(Non)perturbative Spectral Action

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Least Action Principle



- Huge simplification: input symmetries → dynamics
- But . . . How do you choose symmetries?
  - particle physics
  - cosmology (inflation!)

Spectral Action Principle
[Ali Chamseddine, Alain Connes (1997)]

"The physical action only depends upon the spectrum of $\mathcal{D}$."

$$\mathrm{Tr}\left(f(\mathcal{D}/\Lambda)\right) + \tfrac{1}{2}\langle J\psi|\mathcal{D}|\psi\rangle$$

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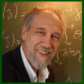

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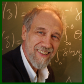

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Choose $\mathcal{A} = C^\infty(M) \otimes \big(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\big)$ and you get ...

$$S_B = \int_M \sqrt{g} d^4x \Bigg(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^\star R^\star + \\ + \frac{1}{4} G^i_{\mu\nu} G^{\mu\nu i} + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \\ + \frac{1}{2} |D_\mu H|^2 - \mu_0^2 |H|^2 + \lambda_0 |H|^4 - \xi_0 R |H|^2 \Bigg) + \mathcal{O}(\Lambda^{-1})$$

- EH term + Weyl term + cosmological constant + topological term,
- dynamical terms of SM bosons,
- Higgs sector, coupling between Higgs and gravity.

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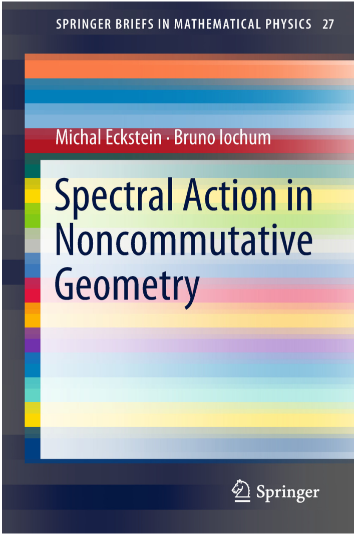
Spectral Action in Noncommutative Geometry



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1. The dwelling of the spectral action
   - Spectral triples
   - OP's
   - pdos
   - Dimension spectrum
   - Noncommutative integral
2. Residues, expansions, and all that
   - Spectral functions and their transforms
   - Asymptotic expansions
   - From heat traces to zeta functions
   - From zeta functions to heat traces
3. Spectral action and its fluctuations
   - Classes of cut-off functions
   - Asymptotic expansion of the spectral action
   - Fluctuations of the spectral action
4. Summary

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$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ - spectral triple

- $\mathcal{A}$ – unital involutive algebra
- $\mathcal{H}$ – separable Hilbert space with $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- $\mathcal{D}$ – a (possibly unbounded) selfadjoint operator on $\mathcal{H}$, such that
  - $[\mathcal{D}, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$,
  - $\mathcal{D}$ has a compact resolvent – i.e. $(\mathcal{D} - \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$ for $\lambda \notin \operatorname{spec}(\mathcal{D})$.

Archetypical example

Let M be a compact spin Riemannian manifold and $\mathcal{S}$ be a spinor bundle over M. Then, $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(M, \mathcal{S})$, $\mathcal{D} = \not{D} := -i\gamma^\mu \nabla^{\mathcal{S}}_\mu$ is a spectral triple.

- Define $D := \mathcal{D} + P_0$, with P_0 – projection on $\ker \mathcal{D}$.

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Finite-dimensionality and regularity

Finite-dimensionality

The triple is said of **dimension** p (or p-**dimensional**) when

$$p := \inf \{q \geq 0 \mid \operatorname{Tr} |D|^{-q} < \infty\} < \infty.$$

- Can have 0-dimensional spectral triples with $\dim \mathcal{H} = \infty$.
  (vide: ST on standard Podleś sphere by L. Dąbrowski & A. Sitarz (2003))

Regularity

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is **regular** if

$$\forall a \in \mathcal{A} \quad a, [\mathcal{D}, a] \in \bigcap_{n \in \mathbb{N}} \operatorname{Dom} \delta'^n, \quad \text{where } \delta' := [|\mathcal{D}|, \cdot].$$

- δ' gives a family of seminorms, commutatively: $\mathcal{A}_{\delta'} \simeq C^\infty(M)$.

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OP's

Define: $\delta(\cdot) := [|D|, \cdot]$ and $\nabla(\cdot) := [D^2, \cdot]$.

- Remark: $\operatorname{Dom}\delta = \operatorname{Dom}\delta' \subset \mathcal{B}(\mathcal{H})$

Introducing

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$\mathcal{P}(\mathcal{A}) :=$ polynomial algebra generated by $\mathcal{A}$, $\mathcal{D}$, $|\mathcal{D}|$ and $J\mathcal{A}J^{-1}$, if J exists.

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Examples:

- $a[D, b]|D|^{-3}[\mathcal{D}^2, c]\mathcal{D}^6 \in \Psi^4(\mathcal{A})$.
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Dimension spectrum – definitions

A regular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension p has a **dimension spectrum** Sd if $\mathrm{Sd} \subset \mathbb{C}$ is discrete and for any $T \in \Psi^0(\mathcal{A})$, the function

$$\zeta_{T,D}(s) := \mathrm{Tr}\ T|D|^{-s}, \quad \text{defined for} \quad \Re(s) > p,$$

extends meromorphically to $\mathbb{C}$ with poles located in Sd.
We say that the dimension spectrum is of **order** $k \in \mathbb{N}^*$ if all of the poles of functions $\zeta_{T,D}$ are of the order at most k and **simple** when $k = 1$.

- If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is not finitely summable, then $\zeta_{1,D}$ does not exist.
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Dimension spectrum – examples

(Almost-)commutative geometry

If $d = \dim M$, then $\mathrm{Sd}(C^\infty(M), L^2(M,S), D\!\!\!/) = d - \mathbb{N}$ and it is simple.

Manifolds with conical singularities [J.-M. Lescure (2001)]

If $d = \dim M$, then $\mathrm{Sd}(\mathcal{A}, \mathcal{H}, \mathcal{D}) = d - \mathbb{N}$ and of *order 2*.

Fractals [A. Connes, M. Marcolli (2008)]

For $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on the ternary Cantor set, Sd is simple and

$$\mathrm{Sd} = \frac{\log 2}{\log 3} - \mathbb{N} + i\frac{2\pi}{\log 3}\mathbb{Z}$$

Standard Podleś sphere [M.E., B. Iochum, A. Sitarz (2014)]

For $(\mathcal{A}_q, \mathcal{H}_q, \mathcal{D}_q)$ on the standard Podleś sphere, Sd is of order 2 and

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(Almost-)commutative geometry

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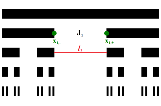

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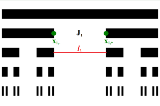

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Noncommutative integral

Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a regular p-dimensional spectral triple with a dimension spectrum. For any $T \in \Psi^{\mathbb{C}}(\mathcal{A})$ and any $k \in \mathbb{Z}$ define

$$\fint^{[k]} T := \operatorname*{Res}_{s=0} s^{k-1}\zeta_{T,D}(s), \qquad \fint T := \fint^{[1]} T = \operatorname*{Res}_{s=0} \zeta_{T,D}(s).$$

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Outline

Spectral functions

$$\mathcal{T}^p := \left\{ H \in \mathcal{L}(\mathcal{H}) \;\middle|\; H > 0 \text{ and } \forall \epsilon > 0 \ \mathrm{Tr}\ H^{-p-\epsilon} < \infty, \text{ but } \mathrm{Tr}\ H^{-p+\epsilon} = \infty \right\}$$

Let $H \in \mathcal{T}^p$ and $K \in \mathcal{B}(\mathcal{H})$:

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$$\zeta_{K,H}(s) := \mathrm{Tr}\ K H^{-s} = \sum_{n=0}^{\infty} \mathrm{Tr}(P_n(H)K)\, \lambda_n(H)^{-s}, \text{ for } \Re(s) > p.$$

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$$\mathrm{Tr}\ K e^{-t\,H} := \sum_{n=0}^{\infty} \mathrm{Tr}(P_n(H)K)\, e^{-t\lambda_n(H)}, \text{ for } t > 0.$$

Use:

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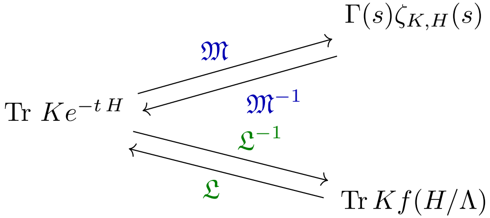

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Heat traces from heat kernels – differential case

Theorem [Gilkey (2004)]

Let be a positive elliptic *differential* op H of order m acting on a vector bundle E over a closed d-dimensional Riemannian manifold M and let $K \in C^\infty(\mathrm{End}(E))$

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$\zeta_{K,H}$ admits a meromorphic extension to $\mathbb{C}$ with (possibly removable) simple poles at $s = (d-k)/m$ for $k \in \mathbb{N}$ and

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$(\varphi_k)_k$ is an **asymptotic scale** at x_0 if, for any k

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(Extended) asymptotic expansion

Let $(\varphi_k)_{k\in\mathbb{N}}$ be an asymptotic scale at x_0 and g a complex function around x_0. The function g has an **asymptotic expansion with respect to** $(\varphi_k)_k$ if there exists $(\rho_k)_{k\in\mathbb{N}}$ such that

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In general, the expansion $\operatorname{Tr} K e^{-t\,H} \underset{t\downarrow 0}{\sim} \sum_{k=0}^{\infty} \rho_k(t)$ *diverges for all* t>0.

- **Convergent expansion** $\operatorname{Tr} K e^{-t\,H} = \sum_{k=0}^{\infty} \rho_k(t) + R_\infty(t)$, for $t \in (0,T)$.
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Examples from spheres

Let $\not{D}_n$ be the standard Dirac operator on S^n, then

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In general, the expansion $\mathrm{Tr}\, K e^{-t\,H} \underset{t\downarrow 0}{\sim} \sum_{k=0}^{\infty} \rho_k(t)$ *diverges for all* t>0.

- **Convergent expansion** $\mathrm{Tr}\ K e^{-t\,H} = \sum_{k=0}^{\infty} \rho_k(t) + R_\infty(t)$, for $t \in (0,T)$.
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Examples from spheres

Let $D\!\!\!/_n$ be the standard Dirac operator on S^n, then

- divergent:

  On S^2 we have $\mathrm{Tr}\, e^{-t\,D\!\!\!/_2^2} \underset{t\downarrow 0}{\sim} 2t^{-1} - 4\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_{2k+2}}{2k+2}\, t^k$.
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  On S^3 we have $\mathrm{Tr}\, e^{-tD\!\!\!/_3^2} = \frac{\sqrt{\pi}}{2} t^{-3/2} - \frac{\sqrt{\pi}}{4} t^{-1/2} + \mathcal{O}_0(e^{-1/t})$, for all $t > 0$.
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Theorem [M.E., B. Iochum (2018)]

Let $H \in \mathcal{T}^p, K \in \mathcal{B}(\mathcal{H})$. *Assume that*

$$\mathrm{Tr}\, K e^{-t\,H} \underset{t \downarrow 0}{\sim} \sum_{k=0}^{\infty} \rho_k(t), \qquad \text{with}$$

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and $\sum$ is absolutely conv. $\forall\, t > 0, k \in \mathbb{N}$.

Then, $\zeta_{K,H}$ has a meromorphic extension to $\mathbb{C}$ with poles of order at most $d+1$ and $\mathfrak{P}(\zeta_{K,H}) \subset X$ and for any $z \in X$ and n,

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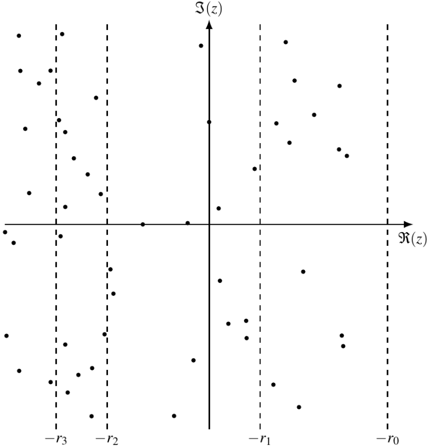

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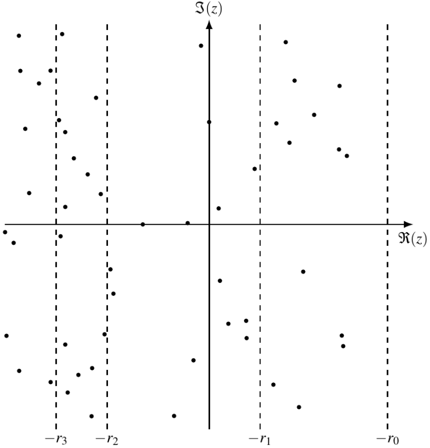

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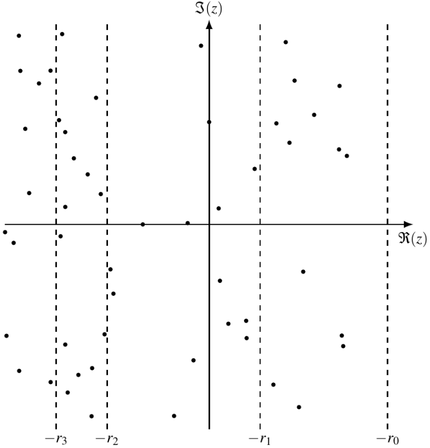

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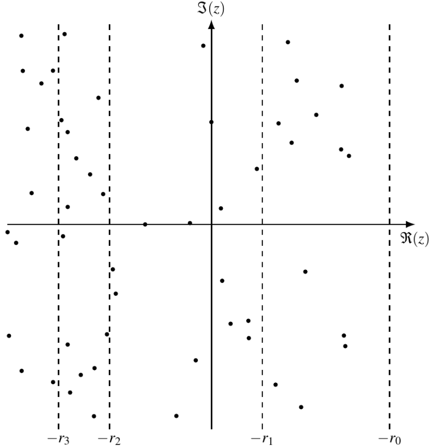

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Assume that $(\mathcal{Z}_{K,H}(s) := \Gamma(s)\zeta_{K,H}(s))$

1. $\zeta_{K,H}$ is meromorphic on $\mathbb{C}$ with poles of order at most d.
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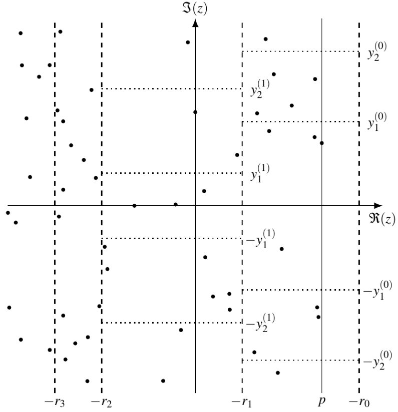

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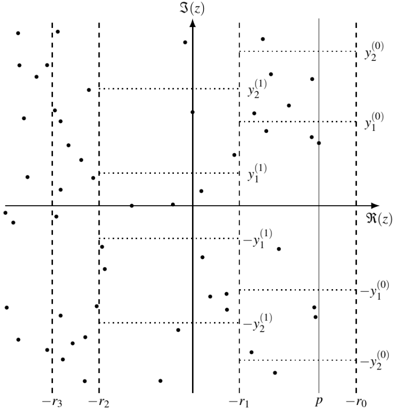

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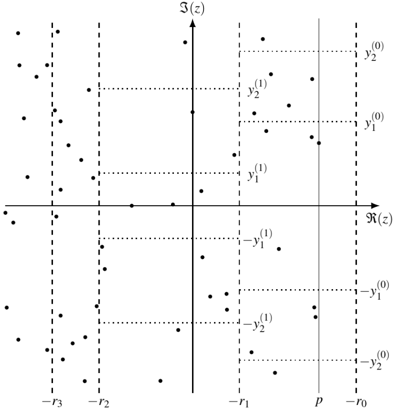

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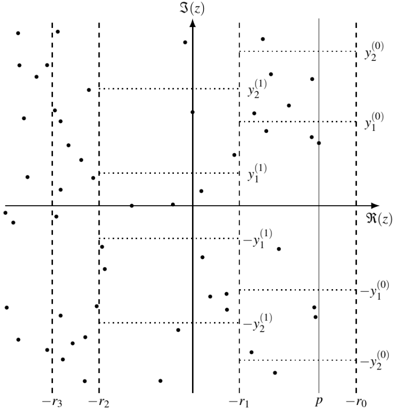

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Outline

Classes of cut-off functions

$$\mathrm{Tr}\, K e^{-t\,H} \xrightarrow[\text{inverse Laplace transform}]{\mathfrak{L}^{-1}} \mathrm{Tr}\, K f(H/\Lambda)$$

Theorem [Bernstein]

f is completely monotone ($f \in C^\infty((0,\infty),\mathbb{R})$ and $(-1)^n f^{(n)}(x) \geq 0$) if and only if $f(x) = \mathfrak{L}[\phi](x)$ for all $x > 0$ for a unique non-negative measure ϕ on $\mathbb{R}^+$.

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$$\mathcal{C} \;:=\; \{f = f^+ - f^-,\, f^{\pm} \in \mathcal{CM} \;\mid\; f(x) \geq 0, \text{ for } x > 0\},$$

$$\mathcal{C}_0 \;:=\; \{f = \mathfrak{L}[\phi] \in \mathcal{C} \;\mid\; \forall\, n \in \mathbb{N},\; \int_0^{\infty} s^n d\,|\phi|\,(s) < \infty\},$$

$$\mathcal{C}^p \;:=\; \{f \in \mathcal{C} \;\mid\; f^{\pm}(x) = \mathcal{O}_{\infty}(x^{-p})\}, \quad \mathcal{C}_0^p \;:=\; \mathcal{C}_0 \cap \mathcal{C}^p, \text{ for } p > 0.$$

- Need $f \in \mathcal{C}$, for f to be a Laplace transform.
- Need $f \in \mathcal{C}_0$ to apply $\mathfrak{L}^{-1}$ term by term in the expansion of $\mathrm{Tr}\; K e^{-t\,H}$.
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Classes of cut-off functions

$$\mathrm{Tr}\; K e^{-t\,H} \xrightarrow[\text{inverse Laplace transform}]{\mathfrak{L}^{-1}} \mathrm{Tr}\, K f(H/\Lambda)$$

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Asymptotic expansion of the spectral action

Theorem [M.E., B. Iochum (2018)]

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a p-dimensional spectral triple and $T \in \mathcal{B}(\mathcal{H})$. *Assume that*,

$$\mathrm{Tr}\; T e^{-t|D|} \underset{t\downarrow 0}{\sim} \sum_{k=0}^{\infty} \rho_k(t), \quad \rho_k(t) = \sum_{z\in X_k} \Big[\sum_{n=0}^{d} a_{z,n}(T,|D|) \log^n t \Big] t^{-z}$$

and the series defining $\rho_k(t)$ is absolutely convergent for any $t > 0$ and any $k \in \mathbb{N}$.

Then, for any $f = \mathfrak{L}[\phi] \in \mathcal{C}_0^r$ with $r > p$,

$$\mathrm{Tr}\; T f(|D|/\Lambda) \underset{\Lambda\to+\infty}{\sim} \sum_{k=0}^{\infty} \psi_k(\Lambda), \qquad \text{w.r.t. the scale } (\Lambda^{-r_k})_k,$$

with $\psi_k(\Lambda) = \sum_{z\in X_k} \Lambda^z \sum_{n=0}^{d} (-1)^n \log^n \Lambda \sum_{m=n}^{d} \binom{m}{m-n} a_{z,m}(T,|D|)\, f_{z,m-n}$,

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Spectral action on the standard Podleś sphere

The Podleś sphere algebra is generated, for some $0 < q < 1$, by $A = A^*$, B, B^* subject to the relations

$$AB = q^2 BA, \quad AB^* = q^{-2} B^* A, \quad BB^* = q^{-2} A(1-A), \quad B^* B = A(1 - q^2 A).$$

Dąbrowski–Sitarz spectral triple is equipped with $\mathcal{D}_q$ with the spectrum

$$\mu_n(\mathcal{D}_q) = c(q^{-n-1} - q^{n+1}) \quad \text{and} \quad M_n(|\mathcal{D}_q|) = 4(n+1), \quad n \in \mathbb{N}.$$

Theorem [M.E., B. Iochum, A. Sitarz (2014)]

Let $f \in C_0^r$ for some $r > 0$ and denote $\kappa := \frac{2\pi i}{\log q}$. Then, for any $t > 0$,

$$S(\mathcal{D}_q, f, \Lambda) = \sum_{k=0}^{\infty} \sum_{j \in \mathbb{Z}} \sum_{n=0}^{2} a_{-2k+\kappa j,n} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \times f_{-2k+\kappa j,m} (\log \Lambda)^{n-m} \Lambda^{-2k+\kappa j}.$$

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Fluctuations of geometry

- The spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D}_{\mathbb{A}})$, with $\mathcal{D}_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$ are equivalent for a *suitable* $\mathbb{A} = \mathbb{A}^* \in \mathcal{B}(\mathcal{H})$.

Theorem [A. Connes, A. Chamseddine (2006)]

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be regular and let $\mathbb{A} \in \Psi^0(\mathcal{A})$. Then, $\forall N \in \mathbb{N}^*, s \in \mathbb{C}$,

$$|D_{\mathbb{A}}|^{-s} = |D|^{-s} + \sum_{n=1}^{N} K_n(Y,s)|D|^{-s} \mod \mathrm{OP}^{-(N+1)-\Re(s)},$$

with $K_n(Y,s) \in \Psi^{-n}(\mathcal{A})$.

Corollary

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a regular p-dimensional spectral triple with a dimension spectrum and let $\mathbb{A} \in \Psi^0(\mathcal{A})$. Then, $(\mathcal{A}, \mathcal{H}, \mathcal{D}_{\mathbb{A}})$ will also be regular, p-dimensional and possessing a dimension spectrum.

- The existence of an asymptotic expansion for $\mathrm{Tr}\, e^{-t|\mathcal{D}|}$ *does not*, in general, imply one for $\mathrm{Tr}\, e^{-t|\mathcal{D}_{\mathbb{A}}|}$.
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Theorem [A. Connes, M. Marcolli (2008), M.E., B. Iochum (2018)]

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be regular p-dimensional with a simple dimension spectrum and let $\mathbb{A} \in \Psi^0(\mathcal{A})$. *Assume*, moreover, that

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where the, possibly infinite, series over α is absolutely convergent for all $t > 0$.

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where the, possibly infinite, series over α is absolutely convergent for all $t > 0$.

Then, for any $f \in \mathcal{C}_0^r$ with $r > p$,

$$\mathrm{Tr}\, f(|\mathcal{D}_{\mathbb{A}}|/\Lambda) = \sum_{\alpha \in \mathrm{Sd}^+} \Lambda^\alpha \int_0^\infty x^{\alpha-1} f(x)dx \sum_{n=0}^{\lfloor p - \Re(\alpha) \rfloor} ⨍ P_n(\alpha, D, D^{-1}, \mathbb{A})\, |D|^{-\alpha}$$

$$+ f(0)\Big[\zeta_D(0) + \sum_{k=1}^{p} \frac{(-1)^k}{k} ⨍ (\mathbb{A}\, D^{-1})^k\Big] + \mathcal{O}_\infty(1),$$

where $P_n \in \Psi^{-n}(\mathcal{A})$ are polynomials in all variables and of degree n in $\mathbb{A}$:

$$P_0 = 1, \quad P_1 = -\alpha \mathbb{A} D^{-1}, \quad P_2 = \tfrac{\alpha}{4}(\alpha+2)(\mathbb{A}D^{-1})^2 + \tfrac{\alpha^2}{4}\mathbb{A}^2 D^{-2}, \quad \ldots$$

Summary

- *Given* an asymptotic expansion for a heat trace one obtains
  - a meromorphic extension of the corresponding zeta function,
  - an asymptotic expansion of the spectral action for a class of cut-offs.
- Such an expansion might
  - have terms log-periodic oscillating with energy,
  - have terms proportional to $(\log \Lambda)^n$ for any $n \in \mathbb{N}$,
  - be convergent and exact for some range of energies.
- One needs to study the fluctuations of geometry
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Thank you for your attention!