Coarse-graining of measurement and quantum-to-classical transition in the bipartite scenario

Sibasish Ghosh

Optics & Quantum Information Group
The Institute of Mathematical Sciences
C. I. T. Campus, Taramani
Chennai - 600 113

[Based on the joint work with Madhav Krishnan V. and Tanmoy Biswas (arXiv:1703.00502 (quant-ph))]
Motivation
Quantum-to-classical transition

- We do believe that every physical world can be effectively described using laws of Quantum Mechanics.

- But, in practice, we don’t need that always – laws of classical physics are sufficient, in plenty of such cases.

- A quantum mechanical system **may** start behaving classically (i) with increase in no. of sub-systems, (ii) with increase in system dimension, (iii) change in interaction under external fields and/or among the constituent sub-systems, (iv) change in observation of the system, etc.

- A pertinent question therefore is: **When does a quantum system start behaving classically?**
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But system-specific boundary may exist!
Quantum-to-classical transition via coarse-graining

- **Microscopic** systems are *generally* described by Quantum Physics while **macroscopic** systems are *generally* described by Classical Physics.

- **Coarse-graining** of measurements may lead to distinguish states of macroscopic systems while we may need **fine-grained** measurements for the corresponding microscopic scenario.

- Here we will study properties of specific states of continuous variable bipartite systems under the action of different coarse-graining measurements to identify quantum-to-classical transition.
Coarse-graining vs. fine-graining of measurement

Normal light is enough to distinguish the colours.

Laser light is required to distinguish the energy transitions.

Distinguishing macroscopic objects

Distinguishing microscopic objects
Two types of coarse-graining measurements: resolution and reference

A brief history about related works

Non-classicality of states in Quantum Optics: our approach

Case 1: On quantum-to-classical transition for
\[ |\psi_\alpha\rangle = \frac{1}{\sqrt{2}} \times (|\alpha_e\rangle \otimes |\alpha_o\rangle + |\alpha_o\rangle \otimes |\alpha_e\rangle) \]

Case 2: On quantum-to-classical transition for
\[ |\psi_N\rangle = \frac{1}{\sqrt{2}} \times (|0\rangle \otimes |N\rangle + |N\rangle \otimes |0\rangle) \]

Case 3: On quantum-to-classical transition for
\[ |\psi_r\rangle = \frac{1}{\sqrt{2}} \times (|\Psi_r^+\rangle \otimes |\Psi_r^-\rangle + |\Psi_r^-\rangle \otimes |\Psi_r^+\rangle) \]

Discussion (with speculations on non-commutative extensions)
Two types of coarse-graining measurements: resolution and reference.
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Rotation of measurement direction leads to rotated position of detectors.
Resolution and reference coarse-graining

Coarse-grained measurements are of two types, according to Jeong et al. [Phys. Rev. Lett. 112, 010402 (2014)]:

• Resolution coarse-graining: Measurement of POVM $\mathcal{M}_k = \{ E_{k\alpha} : E_{k\alpha} \geq 0, \sum_{\alpha=1}^{N} E_{k\alpha} = I \}$ is performed with apriori probability $p_k$.

• Reference coarse-graining: Measurement set-up (i.e., direction) is chosen from a given set with apriori probability.
Resolution coarse-graining

- The $\alpha$-th outcome of the **resolution** coarse-grained POVM occurs $\iff$ $\alpha$-th outcome of the POVM $\mathcal{M}_1$ occurs (with probability $p_1$) or, $\alpha$-th outcome of the POVM $\mathcal{M}_2$ occurs (with probability $p_2$), . . . .

- The resolution coarse-grained POVM is:
  \[ \{ F_\alpha \equiv \sum_k p_k E_{k\alpha} : \alpha = 1, 2, \ldots, N \} \]
Example of resolution coarse-graining (Jeong et al.)

- **Example:**
  
  1. \( \{ |o_n\rangle : n = 0, \pm 1, \pm 2, \ldots \} \) be an ONB for an infinite dimensional quantum system.

  2. For each \( k \in \{0, \pm 1, \pm 2, \ldots \} \), consider the two-outcome observable: \( O^k = O^k_+ - O^k_- \) where \( O^k_+ = \sum_{n=k}^{+\infty} |o_n\rangle \langle o_n| \) and \( O^k_- = \sum_{n=-\infty}^{k-1} |o_n\rangle \langle o_n| \).

  3. Choose now measurement of \( O^k \) with the apriori probability \( P_\delta(k) \), where \( P_\delta(k) = N_\delta \exp[-k^2/(2\delta)] \) – the discrete Gaussian distribution with variance \( \delta^2 \).

  4. The resolution coarse-grained observable:
     \[ O_\delta = \sum_{k=-\infty}^{+\infty} P_\delta(k) O^k. \]
Reference coarse-graining

- Original POVM: \( \{ E_\alpha : \alpha = 1, 2, \ldots N \} \).

- Choose an element \( U \) from a given set \( S = \{ U_1, U_2, \ldots \} \) of unitaries with probability \( P(U) \).

- The \( \alpha \)-th outcome of the reference coarse-grained POVM occurs \( \iff \) the \( \alpha \)-th outcome of the POVM \( \{ U_1^\dagger E_\alpha U_1 : \alpha = 1, 2, \ldots N \} \) occurs (with probability \( P(U_1) \)) or, the \( \alpha \)-th outcome of the POVM \( \{ U_2^\dagger E_\alpha U_2 : \alpha = 1, 2, \ldots N \} \) occurs (with probability \( P(U_2) \)), ....

- The reference coarse-grained POVM is:
  \[
  \{ F_\alpha \equiv \sum_k P(U_k) U_k^\dagger E_\alpha U_k : \alpha = 1, 2, \ldots, N \}
  \]
Example of reference coarse-graining (Jeong et al.)

- **Example:**
  
  - Consider the two-outcome observable: $O^0 = O^0_+ - O^0_-$.  
  - $U(\theta)$ be the unitary acting on the pair $\{|o_n\rangle, |o_{-n}\rangle\}$ (for $n = 1, 2, \ldots$) as:

    \[
    U(\theta)|o_n\rangle = \cos \theta |o_n\rangle + \sin \theta |o_{-n}\rangle,
    \]
    \[
    U(\theta)|o_{-n}\rangle = \sin \theta |o_n\rangle - \cos \theta |o_{-n}\rangle.
    \]

  - $\theta$ is chosen from $[0, 2\pi]$ with probability $P_\Delta(\theta - \theta_0) \equiv N_\Delta \exp[-(\theta - \theta_0)^2/(2\Delta)]$, the Gaussian probability distribution with mean $\theta_0$ and variance $\Delta^2$. 

Sibasish Ghosh  
Optics & Quantum Information Group  
The Institute of Mathematical Sciences  
C. I. T. Campus, Taramani Chennai - 
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Example of reference coarse-graining (continued)

- Example (continued):
  - Choose now measurement of the two-outcome observable
    \( U(\theta) \dagger O^0 U(\theta) \) with the apriori probability \( P_\Delta(\theta - \theta_0) \).
  
  - The reference coarse-grained observable:
    \[
    O_\Delta(\theta_0) = \int_0^{2\pi} P_\Delta(\theta - \theta_0) U(\theta) \dagger O^0 U(\theta) \, d\theta.
    \]
A brief history about related works
Different ways of testing quantumness

- **Quantumness** (more generally, **non-classicality**) of a system may be revealed in different facets:
  - Contextuality
  - Existence of indistinguishable states
  - Violation of some temporal Bell-type inequality
  - Steerability (for bi-partite case)
  - Violation of some local-realistic inequality (for bi-partite/multi-partite case)
  - Singularity and/or negativity of phase-space distribution function
  - . . . . . .
Testing quantumness via violation of local-realistic inequality

- Violation of any local-realistic inequality by a bi-partite/multi-partite state *necessarily* implies ‘non-classicality’ of the state.

- Even in the ‘macroscopic’ limit $J \to +\infty$, Mermin [Phys. Rev. D 22, 356 (1980)] showed violation of a local-realistic inequality by some states of a two spin-$J$ system.

- Unfortunately, this violation demands measurement precision of the order of $J$.

- So, non-classicality of **no** state of the system will be revealed through violation of the inequality if measurements are sufficiently coarse-grained.
Jeong et al.’s work

- Jeong et al. [Phys. Rev. Lett. 112, 010402 (2014)] used the coarse-grained observables $O_\delta$ and $O_\Delta(\theta_0)$ (separately) and showed the violation amounts – in the case of the Bell-CHSH inequality – decrease to zero with the increase in the coarse-graining parameters $\delta$ and $\Delta$.

- But this – **in no way** – indicates approach to ‘classicality’, as non-violation of the BCHSH inequality does not guarantee existence of a local-realistic description of the measurement statistics of the initial state due to coarse-graining of the measurements.

- Is there a way out?
Non-classicality of states in Quantum Optics: our approach
Phase-space distribution functions of quantum optical states

- Corresponding to any classical Hamiltonian system, one can always find its phase-space probability distribution function.

- One can similarly associate different phase-space distribution functions to any state of a quantized electromagnetic field (an infinite dim. quantum system) – depending upon different orderings of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. 
Phase-space distribution functions (continued)

- For normal ordering, any state $\rho$ of the system has the Glauber-Sudarshan $P$-distribution:

$$P_\rho(\gamma) \equiv \frac{1}{\pi^2} \int_{\lambda \in \mathcal{C}} \text{Tr} \left[ \rho e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} \right] e^{\lambda^* \gamma - \lambda \gamma^*} d^2 \lambda$$

- For symmetric ordering, we have the Wigner distribution function for $\rho$:

$$W_\rho(\gamma) \equiv \frac{1}{\pi^2} \int_{\lambda \in \mathcal{C}} \text{Tr} \left[ \rho e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} \right] e^{\lambda^* \gamma - \lambda \gamma^*} d^2 \lambda$$
Phase-space distribution functions (continued)

- **Diagonal representation of $\rho$:**

$$
\rho = \int_{\gamma \in \mathcal{C}} P_\rho(\gamma) |\gamma\rangle \langle \gamma| d^2 \gamma,
$$

|$\gamma\rangle$ being a coherent (classical) state.

- $\rho$ is **classical** iff $P_\rho(\gamma) \geq 0$ for all phase-space points ($x = \sqrt{2} \text{Re}(\gamma)$, $p = \sqrt{2} \text{Im}(\gamma)$).

- $\rho$ is **non-classical** if $W_\rho(\gamma)$ is negative at least at one phase-space point.

- Generalization to multi-mode states is straight-forward.
Our approach (with $P$-distribution)

- We applied measurement of the resolution coarse-grained observable $O_{\delta}$ on certain types of single-mode as well as two-mode states and looked for positivity of the $P$-distributions of the post-measurement states (non-selective case) – to certify classicality of the post-measurement states.

- Non-positivity and/or singularity of the $P$-distributions of the post-measurement states indicate impossibility of quantum-to-classical transition in the concerned case.
Our approach (with Wigner distribution)

- We also applied measurement of the reference coarse-grained observable $O_\Delta$ on certain types of two-mode states and looked for negativity of the Wigner distributions of the post-measurement states (non-selective case) – to certify non-classicality of the post-measurement states.

- Singularities and signatures of phase-space distribution functions in the selective cases are more severe.

- Initial states are chosen with different degrees of presence of ‘non-classicality’ at the single-mode level.
Case 1: On quantum-to-classical transition for
\[ |\psi_\alpha\rangle = \frac{1}{\sqrt{2}} \times (|\alpha_e\rangle \otimes |\alpha_o\rangle + |\alpha_o\rangle \otimes |\alpha_e\rangle) \]
State formulation

- Choose the ONB \( \{|o_n\rangle : n = 0, \pm 1, \pm 2, \ldots \} \) as: \( |o_n\rangle = |2n\rangle \)
  for \( n = 0, 1, 2, \ldots \) and \( |o_{-n}\rangle = |2n - 1\rangle \) for \( n = 1, 2, \ldots \).

- **Coherent state** \( |\alpha\rangle \): For any \( \alpha \in \mathbb{C} \),
  \[
  |\alpha\rangle \equiv e^{-|\alpha|^2/2} \times \left[ \sum_{n=0}^{+\infty} \frac{(\alpha^{2n}/\sqrt{(2n)!})}{\sqrt{2n}} \times |o_n\rangle + \sum_{n=1}^{+\infty} \frac{(\alpha^{2n-1}/\sqrt{(2n-1)!})}{\sqrt{2n-1}} \times |o_{-n}\rangle \right].
  \]

- **Even and odd coherent states**: \( |\alpha_e\rangle = N_e (|\alpha\rangle + | - \alpha\rangle) \),
  \( |\alpha_o\rangle = N_o (|\alpha\rangle - | - \alpha\rangle) \) with \( N_{e/o} = 1/\sqrt{2 \pm 2e^{-2|\alpha|^2}} \).

- We will study quantum-to-classical transition of
  \[
  |\psi_\alpha\rangle = (1/\sqrt{2}) \times (|\alpha_e\rangle \otimes |\alpha_o\rangle + |\alpha_o\rangle \otimes |\alpha_e\rangle).
  \]
Coarse-grained observables for BCHSH inequality

- **Choice of dichotomic observables:**
  \( O^k(\theta_j) \equiv V(\theta_j)^\dagger O^k_+ V(\theta_j) - V(\theta_j)^\dagger O^k_- V(\theta_j) \) with \( j \in \{ a, c \} \) for Alice and \( j \in \{ b, d \} \) for Bob.

- **Action of unitary**
  
  \( V(\theta_j) : \) \( V(\theta_j)|\alpha_e\rangle = \cos \theta_j|\alpha_e\rangle + \sin \theta_j|\alpha_o\rangle \),
  
  \( V(\theta_j)|\alpha_o\rangle = \sin \theta_j|\alpha_e\rangle - \cos \theta_j|\alpha_o\rangle \). For simplicity, we take:
  
  \( U(\theta) = V(\theta) \).

- **Choice of coarse-grained observables:**
  
  \( O_\delta(\theta_j) \equiv \sum_k P_\delta(k) O^k(\theta_j) \),
  
  \( O_\Delta(\theta_j; \theta_0) \equiv \int_0^{2\pi} P_\Delta(\theta_j - \theta_0) U(\theta_j)^\dagger O^0(\theta_0) U(\theta_j) d\theta_j \).

- **Expectation values:** \( E_{ab}(\delta) \equiv \langle \psi_\alpha | O_\delta(\theta_a) \otimes O_\delta(\theta_b) | \psi_\alpha \rangle \), etc.

- **Bell quantity:**
  
  \( B_\delta(\theta_a, \theta_b, \theta_c, \theta_d) \equiv E_{ab}(\delta) + E_{cb}(\delta) + E_{ad}(\delta) - E_{cd}(\delta) \).
The maximum values: $B_\delta$ and $B_\Delta(\theta_0)$

- $B_\delta = \left\{ \max_{\theta_a, \theta_b, \theta_c, \theta_d} \mathcal{F}(\theta_a, \theta_b, \theta_c, \theta_d) \right\} \times \left\{-1 + A + B - \frac{1}{4}(A + B)^2 \right\} + \frac{1}{2}(A - B)^2,$

- with $A \equiv 2C_e^2 \sum_{k=0}^{\infty} \sum_{n=0}^{k} P_\delta(k) \frac{(|\alpha|^2)^{2n}}{(2n)!}, \quad B \equiv 2C_o^2 \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} P_\delta(k) \frac{(|\alpha|^2)^{2n+1}}{(2n+1)!}, \quad C_e^2 = (\cosh |\alpha|^2)^{-1},$ and $C_o = (\sinh |\alpha|^2)^{-1}.$

- $\mathcal{F}(\theta_a, \theta_b, \theta_c, \theta_d) = \cos(2\theta_a + 2\theta_b) + \cos(2\theta_c + 2\theta_b) + \cos(2\theta_a + 2\theta_d) - \cos(2\theta_c + 2\theta_d)$ with max. being $2\sqrt{2}.$

- $B_\Delta(\theta_0) = B_\Delta = 2\sqrt{2}e^{-4\Delta^2}.$
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Coarse-graining of measurement and quantum-to-classical transition in the bipartite scenario
Post-measurement state for reference coarse-graining

- Given the measurement direction $\theta_a$, Alice performs measurement of the dichotomic observable $O^0(\theta_1)$ (chosen with apriori probability $P_{\Delta}(\theta_1 - \theta_a)$) on her system.

- Given the measurement direction $\theta_b$, Bob performs measurement of the dichotomic observable $O^0(\theta_2)$ (chosen with apriori probability $P_{\Delta}(\theta_2 - \theta_b)$) on his system.

- **Post-measurement state:** (w.r.t. $\{|\alpha_e\alpha_e\rangle, |\alpha_e\alpha_o\rangle, \ldots\}$)

$$\rho_{\text{ref}} = \begin{pmatrix}
a & b & c & d \\
b & \frac{1}{2} - a & d & -c \\
c & d & \frac{1}{2} - a & -b \\
d & -c & -b & a
\end{pmatrix}$$
Post-measurement state for reference coarse-graining (continued)

\[ a = \frac{1}{16} \left( 3 - e^{-8\Delta^2} \{ \cos(4\theta_a) + \cos(4\theta_b) \} - e^{-16\Delta^2} \cos(4\theta_a + 4\theta_b) \right) \]

\[ b = \frac{1}{16} \left( e^{-8\Delta^2} \{ \sin(4\theta_a) - \sin(4\theta_b) \} - e^{-16\Delta^2} \sin(4\theta_a + 4\theta_b) \right) \]

\[ c = \frac{1}{16} \left( e^{-8\Delta^2} \{ -\sin(4\theta_a) + \sin(4\theta_b) \} - e^{-16\Delta^2} \sin(4\theta_a + 4\theta_b) \right) \]

\[ d = \frac{1}{16} \left( 1 - e^{-8\Delta^2} \{ \cos(4\theta_a) + \cos(4\theta_b) \} + e^{-16\Delta^2} \cos(4\theta_a + 4\theta_b) \right) \]
$P$-distribution of $\rho_{\text{ref}}$:

- $P_{\text{ref}}(\beta, \gamma) = \sum_{i,j,k,l \in \{e,o\}} \rho_{i,j,k,l} P_{ij}(\beta) P_{kl}(\gamma)$

- $\rho_{i,j,k,l} = \text{Tr}(\rho_{\text{ref}} |i\rangle\langle j| \otimes |k\rangle\langle l|)$, $i, j, k, l \in \{e, o\}$

- $P_{ee}(\beta) = N_e^2 \left\{ 1 + e^{-2|\alpha|^2} \hat{A}(\alpha) \right\} \left[ \delta^{(2)}(\alpha - \beta) + \delta^{(2)}(\alpha + \beta) \right]$ is the $P$-distribution of $|\alpha_e\rangle\langle\alpha_e|$, etc.

- $\hat{A}(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (2\alpha)^n \left( \frac{\partial}{\partial \alpha} \right)^n$

- The $P$-distribution is highly singular function – no classical analogue.
Post-measurement state for resolution coarse-graining

- Given the measurement direction $\theta_a$, Alice performs measurement of the dichotomic observable $O^k(\theta_a)$ (chosen with apriori probability $P_\delta(k)$) on her system.

- Given the measurement direction $\theta_b$, Bob performs measurement of the dichotomic observable $O^m(\theta_b)$ (chosen with apriori probability $P_\delta(m)$) on his system.

- **Post-measurement state:** Difficult to get a closed form analytically.

- We fix our basis by choosing $(\theta_a, \theta_b) = (\pi/4, 3\pi/4)$.

- **Negativity of Wigner distribution:**

$$N_\rho = \frac{1}{2} \int_{\beta,\gamma \in \mathcal{C}} (|\mathcal{W}(\beta,\gamma)| - \mathcal{W}(\beta,\gamma)) \, d^2\beta \, d^2\gamma$$
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Wigner function $\mathcal{W}_{\psi_\alpha}(\beta, \gamma)$ of $|\psi_\alpha\rangle$ with $\gamma$ constant, $N_{\text{max}}$ is the maximum number of photons in each mode (color online).
Wigner function $\mathcal{W}(\beta;\gamma)$ of the post-measurement state of $|\psi_\alpha\rangle$ after sharp measurement with $O^4(\frac{\pi}{4}) \otimes O^4(\frac{3\pi}{4})$, $N_{\max}$ is the maximum number of photons in each mode.
Wigner function negativity of the post-measurement state of $|\psi_\alpha\rangle$ vs. the resolution coarse-graining parameter after measurement with $O_\delta(\frac{\pi}{4}, \frac{3\pi}{4})$ for different truncations $N$, with $\alpha = 2$. Neg. increases with $\delta$. 
Wigner function negativity of the post-measurement state of $|\psi_\alpha\rangle$ after sharp measurement with $O^k(\frac{\pi}{4}) \otimes O^m(\frac{3\pi}{4})$ for different values of $k$ and $m$. $N_{\psi_\alpha}$ has less negative value for $k=m=0$ compared to other values of $k$ and $m$. 
Case 2: On quantum-to-classical transition for

$$|\psi_N\rangle = \frac{1}{\sqrt{2}} \times (|0\rangle \otimes |N\rangle + |N\rangle \otimes |0\rangle)$$
Wigner function negativity of the post-measurement state of $|\psi_N\rangle$ after the reference coarse-grained measurement with $O_\Delta(\frac{13\pi}{100}, \frac{21\pi}{50})$ increases with the photon numbers $N$. 

$N_{\text{pref}}$ increases with the photon number $N$. 
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Case 3: On quantum-to-classical transition for
\[ |\psi_r\rangle = \left(\frac{1}{\sqrt{2}}\right) \times (|\psi^r_+\rangle \otimes |\psi^r_-\rangle + |\psi^r_-\rangle \otimes |\psi^r_+\rangle) \]
Introduction of single-mode non-classicality via squeezing

\[ |\psi_r\rangle = \frac{1}{\sqrt{2}} \left( |\psi^+_r\rangle|\psi^-_r\rangle + |\psi^-_r\rangle|\psi^+_r\rangle \right), \]

- \( |\psi^+_r\rangle \) and \( |\psi^-_r\rangle \) are two-photon-added and one-photon-added squeezed vacuum states, respectively.

- They are defined as:
  \[ |\psi^+_r\rangle \equiv \frac{1}{\cosh^2 r \sqrt{2 + \tanh^2 r}} (\hat{a}^\dagger)^2 S(r) |0\rangle, \]
  \[ |\psi^-_r\rangle \equiv \frac{1}{\cosh r} \hat{a}^\dagger S(r) |0\rangle. \]

- \( S(r) = \exp \left( \frac{r}{2} (\hat{a}^2 - (\hat{a}^\dagger)^2) \right) \) is the squeezing operator with real squeezing parameter \( r \).

- \( B_\Delta = 2\sqrt{2} e^{-4\Delta^2} \).
Coarse-graining of measurement and quantum-to-classical transition in the bipartite scenario

Wigner function negativity of the post-measurement state of $|\psi_r\rangle$ after the reference coarse-grained measurement $O_{\Delta\left(\frac{53\pi}{100},\frac{2\pi}{3}\right)}$. $N_{\rho_{ref}}$ decreases with the non-classicality parameter $r$. 

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For fixed $\Delta$, $N_{\text{pref}}$ does not always behave monotonically with respect to $r$. 

Wigner function negativity of the post-measurement state of $|\psi_r\rangle$ after the reference coarse-grained measurement $O_{\Delta}(\frac{\pi}{4}, \frac{3\pi}{4})$.
Discussion
Conclusion

- Even though the violation of any local realistic inequality does indicate some ‘non-classical’ behaviour of the bipartite state at hand, the non-violation does not guarantee any ‘classicality’ unless we can provide a local realistic model of the state.

- As the notion of non-classicality is a well-established feature in quantum optics, we looked at the P-distribution as well as the negativity of the Wigner function of the post-measurement state in the case of non-selective measurement involving both reference as well as resolution coarse-graining, by starting from some suitably chosen two-mode entangled states.
Contrary to general indication of quantum-to-classical transition via non-violation of the Bell-CHSH inequality, we found the presence of non-classicality in the post-measurement states (of certain classes of states) irrespective of whether we choose reference or resolution coarse-graining.

We hope that our study will help better understand the notion of quantum-to-classical transition.
Open questions

- To compare the three cases presented in this paper in a consistent way, it will be pertinent to study the behavior of non-classicality of the post-measurement state with respect to average photon number in the initial state \( \langle \psi_i | (N_1 \otimes I + I \otimes N_2) | \psi_i \rangle \), where, \( N_1 \) and \( N_2 \) are single mode photon number operators, for a fixed value of the coarse-graining parameter.

- We would like to determine the effect of rotations more general than what has been considered here for reference coarse-graining to study quantum-to-classical transition.
The present study can, in principle, be extended to multipartite systems.

We are also interested in relating our analysis with other approaches to quantum-to-classical transitions such as the one put forth in Gisin and his collaborators [Phys. Rev. Lett. 113, 090403 (2014)].
Perspectives on non-commutative phase-space extension

- Bell-type inequalities have been extended to CV systems using Wigner functions on (commutative) phase-space.
- Extension of Wigner function formalism has been made by Bastos et al. [JMP 49, 072101 (2008); CMP 299, 709 (2010)] in the case of non-commutative phase-space.
- Unfortunately, in the case of CV BCHSH inequality (i.e., with dichotomous observations), better non-locality is **not** revealed with non-commutative Wigner functions – so far as squeezed states are concerned [Bastos et al., PRD 93, 104055 (2016)].
- But there is still a lot of room to see the effect of phase-space non-commutativity for other types of Bell-type inequalities, particularly with infinite-valued measurements [e.g., Ketterer et al., PRA 91, 012106 (2015)].
To study quantum-to-classical transition in non-commutative phase-space setup – in the line of Quantum Optics – it may be necessary to develop the non-commutative version of Glauber-Sudarshan $P$-function, like in the case of Wigner function, mentioned above.

It is also useful to study the effect of phase-space non-commutativity on system dynamics in the context of quantum-to-classical transition.
THANK YOU!