New Results from $SU(2)$ and $SU(3)$ Gauge Matrix Models

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Current Developments in Quantum Field Theory and Gravity
SNBNCBS, Kolkata
3 December 2018

Introduction

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1. Pure Yang-Mills Theory
2. Quantization and Spectrum of YM Matrix Model
3. Variation Estimate of Energies
4. Comparison with Lattice Data
5. Including Quarks
6. Born-Oppenheimer Approximation
7. Fermion Energies
8. Quantum Phases of $SU(2)$ Yang-Mills-Dirac Theory

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Review of YM Theory

- What are the physical states of QCD?
- Wide implications: confinement, chiral symmetry breaking, color superconductivity, hadron masses, ...
- Recall that the $SU(N)$ Yang-Mills action is

$$S_{YM} = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- The gauge symmetry $A_\mu \mapsto uA_\mu u^{-1} + u\partial_\mu u^{-1}, \; u(x) \in SU(N)$ is actually a redundancy.
- The configuration space $\mathcal{C}$ = All gauge fields $\mathcal{A}$ modulo all gauge transformations $\mathcal{G}$.

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Why is Yang-Mills Difficult?

- Gauge symmetry: nonholonomic constraints.
- The configuration space $\mathcal{C}$ has non-trivial topology.
- Non-Abelian makes it non-linear: $[A_\mu, A_\nu]^2$ term.
- It is an infinite-dimensional dynamical system.

Gauge theory is difficult because of all the above!

Approximation by a simpler model? Many suggestions $\cdots$

- Chiral Lagrangians, Nambu-Jona-Lasinio model $\cdots$
- Lattice QCD: Discretize space-time, work with holonomies.
- String theory, AdS/CFT: approximate finite N by infinity.
- Perhaps other approaches, with their own successes/limitations.

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Another Approximation: Matrix Models

- Look at Yang-Mills on $S^3 \times \mathbb{R}$.
- Restrict to a subset $\mathcal{M}$ of gauge fields: keep only the left-invariant ones.
- Remarkably, these form a finite-dimensional space $M_{3,N^2-1}(\mathbb{R})$.
- Gauge group $\mathcal{G}$ is also now finite-dimensional: ad $SU(N)$.
- This approximation captures (some of) the constraints, nonlinearity, and underlying topology!
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The construction is simple and elegant (Narasimhan-Ramadas 1980):

- Start with the Maurer-Cartan form Ω of $SU(N)$.
- Pullback of Ω to to S^3 gives the left-invariant gauge field M_{ia}, $i = 1, 2, 3;\ a = 1, \cdots N^2 - 1$.
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- The $\mathcal{C}$ for pure $SU(N)$ is $M_{3,N^2-1}(\mathbb{R})/Ad\,SU(N)$.
- $\dim(\mathcal{C})$ is $3(N^2-1)-(N^2-1)=2(N^2-1)$ (not so at fixed points).
- Wavefunctions are sections of vector bundles on $\mathcal{C}$ that transform according to representations of $Ad\,SU(N)$.
- Those transforming according to the trivial representation are colorless, while those transforming nontrivially are coloured.

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Quantization of the Matrix Model

- The dynamical variables: M_{ia} and and p_{ia} (the Legendre transform of $\frac{dM_{ia}}{dt} = E_{ia}$).
- Quantisation: $[M_{ia}, p_{jb}] = i\delta_{ij}\delta_{ab}$.
- The Hamiltonian is

$$H = \frac{1}{R}\left(\frac{g^2 p_{ia}p_{ia}}{2} + B_{ia}B_{ia}\right) = \frac{1}{R}\left(-\frac{g^2}{2}\sum_{i,a}\frac{\partial^2}{\partial M_{ia}^2} + V(M)\right)$$

- The overall factor of R comes from dimensional analysis.
- The physical states $|\psi_{phys}\rangle$ are given by $G_a|\psi_{phys}\rangle = 0$.

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Energy Spectrum

- $H = H_0 + \frac{1}{R} V_{int}(M) = \frac{1}{R}\left(-\frac{1}{2}\frac{\partial^2}{\partial M_{ia}^2} + \frac{1}{2} M_{ia} M_{ia}\right) + \frac{1}{R}\left(-\frac{g}{2}\epsilon_{ijk} f_{abc} M_{ia} M_{jb} M_{kc} + \frac{g^2}{4} f_{abc} f_{ade} M_{ib} M_{jc} M_{id} M_{je}\right)$
- Perturbation theory is not analytic at $g = 0$.
- We estimate the energies by variational calculation instead.
- Choose colorless eigenstates of H_0 as trial wavefunctions, organized by this spin.
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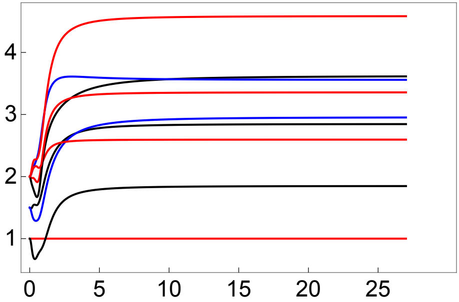
Energy Difference Ratios

- Remarkably, we find that the ratios of energy differences become independent of g for large g.

Ratios of mass differences $\frac{\mathcal{E}(X)-\mathcal{E}(0^+)}{\mathcal{E}(2^+)-\mathcal{E}(0^+)}$ as a function of g. (The black, blue and red curves represent spin-0, spin-1 and spin-2 levels respectively.)



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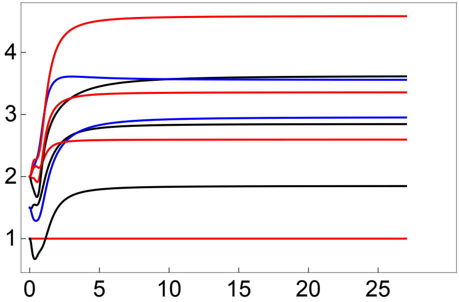

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Renormalization Group Equation

- Neither R nor the bare coupling g are directly measurable.
- For sensible results as $R \to \infty$, make g a function of R such that all energies have well-defined values in this limit.
- Make $g = g(R)$ by fixing $\mathcal{E}_0[2] - \mathcal{E}_0[0]$ to the observed (lattice) value.
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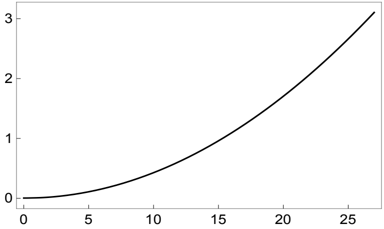
Integrated Renormalization Group Equation

- In practice it is easier to make $R(g) = \frac{\mathcal{E}_0[2]-\mathcal{E}_0[0]}{m(2^+)-m(0^+)}$.

R(*g*) versus *g*.



- Here we have used $m(2^+) - m(0^+) = 460$ MeV.

- Actual numerical values of masses also need asymptotic $c(R)/R$.
- Fix the physical mass of our lowest glueball to be within the range predicted by lattice simulations (1580 – 1840 MeV).
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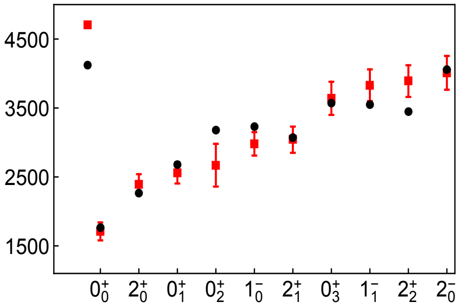
Glueball states J^C	Physical masses from matrix model (MeV)	Physical masses from lattice QCD (MeV)
0^+	$1757.08^\dagger$	1580 - 1840
2^+	$2257.08^\dagger$	2240 - 2540
0^+	2681.45	2405 - 2715
0^{*+}	3180.82	2360 - 2980
1^-	3235.41	2810 - 3150
2^+	3054.97	2850 - 3230
0^{*+}	3568.02	3400 - 3880
1^-	3535.66	3600 - 4060
2^{*+}	3435.75	3660 - 4120
2^-	4050.14	3765 - 4255

$^\dagger \equiv$ (input)

Glueball Masses (MeV)



■ ≡ Lattice • ≡ Matrix Model. 0^{++} and 2^{++} are used in Matrix Model input.

For 0^{*++}, lattice has poor statistics near the continuum limit, so finite volume effects are substantial.

For 2^{*++}, lattice has large errors due to the presence of two other glueball states in the vicinity.

THESE ASYMPTOTIC VALUES AGREE WELL WITH LATTICE PREDICTIONS FOR GLUEBALL MASSES.

Adding Fermions

- But you ask: **What about the quarks?**
- We will consider massless fundamental fermions (quarks!) coupled to the $SU(2)$ matrix model.
- The fundamental fermion $\lambda_{\alpha a} \equiv \lambda_A$ couples to the gauge field via

$$H^{ff} \equiv \left(-(\lambda_A)^\dagger \lambda_A - \frac{1}{2}(\tau_b)_{AC}(\lambda_A)^\dagger \sigma_i \lambda_C M_{ib} \right) = (\lambda_A)^\dagger \mathcal{H}^{ff}_{AB} \lambda_B.$$

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- The total Hamiltonian is $H = H_{YM} + H^{ff}$.
- Solve $H\psi^E = E\psi^E$.
- We look at $g << 1$, but rather than do perturbation theory, quantize in two steps:
- First treat the gauge field as a (background) fixed field and quantize the fermions.
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- g_{IJ} is a Riemannian metric, a measure of distance between pure states represented by projectors $P(x_I)$ and $P(x_I + dx_I)$.
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- Φ (or g_{IJ}) is used to hunt for quantum phase transitions (QPTs), as the latter often defy the standard Landau-Ginzburg paradigm.

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- Φ (or g_{IJ}) is used to hunt for quantum phase transitions (QPTs), as the latter often defy the standard Landau-Ginzburg paradigm.

Φ for YM fermions

- We will compute Φ for fundamental fermions coupled to the Yang-Mills field M_{ia}.
- For the 1-fermion states $|\psi^{(1)}\rangle = \sum_A c_A(M)(\lambda_A)^\dagger|0\rangle$, the equation $H^{ff}|\psi^{(1)}\rangle = E|\psi^{(1)}\rangle$ becomes:

$$\mathcal{H}^{ff}_{AB} c_B = E c_A, \quad \mathcal{H}^{ff} = -\frac{1}{2}\sigma_i \otimes \tau_a M_{ia}$$

- We therefore investigate

$$\det(\mathcal{H}^{ff}_{AB} - \lambda \mathbb{I}) = 0,$$

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Fundamental Fermions

- The characteristic equation (with $x = \frac{E}{(\frac{1}{3}\,\mathrm{Tr}\,M^T M)^{1/2}}$) is

$$x^4 - \frac{3}{2}x^2 - \mathbf{g}x + \mathbf{h} = 0$$

where

$$\mathbf{g} \equiv \frac{\det M}{\left(\frac{1}{3}\mathrm{Tr}(M^T M)\right)^{3/2}}, \qquad \mathbf{h} \equiv \frac{1}{16}\left[\frac{2\mathrm{Tr}(M^T M)^2}{\left(\frac{1}{3}\mathrm{Tr}(M^T M)\right)^2} - 9\right].$$

- Since $\mathcal{H}^{ff}$ is manifestly Hermitian, it has only real roots.
- The conditions for this come from Sylvester's theorem: one condition is that the discriminant Δ of $x^4 - \frac{3}{2}x^2 - \mathbf{g}x + \mathbf{h}$ must be non-negative.
- This gives us an unexpected identity obeyed by 3×3 real matrices:

$$27\mathbf{g}^2 - 54\mathbf{g}^4 + 162\mathbf{h} - 432\mathbf{g}^2 h - 576\mathbf{h}^2 + 512\mathbf{h}^3 \geq 0$$

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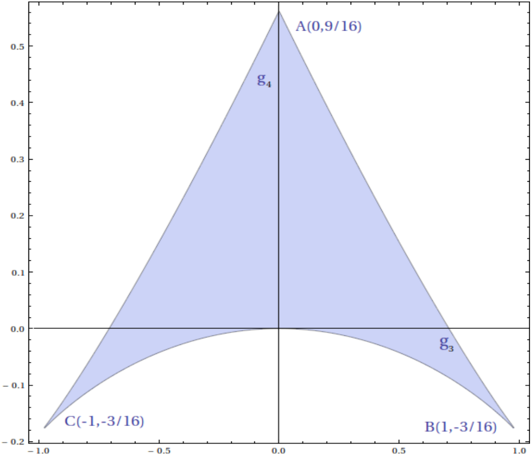
- At the top corner, the degeneracy structure is $(2, 2)$.
- At the two corners at the bottom, the degeneracy structure is $(3, 1)$.



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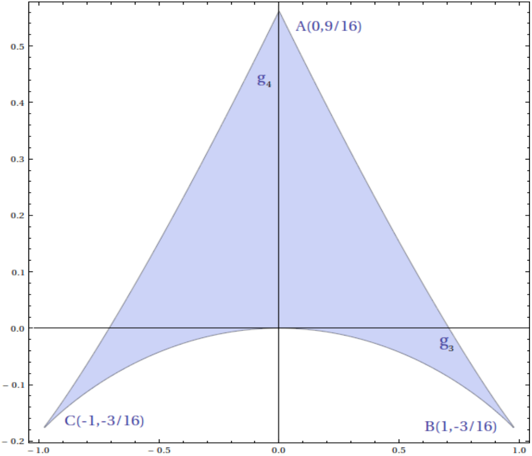

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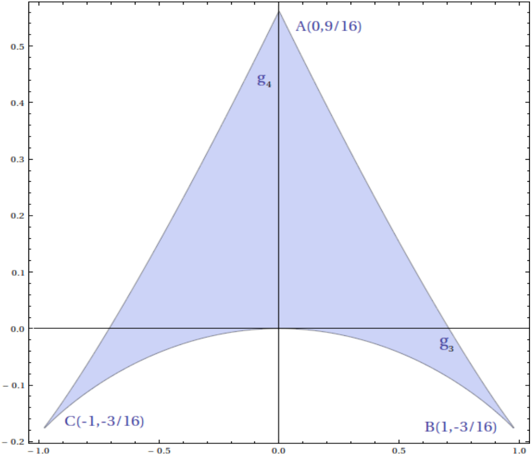

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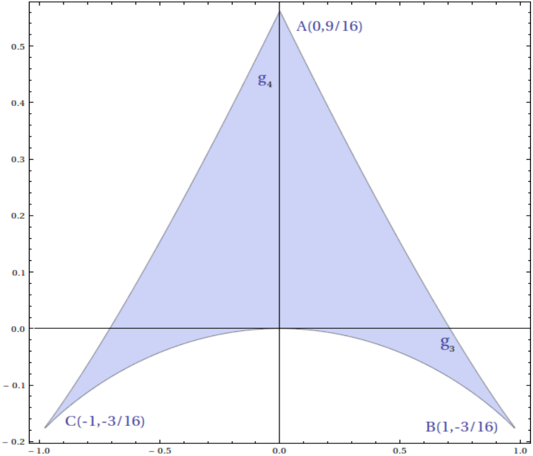

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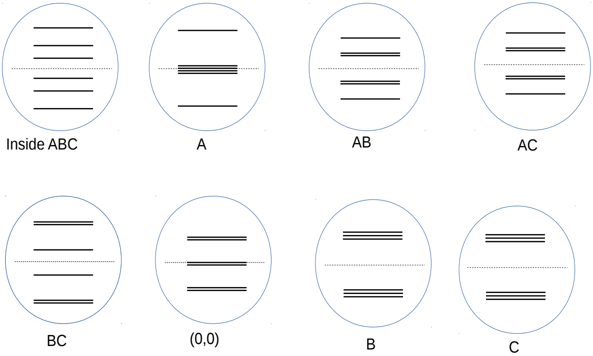
2 Weyl fermions



- The characteristic polynomial of the two fermion Hamiltonian is

$$P_2(x) = x^6 - 3x^4 + 4x^2\left(\frac{9}{16} - \mathbf{h}\right) - \mathbf{g}^2 = 0$$

- This gives us the effective potential

$$\Phi^{(2)}_{bulk} = \frac{6}{\mathbf{f}^2}\frac{-x_1^6 + 5x_1^4 + 4(9/16 - \mathbf{h})(1 - 7x_1^2/3)}{(3x_1^4 - 6x_1^2 + 4(9/16 - \mathbf{h}))^2}, \quad \mathbf{f}^2 = \frac{1}{3}\mathrm{Tr}M^T M.$$

where $x_1(\mathbf{g}, \mathbf{h})$ is the smallest root of P_2.

- The ground state degeneracy changes from 1 to 2 at the edge BC, and to 3 at the corner B. At the edge BC:

$$\Phi^{(2)}_{edge} = \frac{2}{9\mathbf{f}^2}\frac{9 - 6x_1^2 + 5x_1^4}{x_1^2(1 - x_1^2)^2} \to \frac{2}{9a^2}\frac{1}{(1 + x_1)^2}$$

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- We see that the Hilbert space for gauge dynamics has split into 3 regions:
- Inside the bulk, it is governed by $\Phi^{(2)}_{bulk}$, which diverges as we approach the edge BC or the corner B.
- On the edge BC, the dynamics is governed by $\Phi^{(2)}_{edge}$, which diverges as we approach the corner B.
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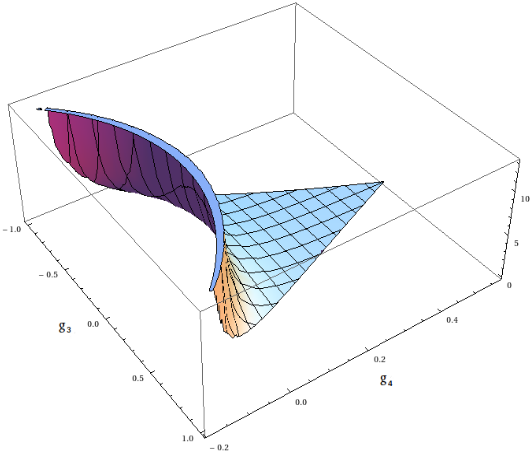
Scalar potential for 2 Weyl fermions



- There are therefore three distinct phases of $SU(2)$ gauge theory (with Weyl fermions).
- These are superselected: states in one phase cannot be obtained as superpositions of states from other sectors.
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Ongoing Work and Outlook

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- Include fermions (quarks), and try to get the masses of light hadrons.
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This is joint work with

- Nirmalendu Acharyya, AP Balachandran, Mahul Pandey, Sambuddha Sanyal, G. Mohankarthik
- Lattice data is taken from
  Morningstar and Peardon, Phys. Rev D **56**, 4043 (1997);
  Chen *et al* Phys. Rev D. **73** 014516 (2006).

- $$A_{ia} = \frac{1}{\sqrt{2}}\left(M_{ia} + \frac{\partial}{\partial M_{ia}}\right), \quad A^\dagger_{ia} = \frac{1}{\sqrt{2}}\left(M_{ia} - \frac{\partial}{\partial M_{ia}}\right) \implies [A_{ia}, A^\dagger_{jb}] = \delta_{ia}\delta_{jb}$$
- The oscillator vacuum is $\langle M|0\rangle = \frac{1}{\pi^6} e^{-\frac{Tr(M^T M)}{2}}$
- Spin-0:

$$
\begin{aligned}
|\psi^0_1\rangle &= |0\rangle\\
|\psi^0_2\rangle &= A^\dagger_{ia}A^\dagger_{ia}|0\rangle\\
|\psi^0_3\rangle &= \epsilon_{ijk} f_{abc} A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{kc}|0\rangle\\
|\psi^0_4\rangle &= A^\dagger_{ia}A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{jb}|0\rangle\\
|\psi^0_5\rangle &= A^\dagger_{ia}A^\dagger_{ib}A^\dagger_{ja}A^\dagger_{jb}|0\rangle\\
|\psi^0_6\rangle &= d_{abe}d_{cde}A^\dagger_{ia}A^\dagger_{ib}A^\dagger_{jc}A^\dagger_{jd}|0\rangle\\
|\psi^0_7\rangle &= \epsilon_{ijk} f_{abc} A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{kc}A^\dagger_{ld}A^\dagger_{ld}|0\rangle\\
|\psi^0_8\rangle &= \epsilon_{ijk} f_{abc} d_{a_1 b_1 e} d_{a_2 c e} A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{ka_1}A^\dagger_{lb_1}A^\dagger_{la_2}|0\rangle\\
|\psi^0_9\rangle &= A^\dagger_{ia}A^\dagger_{ia}A^\dagger_{jb}A^\dagger_{jb}A^\dagger_{kc}A^\dagger_{kc}|0\rangle\\
|\psi^0_{10}\rangle &= A^\dagger_{ia}A^\dagger_{ib}A^\dagger_{jb}A^\dagger_{jc}A^\dagger_{kc}A^\dagger_{ka}|0\rangle\\
|\psi^0_{11}\rangle &= \epsilon_{ijk}\epsilon_{lmn}A^\dagger_{ia}A^\dagger_{la}A^\dagger_{jb}A^\dagger_{mb}A^\dagger_{kc}A^\dagger_{nc}|0\rangle\\
|\psi^0_{12}\rangle &= \epsilon_{i_1 j_1 k_1} f_{a_1 b_1 c_1}\epsilon_{i_2 j_2 k_2} f_{a_2 b_2 c_2} A^\dagger_{i_1 a_1}A^\dagger_{j_1 b_1}A^\dagger_{k_1 c_1}A^\dagger_{i_2 a_2}A^\dagger_{j_2 b_2}A^\dagger_{k_2 c_2}|0\rangle\\
|\psi^0_{13}\rangle &= d_{abc}d_{def}A^\dagger_{ia}A^\dagger_{id}A^\dagger_{jb}A^\dagger_{je}A^\dagger_{kc}A^\dagger_{kf}|0\rangle\\
|\psi^0_{14}\rangle &= d_{b_1 c_1 d}d_{b_2 c_2 d}A^\dagger_{ia}A^\dagger_{ia}A^\dagger_{jb_1}A^\dagger_{jc_1}A^\dagger_{kb_2}A^\dagger_{kc_2}|0\rangle\\
|\psi^0_{15}\rangle &= \epsilon_{i_1 j_1 k_1} f_{a_1 b_1 c_1}\epsilon_{i_2 j_2 k_2} f_{a_2 b_2 c_2} d_{c_1 d_1 e} d_{c_2 d_2 e} A^\dagger_{i_1 a_1}A^\dagger_{j_1 b_1}A^\dagger_{k_1 d_1}A^\dagger_{i_2 a_2}A^\dagger_{j_2 b_2}A^\dagger_{k_2 d_2}|0\rangle\\
|\psi^0_{16}\rangle &= d_{abc}d_{ad_1 e_1}d_{ad_2 e_2}d_{ad_3 e_3}A^\dagger_{id_1}A^\dagger_{ie_1}A^\dagger_{jd_2}A^\dagger_{je_2}A^\dagger_{kd_3}A^\dagger_{ke_3}|0\rangle
\end{aligned}
$$

- f_{abc} and d_{abc} are the structure constants of $SU(3)$.

Spin-1

$$
\begin{aligned}
|\psi_1^1\rangle &= d_{abc} A^\dagger_{jb} A^\dagger_{jc} A^\dagger_{ia} |0\rangle \\
|\psi_2^1\rangle &= \epsilon_{jkl} d_{ab_1c_1} f_{ab_2c_2} A^\dagger_{ib_1} A^\dagger_{jc_1} A^\dagger_{kb_2} A^\dagger_{lc_2} |0\rangle \\
|\psi_3^1\rangle &= d_{ace} A^\dagger_{ia} A^\dagger_{jb} A^\dagger_{jb} A^\dagger_{kc} A^\dagger_{ke} |0\rangle \\
|\psi_4^1\rangle &= d_{ace} A^\dagger_{ib} A^\dagger_{jb} A^\dagger_{ja} A^\dagger_{kc} A^\dagger_{ke} |0\rangle \\
|\psi_5^1\rangle &= d_{ace} A^\dagger_{ia} A^\dagger_{jb} A^\dagger_{jc} A^\dagger_{ke} A^\dagger_{kb} |0\rangle \\
|\psi_6^1\rangle &= d_{abc} f_{bc_1b_2} f_{cc_2b_1} A^\dagger_{ia} A^\dagger_{jb_1} A^\dagger_{jc_1} A^\dagger_{kb_2} A^\dagger_{kc_2} |0\rangle \\
|\psi_7^1\rangle &= \epsilon_{jkl} d_{abc} f_{ade} A^\dagger_{ib} A^\dagger_{jc} A^\dagger_{kd} A^\dagger_{le} A^\dagger_{i_1a_1} A^\dagger_{i_1a_1} |0\rangle \\
|\psi_8^1\rangle &= \epsilon_{jkl} d_{ab_1c_1} f_{aa_2b_2} A^\dagger_{ia_1} A^\dagger_{i_1a_1} A^\dagger_{i_1b_1} A^\dagger_{jc_1} A^\dagger_{ka_2} A^\dagger_{lb_2} |0\rangle \\
|\psi_9^1\rangle &= \epsilon_{ijk} d_{ab_1c_1} d_{aa_2b_2} A^\dagger_{ja_1} A^\dagger_{i_1a_1} A^\dagger_{i_1b_1} A^\dagger_{kc_1} A^\dagger_{la_2} A^\dagger_{lb_2} |0\rangle \\
|\psi_{10}^1\rangle &= \epsilon_{ijk} d_{ab_1c_1} f_{bb_2c_2} A^\dagger_{i_1b_1} A^\dagger_{i_1c_1} A^\dagger_{la} A^\dagger_{lb} A^\dagger_{jb_2} A^\dagger_{kc_2} |0\rangle
\end{aligned}
$$

Spin-2

$$
\begin{aligned}
|\psi^2_1\rangle &= (A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_2\rangle &= A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_1}(A^\dagger_{ia_2}A^\dagger_{ja_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{i_2 a_2}A^\dagger_{j_2 a_2})|0\rangle \\
|\psi^2_3\rangle &= (A^\dagger_{ia_1}A^\dagger_{i_1 a_1}A^\dagger_{i_1 b_1}A^\dagger_{jb_1} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la_1}A^\dagger_{i_1 a_1}A^\dagger_{i_1 b_1}A^\dagger_{lb_1})|0\rangle \\
|\psi^2_4\rangle &= d_{abc}d_{ade}A^\dagger_{i_1 b}A^\dagger_{i_1 c}(A^\dagger_{id}A^\dagger_{je} - \tfrac{1}{3}\delta_{ij}A^\dagger_{ld}A^\dagger_{le})|0\rangle \\
|\psi^2_5\rangle &= A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_1}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_6\rangle &= \tfrac{1}{2}d_{abc}(\epsilon_{ikl}A^\dagger_{ja_1}A^\dagger_{ka_1} + \epsilon_{jkl}A^\dagger_{ia_1}A^\dagger_{ka_1})A^\dagger_{la}A^\dagger_{mb}A^\dagger_{mc}|0\rangle \\
|\psi^2_7\rangle &= \tfrac{1}{2}d_{abc}(\epsilon_{ikl}A^\dagger_{ja} + \epsilon_{jkl}A^\dagger_{ia})A^\dagger_{kb}A^\dagger_{la_1}A^\dagger_{ma_1}A^\dagger_{mc}|0\rangle \\
|\psi^2_8\rangle &= \epsilon_{klm}f_{abc}d_{da_1 a}d_{da_2 b_2}A^\dagger_{ka_1}A^\dagger_{lb}A^\dagger_{mc}(A^\dagger_{ia_2}A^\dagger_{jb_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{i_2 a_2}A^\dagger_{i_2 b_2})|0\rangle \\
|\psi^2_9\rangle &= A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_1}A^\dagger_{i_2 a_2}A^\dagger_{i_2 a_2}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_{10}\rangle &= A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_1}A^\dagger_{i_2 a_2}A^\dagger_{i_2 a_1}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_{11}\rangle &= d_{ab_1 c_1}d_{ab_2 c_2}A^\dagger_{i_1 b_1}A^\dagger_{i_1 c_1}A^\dagger_{i_2 b_2}A^\dagger_{i_2 c_2}(A^\dagger_{ia}A^\dagger_{ja} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{la})|0\rangle \\
|\psi^2_{12}\rangle &= A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_1}(A^\dagger_{ia_2}A^\dagger_{i_2 a_2}A^\dagger_{i_2 b_2}A^\dagger_{jb_2} - A^\dagger_{la_2}A^\dagger_{i_2 a_2}A^\dagger_{i_2 b_2}A^\dagger_{lb_2})|0\rangle \\
|\psi^2_{13}\rangle &= d_{aa_2 b_2}d_{ac_2 e_2}A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_1}A^\dagger_{i_2 a_2}A^\dagger_{i_2 b_2}(A^\dagger_{ic_2}A^\dagger_{jd_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{lc_2}A^\dagger_{ld_2})|0\rangle \\
|\psi^2_{14}\rangle &= \tfrac{1}{2}(\epsilon_{ikl}A^\dagger_{jb}A^\dagger_{kb} + \epsilon_{jkl}A^\dagger_{ib}A^\dagger_{kb})\epsilon_{mnp}d_{ab_1 c_1}f_{bb_2 c_2}A^\dagger_{lb_1}A^\dagger_{mc_1}A^\dagger_{nb_2}A^\dagger_{pc_2}|0\rangle \\
|\psi^2_{15}\rangle &= d_{ab_1 c_1}d_{ab_2 c_2}A^\dagger_{lb_1}A^\dagger_{lc_1}A^\dagger_{mb_2}A^\dagger_{mc_2}(\tfrac{1}{2}(A^\dagger_{ia}A^\dagger_{jb} + A^\dagger_{ja}A^\dagger_{ib}) - \tfrac{1}{3}\delta_{ij}A^\dagger_{la_2}A^\dagger_{lc_2})|0\rangle \\
|\psi^2_{16}\rangle &= d_{ab_1 c_1}d_{bb_2 c_2}A^\dagger_{i_1 a}A^\dagger_{i_1 b}A^\dagger_{j_1 b_1}A^\dagger_{j_1 c_1}(A^\dagger_{ia}A^\dagger_{jb} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{lb})|0\rangle \\
|\psi^2_{17}\rangle &= d_{aa_2 b_2}d_{bc_2 a_1}A^\dagger_{i_1 a_1}A^\dagger_{i_1 a_2}A^\dagger_{j_1 b_2}A^\dagger_{j_1 c_2}(A^\dagger_{ia}A^\dagger_{jb} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la}A^\dagger_{lb})|0\rangle \\
|\psi^2_{18}\rangle &= d_{ab_1 c_1}d_{aa_2 b_2}f_{bb_2 c_2}A^\dagger_{i_1 b_1}A^\dagger_{i_1 c_1}A^\dagger_{i_2 c_2}A^\dagger_{i_2 d_2}(A^\dagger_{ia_2}A^\dagger_{je_2} - \tfrac{1}{3}\delta_{ij}A^\dagger_{la_2}A^\dagger_{le_2})|0\rangle
\end{aligned}
$$

New Identities

We discovered some (new?) identities involving 3×8 matrices:

$$\begin{aligned}\mathrm{Tr}(M^T M D_a M^T M D_a) &= -\frac{1}{2}\mathrm{Tr}(M^T M D_a)\,\mathrm{Tr}(M^T M D_a) \\ &+ \frac{2}{3}\mathrm{Tr}(M^T M M^T M) + \frac{1}{3}\mathrm{Tr}(M^T M)^2 \\ \epsilon_{ijk} f_{abc} M_{ia} M_{jb} (M M^T M)_{kc} &= \frac{1}{3}\epsilon_{ijk} f_{abc} M_{ia} M_{jb} M_{kc}\,\mathrm{Tr}(M^T M)\end{aligned}$$

where $(D_a)_{bc} \equiv d_{abc}$.