Spectral Distances on Non-commutative spaces using Dirac Eigen Spinors

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Types of spaces to be considered :

(1) Fuzzy sphere ($\mathbb{S}^2_*$) : $[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k$
(2)Moyal Plane ($\mathbb{R}^2_*$): $[\hat{x}_1, \hat{x}_2] = i\theta$
(3)Doubled Moyal Plane

Basic Idea

For either kind of spaces ($\mathbb{S}^2_*$,$\mathbb{R}^2_*$) we look for an auxiliary Hilbert space $\mathcal{H}_c$ furnishing a representation of just the coordinate algebra.

Then introduce $\mathcal{H}_q$, the so called 'Quantum Hilbert space',as the space of Hilbert- Schmidt operators $\psi(\hat{x}_i)$ acting on $\mathcal{H}_c$ having finite Hilbert-Schmidt norm:

$$||\psi(\hat{x}_i)||_{H.S} = \sqrt{tr(\psi^\dagger(\hat{x}_i)\psi(\hat{x}_i))} < \infty;$$

$$(\psi, \phi) := tr(\psi^\dagger \phi)$$

We identify $\mathcal{A} = \mathcal{H}_q =$ span $\{|.\rangle\langle.|\} \longrightarrow$ a $*$ algebra and

$\mathcal{H} = \mathcal{H}_c \bigotimes \mathbb{C}^2 \longrightarrow$ The Hilbert space of "spinors"

They are the two of the three ingredients to construct the respective spectral triples $(\mathcal{A}, \mathcal{D}, \mathcal{H})$. The second one is the Dirac operator.

The Dirac operator for $\mathbb{R}^2_*$

Here $\mathcal{H}_c$=Span$\{|n\rangle := \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle\}$; $b = \frac{\hat{x}_1 + i\hat{x}_2}{\sqrt{2\theta}}$ satisfying $[b, b^\dagger] = 1$

Start with Non-commutative Heisenberg algebra (NCHA)
$[\hat{X}_i, \hat{X}_j] = i\theta\epsilon_{ij}$; $[\hat{X}_i, \hat{P}_j] = i\delta_{ij}$; $[\hat{P}_i, \hat{P}_j] = 0$

With the actions $\hat{X}_i\psi(\hat{x}_i) = \hat{x}_i\psi(\hat{x}_i)$; $\hat{P}_i\psi(\hat{x}_i) = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j, \psi(\hat{x}_i)]$

Then $\mathcal{D}_M = \sigma_1 P_1 + \sigma_2 P_2 = \begin{pmatrix} 0 & P_1 - iP_2 \\ P_1 + iP_2 & 0 \end{pmatrix}$ acts on $\Phi = \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix} \in \mathcal{H}_q \otimes \mathbb{C}^2$, by default

$$\Rightarrow [\mathcal{D}_M, \pi(a)]\Phi = \sqrt{\frac{2}{\theta}}\left[\begin{pmatrix} 0 & i\hat{b}^\dagger \\ -i\hat{b} & 0 \end{pmatrix}, \pi(a)\right]\Phi$$

Spectral triple of Moyal plane

Thus one identifies

$$\mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & i\hat{b}^\dagger \\ -i\hat{b} & 0 \end{pmatrix} \xrightarrow{\text{SO(2)}} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix}$$

This can also act on $\mathcal{H}_c \otimes \mathbb{C}^2$ from the left so that finally one has the spectral triple

$$\mathcal{A} = \mathcal{H}_q;\ \mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2;\ \mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix} \tag{1}$$

Pure states that we shall consider are

1. $\rho_m := |m\rangle\langle m|,\ m = 0, 1, 2 \ldots$ harmonic oscillator states
2. $\rho_z := |z\rangle\langle z| = |z),\ z \in \mathbb{C}$

<u>Eigen-spinors of $\mathcal{D}_M$</u>

$$|0\rangle\rangle := \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix},\ |m\rangle\rangle_\pm := \frac{1}{\sqrt{2}} \begin{pmatrix} |m\rangle \\ \pm|m-1\rangle \end{pmatrix};\ m = 1, 2, 3, \ldots \tag{2}$$

Continued...

Eigen-value equation of $\mathcal{D}$:

$$\mathcal{D}\|m\rangle\rangle_{\pm} = \lambda_m^{\pm}|m\rangle\rangle_{\pm}; \;\; \lambda_m^{\pm} = \pm\sqrt{\frac{2m}{\theta}}, \;\; m = 0, 1, 2, \ldots$$

along-with orthogonality : ${}_{\pm}\langle\langle m|n\rangle\rangle_{\pm} = \delta_{mn}; \quad {}_{\pm}\langle\langle m|n\rangle\rangle_{\mp} = 0$

as well as completeness relation

$$|0\rangle\rangle\langle\langle 0| + \sum_{m=1}^{\infty}\Big(|m\rangle\rangle_{+\,+}\langle\langle m| + |m\rangle\rangle_{-\,-}\langle\langle m|\Big) = \mathbb{1}_{\mathcal{H}_q \otimes M_2(\mathbb{C})} \tag{3}$$

We can introduce a projection operator:

$$\mathbb{P}_N = |0\rangle\rangle\langle\langle 0| + \sum_{n=1,\pm}^{N} |n\rangle\rangle_{\pm\,\pm}\langle\langle n| = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \in \mathcal{H}_q \otimes M_2^d(\mathbb{C}), \tag{4}$$

where $P_N = \sum_{m=o}^{N} |m\rangle\langle m| \in \mathcal{H}_q$

Spectral triple of fuzzy sphere ($\mathbb{S}^2_*$)

Coordinate algebra : $[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k$; $i, j, k = 1, 2, 3.$

$$\hat{\vec{x}}^2|n, n_3\rangle = r_n^2|n, n_3\rangle = \lambda n(n+1)|n, n_3\rangle;$$
$$\hat{x}_3|n, n_3\rangle = \lambda n_3|n, n_3\rangle$$

Auxiliary space for the entire $\mathbb{R}^3_*$;

$$\mathcal{H}_c = \bigoplus_n \mathcal{H}_c^{(n)} \; ; \quad \mathcal{H}_c^{(n)} = Span\{|n, n_3\rangle \mid \text{n is fixed}, \; -n \leq n_3 \leq n\}$$

Quantum Hilbert space for the fuzzy sphere of radius r_n:

$$\mathcal{H}_q = \bigoplus_n \mathcal{H}_q^{(n)}; \; \mathcal{H}_q^{(n)} = Span\{|n, n_3\rangle\langle n, n_3'| \mid \text{n fixed}, \; -n \leq n_3, n_3' \leq n\}$$

Spectral triple is $\mathcal{A} = \mathcal{H}_q^{(n)}; \mathcal{H} = \mathcal{H}_c^{(n)} \otimes \mathbb{C}^2, \mathcal{D} = \frac{1}{r_n}\vec{J} \otimes \vec{\sigma}$

Continued...

Eigen-spinors:

$$|n, n_3\rangle\rangle_+ := f(n, n_3)\,|n, n_3\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g(n, n_3)\,|n, n_3 + 1\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|n, n_3'\rangle\rangle_- := -g(n, n_3')\,|n, n_3\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + f(n, n_3')\,|n, n_3 + 1\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $f(n, n_3) = \sqrt{\frac{n+n_3+1}{2n+1}}$, $\quad g(n, n_3) = \sqrt{\frac{n-n_3}{2n+1}}$.

- $\lambda^+_{n_3} = \frac{n}{r_n}$, $\quad -n - 1 \leq n_3 \leq n$, yielding (2n+2)-fold degeneracy
- $\lambda^-_{n_3'} = -\frac{(n+1)}{r_n}$, $\quad -n \leq n_3' \leq n - 1$, yielding 2n-fold degeneracy.

Perelemov coherent state

$$|z\rangle = e^{\frac{\theta}{2\lambda}(\hat{x}_- - \hat{x}_+)}|n, n\rangle \quad (\varphi = 0)$$

$|z| = \tan\left(\frac{\theta}{2}\right)$ is stereographically projected coordinate.

Connes spectral distance

$$d(\omega_1,\omega_2) = \sup_{a\in B} |\omega_1(a) - \omega_2(a)|; \;\; B = \{a \in \mathcal{A} : \|[\mathcal{D},\pi(a)]\|_{op} \leq 1\}$$

Also $\left|\omega(a) - \omega'(a)\right| = |\mathrm{Tr}((\rho_\omega - \rho_{\omega'})a)| = |(\Delta\rho, a)|; \Delta\rho \in \mathcal{H}_q = \mathcal{A}$

- We take ω, ω' to be normal states, so that they can be represented by density matrices $\omega \to \rho_\omega$
- Let $V_0 = \{a \in \mathcal{A} : \|[\mathcal{D},\pi(a)]\|_{op} = 0\}$, then $\omega(a) - \omega'(a) = 0$, $\forall\ a \in V_0$, (certain irreducibility condition)
- The optimal element a_s should attain the supremum value:

$$d(\omega,\omega') = |\omega(a_s) - \omega'(a_s)|; \;\; \|[\mathcal{D},\pi(a_s)]\|_{op} = 1$$

Towards an algorithm to compute finite distances

Basic Idea:-

Start with the Ball condition $||[\mathcal{D}, \pi(a)]||_{op} \leq 1$

Then $||a||_{HS} \leq \frac{1}{||[\mathcal{D},\pi(\hat{a})]||_{op}}$; $||\hat{a}||_{HS} = 1$

$\Rightarrow Sup_{a \in B'} ||a||_{HS} \leq \frac{1}{s}$; $s = Inf_{a \in B'} ||[\mathcal{D}, \pi(\hat{a})]||_{op}$,

Here $B' = \{a \in \mathcal{A} | 0 < ||[\mathcal{D}, \pi(a)]||_{op} \leq 1\} \subset B$ (A dense subset)

Now splitting $\hat{a} = cos\theta \hat{\Delta\rho} + sin\theta \hat{\Delta\rho_\perp}$; $(\hat{\Delta\rho}, \hat{\Delta\rho_\perp}) = 0$, we have

$s \leq Inf_{\theta \in [0, \frac{\pi}{2})} [|cos\theta| \, ||[\mathcal{D}, \pi(\hat{\Delta\rho})]||_{op} + |sin\theta| \, ||[\mathcal{D}, \pi(\hat{\Delta\rho_\perp})]||_{op}]$

Towards an algorithm to compute finite distances

$$d(\rho, \rho') = N \|\Delta\rho\|^2_{\mathrm{HS}}; \ \ N = \frac{1}{\inf\limits_{\Delta\rho_\perp} \|[\mathcal{D}, \pi(\Delta\rho)] + [\mathcal{D}, \Delta\rho_\perp]\|_{\mathrm{op}}} \tag{5}$$

A lower bound is reached when $a_s \propto \Delta\rho$

$$d(\rho, \rho') \geq \frac{\|\Delta\rho\|^2_{\mathrm{H.S}}}{\|[\mathcal{D}, \pi(\Delta\rho)]\|_{op}}; \ \text{where } a_s = \frac{\Delta\rho}{\|[\mathcal{D}, \pi(\Delta\rho)]\|_{op}} \tag{6}$$

In the following we shall be computing distances between pure states given by coherent states and the discrete states.

Distances on $\mathbb{S}^2_*$ (discrete basis)

Infinitesimal distance (In n representation): For $\rho_n := |n_3\rangle\langle n_3|$

$$
\begin{aligned}
d_n(\rho_{n_3+1}, \rho_{n_3}) &= \sup_{a \in B} |\mathrm{tr}(\rho_{n_3+1} a) - \mathrm{tr}(\rho_{n_3} a)| \\
&\leq \frac{\|[J_-, a]\|_{op}}{\sqrt{n(n+1) - n_3(n_3+1)}} \text{ (By Bessels Inequality)} \\
&\leq \frac{r_n}{\sqrt{n(n+1) - n_3(n_3+1)}} \; (By \; \|[J_\pm, a]\|_{op} \leq r_n)
\end{aligned}
$$

This is also the lower bound! $[\mathcal{D}, \pi(d\rho)] = \frac{1}{r_n}\left(\begin{array}{c|c} 0 & A \\ \hline -A^\dagger & 0 \end{array}\right)$

$$
A = \begin{pmatrix} -\sqrt{n(n+1) - n_3(n_3-1)} & 0 & 0 \\ 0 & 2\sqrt{n(n+1) - n_3(n_3+1)} & 0 \\ 0 & 0 & -\sqrt{n(n+1) - (n_3+1)(n_3+2)} \end{pmatrix}
$$

Continued...

$\Rightarrow \|[\mathcal{D}, \pi(d\rho)]\|_{op} = \frac{2}{r_n}\sqrt{n(n+1) - n_3(n_3+1)}$.
Further $\text{Tr}(d\rho)^2 = 2$, which yields

$$d_n(\omega_{n_3+1}, \omega_{n_3}) = \frac{\lambda\sqrt{n(n+1)}}{\sqrt{n(n+1) - n_3(n_3+1)}}. \qquad (7)$$

Finite distance $(m_3 - n_3 \geq 2)$:

$$\begin{aligned} d_n(\omega_{m_3}, \omega_{n_3}) &= \sup_{a \in B} \left|\text{tr}(\rho_{n_3+k}a) - \text{tr}(\rho_{n_3}a)\right| \quad ; \quad \text{where } k = m_3 - n_3 \\ &= \sup_{a \in B} \left| \sum_{i=1}^{k} \text{tr}\left((\rho_{n_3+i} - \rho_{n_3+(i-1)}), a\right) \right| \\ &\leq \sum_{i=1}^{k} \frac{r_n}{\sqrt{n(n+1) - (n_3+i)(n_3+i-1)}}. \end{aligned}$$

Continued...

Again this upper bound is reached by

$$a_s = \sum_{p=n_3}^{m_3-1} \left(\sum_{i=1}^{m_3-p} \frac{r_n}{\sqrt{n(n+1)-(p+i)(p+i-1)}} |p\rangle\langle p| \right)$$

yielding

$$d_n(\omega_{m_3}, \omega_{n_3}) = \sum_{i=1}^{k} \frac{r_n}{\sqrt{n(n+1)-(n_3+i)(n_3+i-1)}}. \tag{8}$$

Here the triangle inequality is saturated as

$$d_n(\omega_{m_3}, \omega_{n_3}) = d_n(\omega_{m_3}, \omega_{l_3}) + d_n(\omega_{l_3}, \omega_{n_3}) \quad \text{for } n_3 \leq l_3 \leq m_3$$

In particular, $d_n(\mathrm{N}, \mathrm{S}) = d_n(\rho_n, \rho_{-n}) = \sum_{k=1}^{2n} \frac{r_n}{\sqrt{k(2n+1-k)}}$.

Examples:

$d_{1/2}(\mathrm{N}, \mathrm{S}) = r_{1/2}$; $d_1(\mathrm{N}, \mathrm{S}) = \sqrt{2}\, r_1$; $d_{3/2}(\mathrm{N}, \mathrm{S}) = \left(\frac{1}{2} + \frac{2\sqrt{3}}{3}\right) r_{3/2}$.

Only in the limit $n \to \infty$ one gets $\lim_{n\to\infty} \frac{d_n(\mathrm{N},\mathrm{S})}{r_n} = \pi$

Distances on $\mathbb{S}^2_*$ (coherent state basis)

Upper bound of finite distance:

Introduce a one parameter family of pure states

$$\rho_\theta \equiv |\theta\rangle\langle\theta| = U_F(\theta)|n\rangle\langle n|U_F^\dagger(\theta) \in \mathcal{H}_n \; ; \; U_F(\theta) = e^{-i\theta J_2} \qquad (9)$$

- In terms of stereographic variable z, $\rho_z = \rho_\theta$;
- $\omega_z(a) = tr\left(\rho_z a\right); \; a^\dagger = a \in \mathcal{H}_q^{(n)}$
- Define $W(t) = \omega_{zt}(a) = tr\left(\rho_{zt} a\right)$, with $t \in [0,1]$ then

$$|\omega_z(a) - \omega_0(a)| = \left|\int_0^1 \frac{\mathrm{d}W(t)}{\mathrm{d}t} dt\right| \leq \int_0^1 \left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| dt \leq r_n\theta. \quad (10)$$

The RHS is the geodesic distance of commutative sphere.

And $\nexists$ any $a \in \mathcal{A} = \mathcal{H}_q^{(n)}$ (for n-finite) saturating the upper bound.

Towards an actual computation

Ball condition in eigen-spinor basis:

$$[\mathcal{D}, \pi(a)] = \frac{1}{r_n} \left(\begin{array}{c|c} 0_{(2n+2)\times(2n+2)} & A_{(2n+2)\times 2n} \\ \hline -\ A^{\dagger}_{2n\times(2n+2)} & 0_{(2n)\times(2n)} \end{array} \right), \qquad (11)$$

where $A_{(2n+2)\times 2n} = (2n+1)_{+}\langle\langle n, n_3|\pi(a)|n, n'_3\rangle\rangle_{-}$ with $-n-1 \leq n_3 \leq n$ and $n-1 \leq n'_3 \leq n-1$. Rectangular null matrices stem from the degeneracy of the spectrum. $\Rightarrow$

$$\|[\mathcal{D}, \pi(a)]\|^2_{\rm op} = \|[\mathcal{D}, \pi(a)]^{\dagger}[\mathcal{D}, \pi(a)]\|_{\rm op} = \frac{1}{r_n^2}\|AA^{\dagger}\|_{\rm op} = \frac{1}{r_n^2}\|A^{\dagger}A\|_{\rm op}.$$

Clearly, it is convenient to deal with $\|AA^{\dagger}\|_{\rm op}$ as it is of lower dimension $(2n \times 2n)$.

$n = \frac{1}{2}$ fuzzy sphere

The algebra element can be taken to be element of $su(2)$ algebra. $a = \vec{a} \cdot \vec{\sigma} \in su(2)$; $\vec{a} \in \mathbb{R}^3$. Here $A^\dagger A$ is just a number.

$$\|[\mathcal{D}, \pi(a)]\|_{op} = \frac{2}{r_{1/2}} |\vec{a}| \leq 1 \Rightarrow |\vec{a}| \leq \frac{r_{1/2}}{2}, \text{ a solid sphere}$$

Take two states $\rho_N = \rho_{\theta=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$\rho_\theta = U(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} U^\dagger(\theta) = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta & \sin\theta \\ \sin\theta & 1 - \cos\theta \end{pmatrix}$$

$U(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \Rightarrow \Delta\rho^\dagger = \Delta\rho = \rho_\theta - \rho_0 \in su(2)$;
$\mathrm{Tr}(\Delta\rho) = 0$. Thus $\Delta\rho = \overrightarrow{\Delta\rho} \cdot \vec{\sigma}$; $\overrightarrow{\Delta\rho} = \frac{1}{2}(\sin\theta, 0, \frac{\cos\theta - 1}{2}) \in \mathbb{R}^3$

Continued...

Finally

$$d_{\frac{1}{2}}(\omega_\theta, \omega_0) = \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} |\omega_\theta(a) - \omega_0(a)| = \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} \left| \mathrm{Tr}_{\mathcal{H}_q^{(n)}}(\Delta\rho a) \right|$$

$= \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} \left| 2\vec{a} \cdot \vec{\Delta\rho} \right|$ and the supremum is reached when $\vec{a} \propto \vec{\Delta\rho}$

$$d_{\frac{1}{2}}(\omega_{\theta_0}, \omega_0) = r_{\frac{1}{2}} \sqrt{(\Delta\rho)_1^2 + (\Delta\rho)_3^2} = r_{\frac{1}{2}} \sin \frac{\theta_0}{2}, \text{No role for} \Delta\rho_\perp$$

A family of $\rho_t = (1-t)\rho_0 + t\rho_\theta;\ 0 \leq t \leq 1$ of mixed states can be thought of interpolating ρ_0 and ρ_θ.
$d_{1/2}(\rho_0, \rho_t) = t r_{1/2} \sin\left(\frac{\theta}{2}\right)$ and $d_{1/2}(\rho_t, \rho_\theta) = (1-t) r_{1/2} \sin\left(\frac{\theta}{2}\right)$ satisfying

$$d(\rho_0, \rho_t) + d(\rho_t, \rho_\theta) = d(\rho_0, \rho_\theta) \tag{12}$$

$n = 1$ fuzzy sphere

Here $\|[\mathcal{D}, \pi(a)]\|_{op} = \frac{1}{r_n}\sqrt{\|A^\dagger A\|_{op}}$. Writing
$M := A^\dagger A = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}$, with matrix entries

$$M_{11} = 3|a_{0,1}|^2 + 2(a_{0,0} - a_{1,1})^2 + |a_{0,-1} - 2a_{1,0}|^2 + 6|a_{1,-1}|^2,$$
$$M_{22} = 3|a_{0,-1}|^2 + 2(a_{0,0} - a_{-1,-1})^2 + |a_{1,0} - 2a_{0,-1}|^2 + 6|a_{1,-1}|^2,$$
$$M_{12} = \sqrt{2}\big[3a_{1,-1}(a_{0,1} + a_{-1,0}) + (a_{0,0} - a_{1,1})(2a_{0,-1} - a_{1,0}) + (a_{0,0} - a_{-1,-1})(2a_{1,0} - a_{0,-1})\big]$$

The two eigen-values are
$E_\pm = \frac{1}{2}\left((M_{11} + M_{22}) \pm \sqrt{(M_{11} - M_{22})^2 + 4|M_{12}|^2}\right)$.
Clearly, $E_+ \geq E_- \ \forall\ a \in B$ which means
$\inf_{a \in B} \|[\mathcal{D}, \pi(a)]\|_{\text{op}} = \frac{1}{r_1}\sqrt{\min(E_+)}$; $a = \Delta\rho + \Delta\rho_\perp$

Continued...

- Writing $a_s = \Delta\rho + \Delta\rho_\perp \in su(3)$

  with $\Delta\rho = e^{i\theta\hat{J}_2}|1\rangle\langle 1|e^{-i\theta\hat{J}_2} - |1\rangle\langle 1|$.
- We write $\Delta\rho_\perp = \sum_{i=1}^{8} c_i\lambda_i$; $\lambda_i' s$ are Gell-Mann matrices.
- But orthogonality condition $(\Delta\rho, \Delta\rho_\perp) = 0$ leaves us with 7 independent parameters.
- On computation, the distances for various angles of θ gives the following table obtained numerically and compared with $d_1^* := \sqrt{2} r_1 \sin\left(\frac{\theta}{2}\right))$

Data set for various distances corresponding to different angles

Angle (degree)	d_1^*/r_1	d_1/r_1
10	0.1232568334	0.1232518539
20	0.2455756079	0.2455736891
30	0.3660254038	0.3660254011
40	0.4836895253	0.4836894308
50	0.5976724775	0.5976724773
60	0.7071067812	0.7071067811
70	0.8111595753	0.8111595752
80	0.9090389553	0.9090389553
90	1	0.9999999998
100	1.0833504408	1.0833504407
110	1.1584559307	1.1584559306
120	1.2247448714	1.2247448713
130	1.2817127641	1.2817127640
140	1.3289260488	1.3289260487
150	1.3660254038	1.3660254037
160	1.3927284806	1.3927284806
170	1.4088320528	1.4088320527

Moyal plane $\mathbb{R}^2_*$

• Upper bound $d(\rho_0, \rho_z) \leq \sqrt{2\theta}|z|$; $\rho_z = |z\rangle\langle z|$ is obtained by considering a 1-parameter family of pure states $\rho_{zt} = |zt\rangle\langle zt|$; $0 \leq t \leq 1$, interpolating ρ_0 and ρ_z.

• In contrast to fuzzy sphere, this upper bound is reached by $a_s = \sqrt{\frac{\theta}{2}}\left(be^{-i\alpha} + b^\dagger e^{i\alpha}\right) \in$ Multiplier algebra.

• It's enough to show

$$d(\rho_0, \rho_{dz}) = \sqrt{2\theta}|dz|(trans.inv.) \tag{13}$$

by taking $d\rho = |dz\rangle\langle dz| - |0\rangle\langle 0| = d\bar{z}|0\rangle\langle 1| + dz|1\rangle\langle 0|$.

Continued..

By observing that $\pi(d\rho) = \begin{pmatrix} d\rho & 0 \\ 0 & d\rho \end{pmatrix}$ is a 5D matrix spanned by $|0\rangle\rangle, |1\rangle\rangle_{\pm}, |2\rangle\rangle_{\pm}$.

and $\|[\mathcal{D}_M, \mathbb{P}_N \pi(a_s)]\mathbb{P}_N\|_{op} = 1$ with $N \geq 2$
the corresponding optimal element is obtained by the lower bound itself

$$d(\rho, \rho') \geq \frac{\|\Delta\rho\|^2_{\mathrm{H.S}}}{\|[\mathcal{D}, \pi(\Delta\rho)]\|_{op}} = \sqrt{2\theta}|dz|$$

See (13) above..

Distances on doubled Moyal plane

Spectral triple:

$\mathcal{A}_T = \mathcal{H}_q \otimes M_2^d(\mathbb{C}), \mathcal{H}_T = (\mathcal{H}_c \otimes \mathbb{C}^2) \otimes \mathbb{C}^2, \mathcal{D}_T = \mathcal{D}_M \otimes \mathbb{1}_2 + \sigma_3 \otimes \mathcal{D}_2.$

Pure states: $\Omega_i^{(z)} = \rho_z \otimes \omega_i; \omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \omega_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$

- One can construct orthonormal eigen-spinors for $\mathcal{D}_T$ and verify Pythagoras theorem, reproducing earlier results of

$$d_t\left(\Omega_1^{(z)}, \Omega_2^{(z)}\right) = \frac{1}{|\Lambda|},\ d_l\left(\Omega_i^{(z)}, \Omega_i^{(0)}\right) = d_M(\rho_z, \rho_0) = \sqrt{2\theta}|z|$$

$$\left\{d_h(\rho_0 \otimes \omega_1, \rho_z \otimes \omega_2)\right\}^2 = \left\{d_t(\rho_0 \otimes \omega_1, \rho_0 \otimes \omega_2)\right\}^2 + \left\{d_l(\rho_0 \otimes \omega_1, \rho_z \otimes \omega_1)\right\}^2$$

Restricted Spectral Triple

For $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the restricted spectral $(\mathcal{A}^{\rho}, \mathcal{H}^{(\rho)}, \mathcal{D}^{(\rho)}) = (\alpha_{(\rho)}(\mathcal{A}), \pi(\rho)\mathcal{H}, \pi(\rho)\mathcal{D}\pi(\rho))$ is obtained by the self adjoint projector $\rho^2 = \rho = \rho^* \in \mathcal{A}$ through the map $\alpha_\rho : \mathcal{A} \to \mathcal{A}; \;\; \mathcal{A} \ni a \longmapsto \alpha_\rho(a) = \rho a \rho$

Here $\pi(\rho)$ indicates that the domain of π is restricted to $\pi|_{\mathcal{H}^{(\rho)}}$

Now $\forall \omega_1, \omega_2 \in \mathcal{P}(\mathcal{A}^{(\rho)}), \;\; d^{(\rho)}(\omega_1, \omega_2) = d(\omega_1 \circ \alpha_\rho, \omega_2 \circ \alpha_\rho)$, provided $[\mathcal{D}, \pi(\rho)] = 0$

This indicates that $\pi(\rho)$ should be built out of the eigen spinors of $\mathcal{D}$.

Example

$$P^{(trans)}_{T(0)} := \mathbb{P}_0 \otimes \mathbb{1}_2 = \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_2 \ \in \ \mathcal{A}_T$$

$$P^{(long)}_{T(i)}(N) := \mathbb{P}_N \otimes \omega_i = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \otimes \omega_i$$

Here $\omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \omega_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

With the first one , we reproduce the spectral triple for 2-point space:

$$P^{(trans)}_{T(0)} \mathcal{A}_T P^{(trans)}_{T(0)} = \begin{pmatrix} |0\rangle\langle 0| & 0 \\ 0 & 0 \end{pmatrix} \otimes M_2^d(\mathbb{C})$$

$$P^{(trans)}_{T(0)} \mathcal{H}_T = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \otimes \mathbb{C}^2$$

$$P^{(trans)}_{T(0)} \mathcal{D}_T P^{(trans)}_{T(0)} = \mathbb{P}_0 \otimes \mathcal{D}_2$$

With the second one , likewise , we reproduce the spectral triple for one of the Moyal planes in the limit N$\to \infty$.

Impact of Higgs field

Change the triplet $T \to \tilde{T}$ where

$$\widetilde{\mathcal{H}}_T = (\mathcal{H}_q \otimes M_2(\mathbb{C})) \otimes M_2^d(\mathbb{C}) \ni \widetilde{\Psi}; \widetilde{\mathcal{D}}_T \widetilde{\Psi} = \mathcal{D}_T \widetilde{\Psi} + \widetilde{\Psi} \mathcal{D}_T$$

so that the Dirac operator can be fluctuated. This gives rise to gauge fields, along with Higgs field $\mathcal{D}_T \to \mathcal{D}_T + H$;

$$H = c\sigma_3 \otimes a_2[\mathcal{D}_2, b_2]; \;\; c = ab \in \mathcal{H}_q \tag{14}$$

If c is such that $[c, \rho_z] = 0$ this gives rise to variation in the transverse distance.

$$d_t(\rho_z \otimes \omega_1, \rho_z \otimes \omega_2) = \frac{1}{|\Lambda(x_1, x_2)|} \tag{15}$$

References

The talk is based on following two publications:

(1) "Revisiting Connes' finite spectral distance on Non-commutative spaces: Moyal plane and Fuzzy sphere" , Int.J.Geom.Meth.Mod.Phys., 15 (2018) 1850204
Yendrenbam Chaoba Devi, Kaushlendra Kumar, BC, Fredrik G. Scholtz

(2)"Spectral distances on the doubled Moyal plane using Dirac eigenspinors", Phys.Rev.D 97,(2018) 086019
Kaushelendra Kumar, BC

THANK YOU