(t_1 - t_2)\Big(T_1(t_1)T_1(t_2) - T_1(t_2)T_1(t_1)\Big) + T_1(t_2)T_1(t_1) \\ &= \theta(t_1 - t_2)T_1(t_1)T_1(t_2) + (1 - \theta(t_1 - t_2))T_1(t_2)T_1(t_1) \\ &= T\Big(T_1(t_1)T_1(t_2)\Big). \end{aligned}$$

Renormalization

BPHZ vs. Dim.Reg.

1-loop contribution to 4-point function in Φ^4 theory:

Dimensional regularization:

$$\begin{aligned} J_\Gamma(p) &= \frac{g^2(\mu^2)^{4-D}}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{((p-k)^2 - m^2 + i\,0^+)} \frac{1}{(k^2 - m^2 + i\,0^+)} \\ &= \frac{ig^2\mu^{4-D}}{32\pi^2}\,\Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dz \left[\frac{4\pi\mu^2}{m^2 - p^2 z(1-z)}\right]^{2-D/2} \end{aligned}$$

$\Rightarrow$

$$J_\Gamma^{\text{(dim.reg)}}(p) = \frac{ig^2}{32\pi^2}\left[-\gamma_E + \int_0^1 dz \ln\left(\frac{4\pi\mu^2}{m^2 - z(1-z)p^2}\right)\right]$$

1-loop contribution to 4-point function in Φ^4 theory:

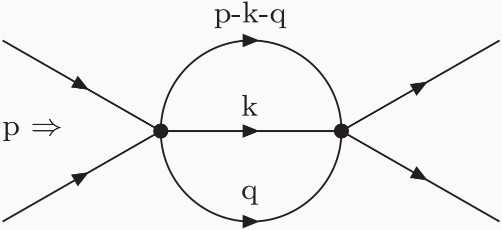
BPHZ:

$$J_\Gamma^{\text{(BPHZ)}}(p) = \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} (1 - t_p^0) \frac{1}{[k^2 - m^2]\,[(p-k)^2 - m^2]}$$

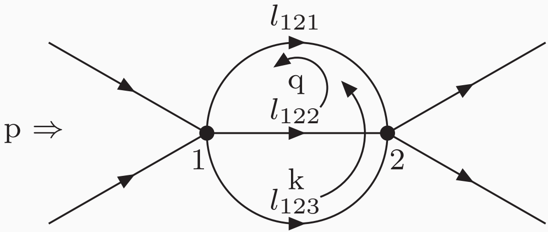
$\Rightarrow$

$$J_\Gamma^{\text{(BPHZ)}}(p) = \frac{ig^2}{32\pi^2} \int_0^1 dz \ln\left(\frac{m^2}{m^2 - z(1-z)p^2}\right)$$

2-loop contribution to 4-point function in Φ^4 theory:



2-loop contribution to 4-point function in Φ^4 theory:



$$\mathcal{F} = \{\emptyset, \{\Gamma\}, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\Gamma, \gamma_1\}, \{\Gamma, \gamma_2\}, \{\Gamma, \gamma_3\}\}$$

$$
\begin{aligned}
\mathcal{R}_\Gamma(p,k,q) &= (1-t_p^2)S_\Gamma\left(1-\sum_{i=1}^{3} t^0_{p^{\gamma_i}}S_{\gamma_i}\right)I_\Gamma(p,k,q) \\
&= (1-t_p^2)\left(\frac{1}{(p-k-q)^2-m^2}\,\frac{1}{k^2-m^2}\,\frac{1}{q^2-m^2}\right. \\
&\quad -\frac{1}{[k^2-m^2]^2}\,\frac{1}{q^2-m^2}-\frac{1}{[q^2-m^2]^2}\,\frac{1}{k^2-m^2} \\
&\qquad \left.-\frac{1}{(p-k-q)^2-m^2}\,\frac{1}{[(\lambda q-\mu k)^2-m^2]^2}\right).
\end{aligned}
$$

Original, divergent integral: $J_\Gamma^{\text{div}}(p) = \int d\underline{k}\; I_\Gamma(\underline{k}, p).$

Regularized expression (BPHZ): $J_\Gamma(p) = \int d^4k\, d^4q\; \mathcal{R}_\Gamma(p, k, q)$

Result:

$$J_\Gamma(p) \;=\; \frac{g^2}{6(4\pi)^4} \int_0^1 dz \int_0^1 dx \frac{(1-2z)(1-2x)p^2}{(z-1)(1-z+z^2)} \times \\ \times \ln\left(\frac{[z(1-z)(1-x)+x]m^2}{-xz(1-z)(1-x)p^2 + z(1-z)(1-x)m^2 + xm^2} \right).$$

QFT on DFR spacetime

Scalar quantum field on quantum spacetime

$$\phi(q) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{2E_k} \left(e^{-iq_\mu k^\mu} \otimes a_{\vec{k}} + e^{iq_\mu k^\mu} \otimes a^\dagger_{\vec{k}} \right),$$

$$\mathfrak{H} = \mathscr{H}_q \otimes \mathcal{F}_\phi .$$

Renormalization

Taylor subtraction for QFT on DFR spacetime

BPHZ and noncommutative QFT

▷ The breakdown of (Poincaré) covariance/causality/locality at the Planck scale is a common feature of QFT most models on noncommutative spacetimes.

▷ Given the strong implications of covariance and causality for perturbative renormalization (Epstein-Glaser, BPHZ), one is naturally led to question whether direct application of, say, Zimmermann's forest formula, is justified from a physical point of view.

▷ Previous work by Blaschke, Garschall, Gieres, Heindl, Schweda and Wohlgenannt (2013) show that straightforward application of the Taylor subtraction operator in scalar QFT based on Moyal product (Euclidean signature) does not work, due to UV/IR mixing.

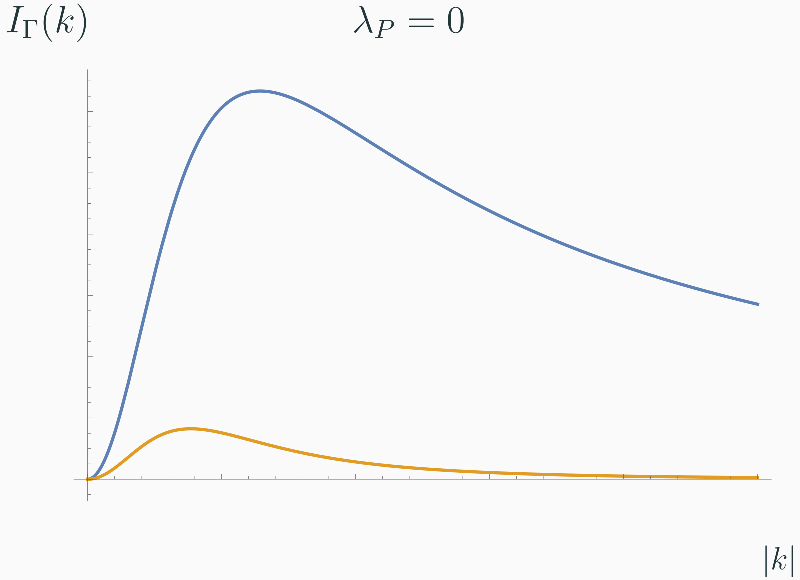
▷ In contrast, our calculations appear to suggest that in the original version of scalar QFT on DRF quantum spacetime the Taylor subtraction works, and leads to the correct large scale limit.

Examples

Examples

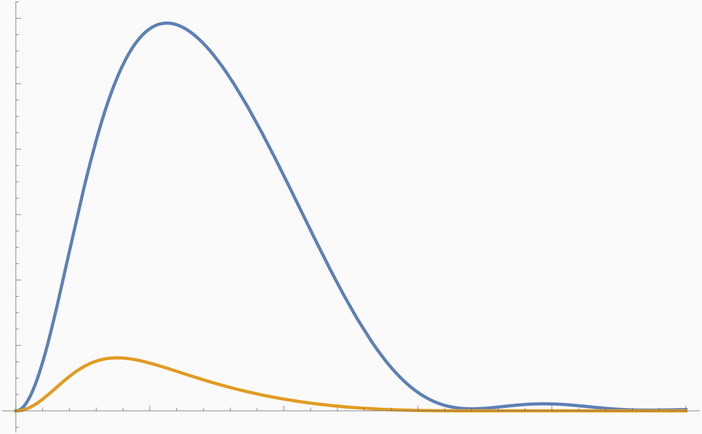
Fish diagram

2nd order 2-point function (Φ^3)



2nd order 2-point function (Φ^3)

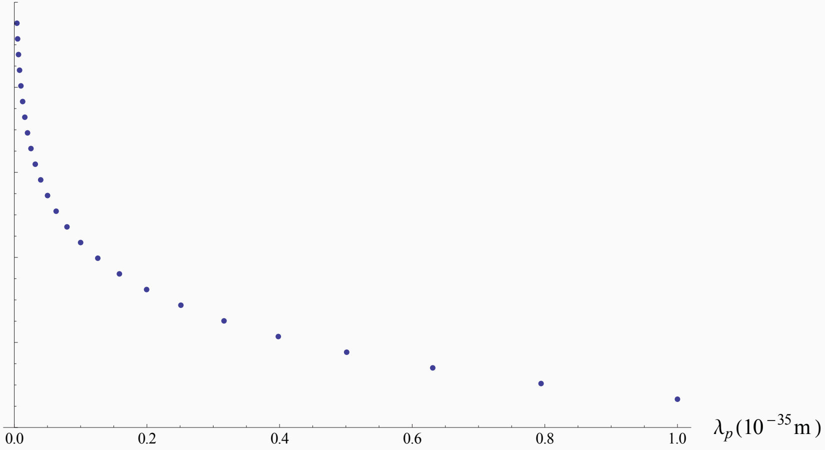
$I_\Gamma(k)$ λ_P set at m_ϕ-scale



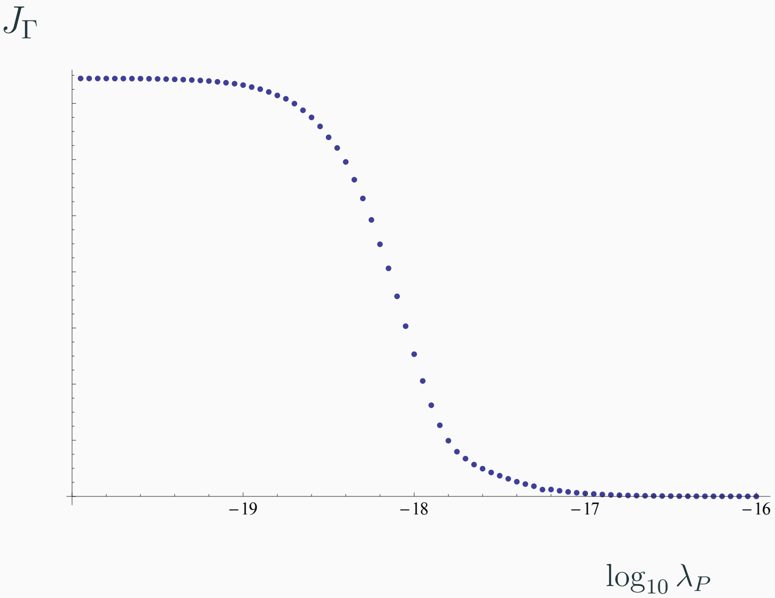
$|k|$

2nd order 2-point function (Φ^3)

$J_\Gamma(p) = \int d\underline{k}\ I_\Gamma(\underline{k}, p)$



Large scale limit

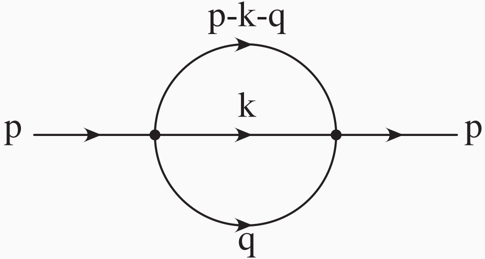


Examples

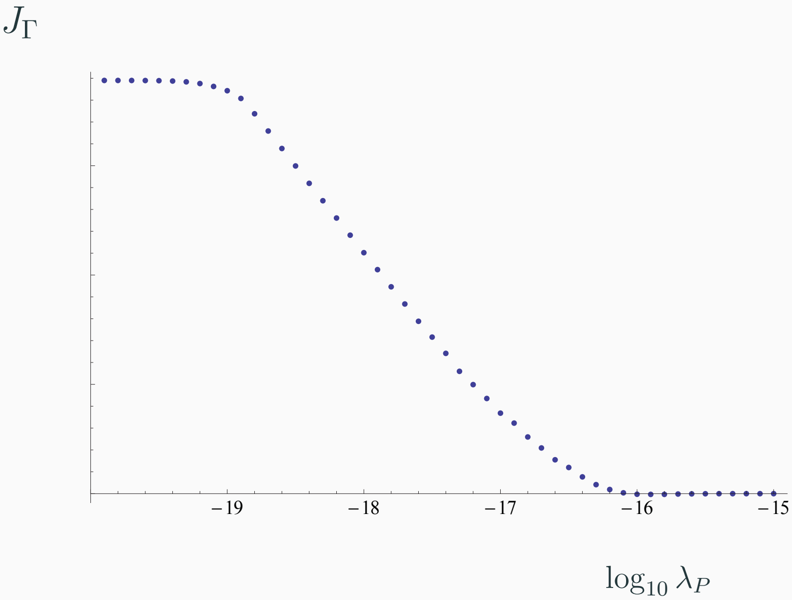
Sunrise diagram

2nd order 2-point function (Φ^4)

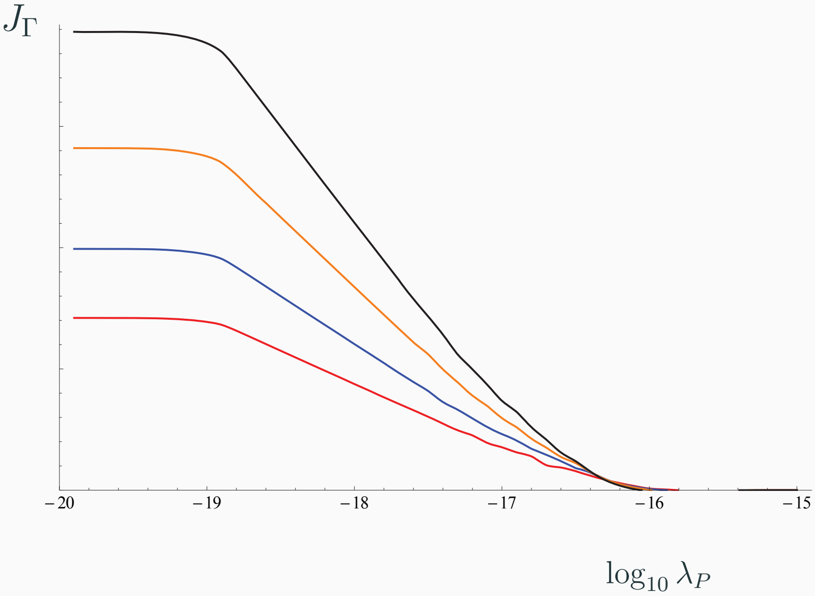
Sunrise diagram



Large scale limit



Large scale limit, different masses



Final remarks

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▷ Rigorous approach to renormalization: see next talk by G. Morsella!