The Role of Raychaudhuri equation in a Self Similar Collapse

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Overview

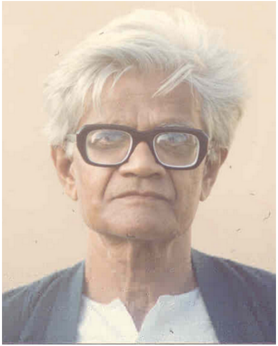
1. Raychaudhuri equation
2. Metric and Self-similarity
3. Analysis of the behavior
4. Approximate analytic study

Raychaudhuri equation for a timelike congruence having four velocity u^μ is given by,

[A. K. Raychaudhuri (1955), J. Ehlers (1993)]

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 + \nabla_\mu a^\mu - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu$$

$\theta = \nabla_\mu u^\mu$	expansion scalar
$\sigma_{\mu\nu} = \nabla_{(\nu}u_{\mu)} - \frac{1}{3}h_{\mu\nu}\theta + a_{(\nu}u_{\mu)}$	shear tensor
$\omega_{\mu\nu} = \nabla_{[\nu}u_{\mu]} - a_{[\nu}u_{\mu]}$	rotation tensor
$a_\mu = u^\nu\nabla_\nu u_\mu$	four acceleration
$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$	spatial metric



Raychaudhuri equation

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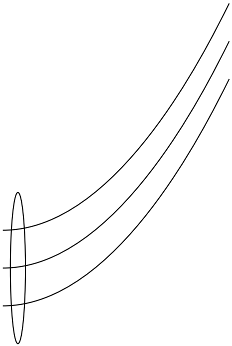

Figure : Area enclosing a congruence

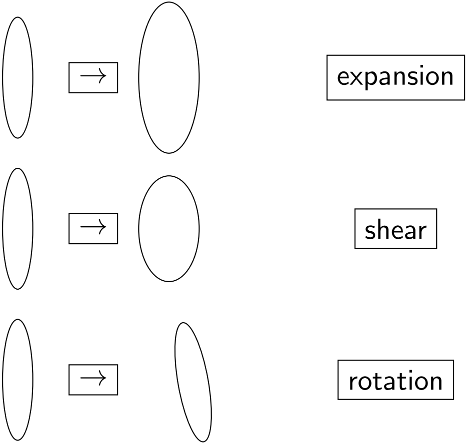

Figure : Expansion, Shear and Rotation

[**S. Kar & S. Sengupta (2007)**]

Metric and Self-Similarity

Metric and Self-Similarity

- The metric is assumed to be conformally flat -

$$ds^2 = \frac{1}{A(r,t)^2}\left[- dt^2 + dr^2 + r^2 d\Omega^2 \right],$$

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- The energy-momentum tensor is that of a massless scalar -

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- A spacetime is assumed to be self-similar such that,

  [**A. H. Cahill & M. E. Taub, (1971)**],
  [**B.J. Carr & A.A. Coley, (1999)**]

,

$$A(r,t) = B(z), \;\; \phi(r,t) = \phi(z), \;\; z = {}^{t}/_{r}$$

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Analysis of the behavior

- For a comoving observer- $u^\alpha = A\delta_0^\alpha$, with the energy-momentum tensor of a massless scalar field, Raychaudhuri equation can be written as,

$$\frac{d\theta}{d\lambda} + \frac{1}{3}\theta^2 + \left(-3A'^2 + \frac{2AA'}{r} + AA''\right) = A^2\dot{\phi}^2$$

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- Writing $A(r,t) = B(z)$, where $z = {}^{t}/_{r}$, for self similar case, the above condition becomes,

$$\left(\frac{d\phi}{dz}\right)^2 \geq \frac{3z^2}{B^2}\left(\frac{dB}{dz}\right)^2 - \frac{z^2}{B}\frac{d^2B}{dz^2}$$

Analysis of the behavior

- Using information from field equations, the condition for focusing can be written as,

$$\frac{1}{B}\frac{d^2B}{dz^2} + P(z)\frac{1}{B^2}\left(\frac{dB}{dz}\right)^2 + Q(z)\frac{1}{B}\frac{dB}{dz} \begin{cases} \geq 0 & \text{for} \quad 0 < z < 1 \\ \leq 0 & \text{for} \quad 1 < z < 2 \\ \geq 0 & \text{for} \quad z > 2 \end{cases}$$

$$P(z) = \frac{3z^3 - 3z^2 - 6z - 6}{4 + z^2 - z^3}, \quad Q(z) = \frac{4(2z-1)}{(1-z)(4+z^2-z^3)}$$

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- This condition is studied for high z values which is chosen in the limit $t \rightarrow \mathrm{finite}$ and $r \rightarrow 0$ (Central Singularity).

Analysis of the behavior

- For high values of z, the condition for focusing can be approximated as,

$$\left(\frac{3}{z} + \frac{8}{z^3}\right) \frac{1}{B} \frac{dB}{dz} \geq 0$$

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- Thus, for focusing $\frac{dB}{dz}$ has to be non-negative.
- The situation is further analyzed by solving the condition -

$$\frac{1}{B}\frac{d^2B}{dz^2} + P(z)\frac{1}{B^2}\left(\frac{dB}{dz}\right)^2 + Q(z)\frac{1}{B}\frac{dB}{dz} \geq 0,$$

for the limiting case.

Analysis of the behavior

- Variation of $\frac{dB(z)}{dz}$ with a positive initial $\frac{dB}{dz}$:

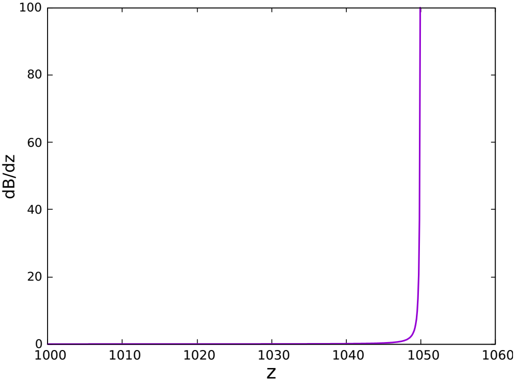

Figure : Plot of $\frac{dB}{dz}$ with z $(B_0 = 1, \left(\frac{dB}{dz}\right)_0 = 0.01)$

Analysis of the behavior

- Variation of $B(z)$ with a positive initial $\frac{dB}{dz}$:

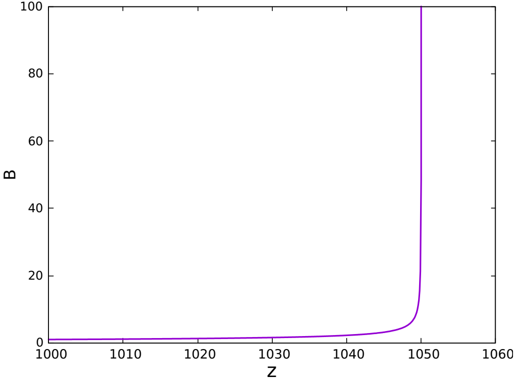

Figure : Plot of B with z $(B_0 = 1, \left(\frac{dB}{dz}\right)_0 = 0.01)$

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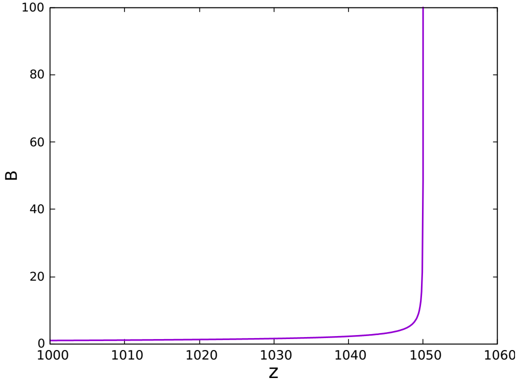

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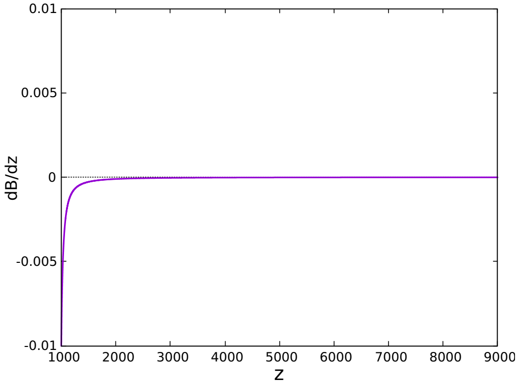
- For this kind of initial condition the scalar field collapses and a zero proper volume singularity forms.

Analysis of the behavior

- Variation of $\frac{dB(z)}{dz}$ with a negative initial $\frac{dB}{dz}$:



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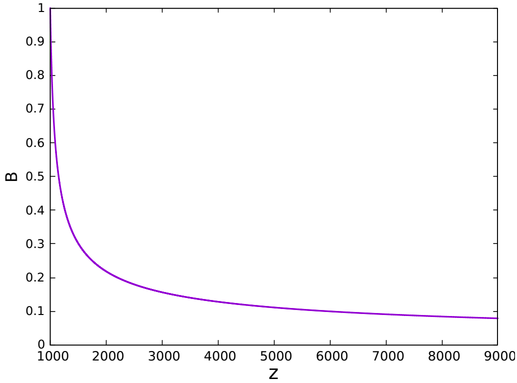

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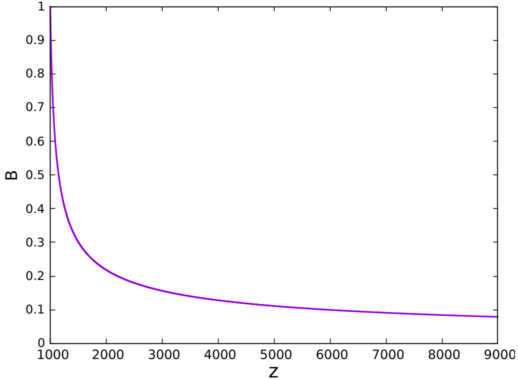

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- The solution is non-collapsing for this kind of initial condition.

Approximate analytic study

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- The condition which ensures focusing is -

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- If $u_{\rm eq}$ and $B_{\rm eq}$ are the solutions for $\frac{du}{dz} = 2u^2$,

$$\frac{1}{u_{\rm eq}} = C - 2z, \quad B_{\rm eq} = \frac{D}{\sqrt{|C-2z|}}, \quad R_{\rm eq}(r,t) = \frac{r^{\frac{1}{2}}|Cr-2t|^{\frac{1}{2}}}{D}.$$

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- If initially $u = u_{\rm eq}$ and $u, u_{\rm eq} > 0$, $u \geq u_{\rm eq}$ always. Thus $u_{\rm eq} \to \infty$ ensures $u \to \infty$. Similarly, $B \geq B_{\rm eq}$ always and the solution is collapsing.

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- For second set of initial data $u, u_{\mathrm{eq}} < 0$ initially. Thus $u \leq u_{\mathrm{eq}}$ and $B \leq B_{\mathrm{eq}}$ always.
- It should be noted that the condition for singularity formation has already been established using Raychaudhuri equation as,

$$\frac{dB}{dz} \geq 0$$

Approximate analytic study

- For the collapsing solution, few more investigations are performed using,

$$B = B_{\rm eq} = \frac{B_0}{\sqrt{C - 2z}}$$

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- Singularity forms at $z = {}^{C}\!/_{2}$. Thus, innermost shells become singular before the outer ones.
- Using the condition for shell crossing singularity,

  [**P. Yodzis et al. (1973)**, **B. Waugh & K. Lake (1986)**]

$$\frac{\partial g_{\theta\theta}}{\partial r} = 0, \qquad g_{\theta\theta} \neq 0, \qquad r > 0$$

  it is proved that this is not a shell crossing singularity, rather it is shell focusing.

Approximate analytic study

- The study of apparent horizon ($g^{\mu\nu} R_{,\mu} R_{,\nu} = 0\,,\ R = {}^{r}\!/_{A(r,t)}$) shows that the singularity forms at a smaller z value than the apparent horizon which is the signature of a Naked singularity.

Approximate analytic study

- The study of apparent horizon ($g^{\mu\nu} R_{,\mu} R_{,\nu} = 0\,,\ R = {}^{r}/_{A(r,t)}$) shows that the singularity forms at a smaller z value than the apparent horizon which is the signature of a Naked singularity.
- The expression for the scalar field is given by,

$$\phi = \ln\left[\frac{F}{(C-2z)^{E/2}}\right] \ \text{for} \ \ C > 2z$$
$$\phi = \ln\left[F\,(2z-C)^{E/2}\right] \ \text{for} \ \ C < 2z$$

where $E > 0$.

Conclusions and Future Work

- We have studied the evolution of conformally flat self similar spacetime minimally coupled to a massless scalar field using Raychaudhuri equation.
- We have found the dependence of the evolution on the different type of initial conditions which may give rise to collapsing or regular solution.
- Different properties and nature of the final outcome have been analyzed for the collapsing case.
- Using this approach useful information about the spacetime can be extracted without explicitly solving the field equations.
- Our immediate future goal is to consider the massive scalar field case.
- We want to analyze more general spacetimes using this approach to see its usefulness.

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Thank You