Noncommutative Fluid Dynamics and its Possible Cosmological Consequences

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Collaborators

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- The talk is based on the papers:
  P. Das and S.G.; PRD (2018); PRD(Rap.Com.) (2017); EPJC (2016) , Erratum: EPJC (2017) .
  A. K. Mitra, R. Banerjee, S. G., JCAP (2018); IJMPA (2017); EPJC (2015).

- A very exhaustive review of fluid dynamics from high energy physics perspective (that we have followed closely):
  R. Jackiw, V.P. Nair, S.-Y. Pi, A.P. Polychronakos, J.Phys. A37 (2004) R327-R432 .

Plan of talk

- Introduction
- Lagrangian and Hamiltonian formulation of canonical fluid
- Noncommutative generalization of fluid
- Dynamics, symmetries and conservation laws for Noncommutative fluid
- Action formulation of Noncommutative fluid
- Multiple Central Extension in Noncommutative fluid Galilean algebra
- Cosmological consequences of Noncommutative fluid
- Conclusion

Introduction

- Success of fluid models: at sufficiently high energy densities local equilibrium prevails in an interacting theory $\rightarrow$ local inhomogeneities are smoothed out $\rightarrow$ Discrete (Lagrangian) to continuous (Eulerian) fluid variables.
  It applies at microscopic scales (liquid drop model in early nuclear physics and quark-gluon-plasma produced RHIC/LHC) and at macroscopic scales (generic fluid models in cosmology).
- Topical areas of interest: (i) fluid-gravity correspondence
  (ii) cosmology
- Fluid dynamics (at least for simplest ideal (no viscosity) non-relativistic fluid)):
  (i) continuity equation
  (ii) Euler force equation
- In this talk we develop a generalized fluid dynamics model in Non-Commutative (NC) space and show its possible impact via cosmological perturbations.
  NC fluid model is constructed from first principles, based on the map between the Lagrangian and Hamiltonian (Euler) formulation of fluid dynamics (Jackiw et.al, 2004).

- Noncommutative (NC) effects in physics
  Reviews: M.R.Douglas and N.A.Nekrasov, Rev. Mod. Phys. 2001; R. J. Szabo, Phys. Rep. 2003; R. Banerjee, B. Chakraborty, S. G., P. Mukherjee, S. Samanta, Found.Phys. 2009 .....
  Generic NC extension:
- In NC quantum mechanics it is customary to consider NC space as

$$[\dot{X}^i, X^j] = (i/m)\delta^{ij} + i\alpha^{ij}(X, \dot{X}), \quad [X^i, X^j] = i\theta^{ij}(X, \dot{X}),$$

$$[\dot{X}^i, \dot{X}^j] = i\beta^{ij}(X, \dot{X}). \qquad (1)$$

If $\alpha^{ij}, \theta^{ij}, \beta^{ij}$ depend on $X^i, \dot{X}^i$ $\rightarrow$ (algebraic) restrictions on their structure from Jacobi identity.
Originally NC phase space, with operatorial extensions, first appeared in Snyder (PR, 1947) as a regularization for short distance singularity. Even though the original motivation was not successful it is still a very active area of research, leading to
Generalized Uncertainty Principle framework (Kemp, J.Ph.A 1997), Doubly Special Relativity (G. Amelino-Kamelia, Nature 2002), among others.

- For constant (non-operatorial) $\alpha^{ij}, \theta^{ij}, \beta^{ij}$ there is no such Jacobi identity restriction. NC space with $\theta^{ij} \neq 0$, first appeared in the work of Seiberg and Witten (JHEP 1999), where in a background two-form field open string endpoints (in certain low energy limits) resided on NC branes.
- From High Energy Physics perspective bounds on θ, ($\theta \leq (10 TeV)^{-2}$, relevant at $> TeV$ scale), are available (M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, PRL 2001; S.Carroll, J.A.Harvey, V.A.Kostelecky, C.D.Lane, T.Okamoto PRL 87 (2001).).

- In our works we have used constant NC (or deformation) parameter in a classical framework,

$$[\dot{X}^i, X^j] = (1/m)\delta^{ij}, \quad [X^i, X^j] = \theta^{ij}, \quad [\dot{X}^i, \dot{X}^j] = 0. \tag{2}$$

Strictly speaking we might refer to it as Noncanonical fluid (instead of Noncommutative fluid). The name Noncommutative fluid has stuck because the way noncanonical effect is introduced is a classical analogue of NC quantum mechanics.
We also use "operators" to mean classical variables with non-zero Poisson brackets.

- Incorporating NC effects in field theory
- Two parallel ways of constructing NC extension of a fluid:
  (i) directly apply the Groenwald-Moyal $*$-product in action of conventional fluid $\rightarrow$ NC-extended field theory action and proceed.
  M. C. B. Abdalla, L. Holender, M. A. Santos, and I. V. Vancea, PRD 2012; L. Holender, M. A. Santos, M. T. D. Orlando, and I. V. Vancea, PRD 2011; M.V. Marcial, A.C.R. Mendes, C. Neves, W. Oliveira and F.I. Takakura, PLA 2010.
- (ii) Our approach: Lagrangian (discrete) fluid model $\rightarrow$ introduce the NC coordinates $\rightarrow$ exploit the map connecting Lagrangian to Euler Hamiltonian (continuum) framework $\rightarrow$ NC effects in fluid field theory (P. Das and SG, EPJC 2016; Jackiw et.al., review for canonical fluid).

- **Lagrangian description of fluid dynamics**

  Microscopic picture where the fluid is treated as a large collection of point particles obeying canonical Newtonian dynamics. The d.o.f.s consist of the particle coordinate and velocity, $\mathbf{X}(\mathbf{x}, t), d\mathbf{X}(\mathbf{x}, t)/dt$, respectively.

- Newton's law for each particle (Lagrangian) coordinate $X_i(t)$ and velocity $u_i(t) = \dot{X}_i$ is given by,

$$m\ddot{X}_i(t) = m\dot{u}_i(t) = F_i(X(t)), \quad (3)$$

  where m is the particle mass and $F_i(X(t))$ is the applied force.

- Canonical Poisson brackets

$$[\dot{X}^i, X^j] = (1/m)\delta^{ij}, \quad [X^i, X^j] = [\dot{X}^i, \dot{X}^j] = 0. \quad (4)$$

Hamiltonian (Eulerian) description of fluid dynamics

- Macroscopic picture as a field theory Hamiltonian $H = \int d^3r \mathcal{E}(\rho, \mathbf{u})$ and a set of Poisson brackets between the fluid d.o.f, density and velocity fields $\rho(\mathbf{r}, t),\ \mathbf{u}(\mathbf{r}, t)$,

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0, \tag{5}$$

$$\{u^i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i \delta(\mathbf{r} - \mathbf{r}'), \tag{6}$$

$$\{u^i(\mathbf{r}), u^j(\mathbf{r}')\} = -\frac{\omega_{ij}(\mathbf{r})}{\rho(\mathbf{r})} \delta(\mathbf{r} - \mathbf{r}'), \tag{7}$$

  where

$$\omega_{ij}(\mathbf{r}) = \partial_i\ u_j(\mathbf{r}) - \partial_j u_i(\mathbf{r}) \tag{8}$$

  is called the fluid vorticity.

- The fluid equations of motion are derived from the above as Hamilton's equation of motion.

- Positing the Hamiltonian for a barotropic fluid (pressure depending only on density ρ) with potential energy $U(\rho)$ as

$$H = \int dr \left(\frac{1}{2} \rho \mathbf{u}^2 + U(\rho) \right), \quad (9)$$

the equations of motion turn out to be

$$\dot{\rho} = \{H, \rho\} = -\vec{\nabla} \cdot (\mathbf{u}\rho), \quad (10)$$

$$\dot{\mathbf{u}} = \{H, \mathbf{u}\} = -(\mathbf{u} \cdot \vec{\nabla})\mathbf{u} - \vec{\nabla} U'(\rho). \quad (11)$$

where $U'(\rho) = \partial U / \partial \rho$.
The above equations (10), (11) constitute respectively continuity equation and Euler (force) equation, the two central equations governing perfect fluid dynamics.

Map between Lagrangian and Eulerian d.o.f.

Field theoretic Poisson algebra between Euler variables is derived from the canonical point mechanics Poisson brackets between Lagrangian variables. We exploit this formalism to generate NC extended algebra among the Euler degrees of freedom, ie. density and velocity field variables.

- The Eulerian density field for the single particle is,

$$\rho(t,\mathbf{r}) = m\delta(\mathbf{X}(t) - \mathbf{r}). \tag{12}$$

Hence the density and velocity fields for a collection of particles are given by,

$$\rho(t,\mathbf{r}) = m\sum_{n=1}^{N}\delta(\mathbf{X}_n(t) - \mathbf{r}), \tag{13}$$

$$\mathbf{u}(t,\mathbf{r}) = \frac{m}{\rho(t,\mathbf{r})}\sum_{n=1}^{N}\dot{\mathbf{X}}_n(t)\delta(\mathbf{X}_n(t) - \mathbf{r}). \tag{14}$$

- For fluid field theory the discrete particle labels $\rightarrow$ continuous variables,

$$\rho(r) = \rho_0 \int \delta(X(x) - r)\ dx, \tag{15}$$

$$u_i(r) = \frac{\rho_0 \int \dot{X}_i(x)\delta(X(x) - r)\ dx}{\rho(r)}. \tag{16}$$

- Poisson brackets

$$[\dot{X}^i, X^j] = (1/m)\delta^{ij}, \quad [X^i, X^j] = 0, \quad [\dot{X}^i, \dot{X}^j] = 0. \tag{17}$$

- are generalized for continuum,

$$\{\dot{X}^i(\mathbf{x}), X^j(\mathbf{x}')\} = \frac{1}{\rho_0}\delta^{ij}\delta(\mathbf{x} - \mathbf{x}'); \quad \{X^i(\mathbf{x}), X^j(\mathbf{x}')\} = \{\dot{X}^i(\mathbf{x}), \dot{X}^j(\mathbf{x}')\} = 0. \tag{18}$$

- Using the defining equations for Euler variables from (15, 16), we need to compute,

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = \rho_0^2 \left[\int \delta(X(x) - r)dx, \int \delta(X(y) - r')dy \right], \qquad (19)$$

$$\{u_i(\mathbf{r}), \rho(\mathbf{r}')\} = \left[\frac{\int dx \dot{X}_i(x)\delta(X(x) - r)}{\int dx \delta(X(x) - r)}, \rho_0 \int \delta(X(y) - r')dy \right], \qquad (20)$$

$$\{u_i(\mathbf{r}), u_j(\mathbf{r}')\} = \left[\frac{\int dx \dot{X}_i(x)\delta(X(x) - r)}{\int dx \delta(X(x) - r)}, \frac{\int dy \dot{X}_j(y)\delta(X(y) - r')}{\int dy \delta(X(y) - r')} \right], \qquad (21)$$

where the NC algebra between the Lagrange particle coordinates have to be used. $X_i, \dot{X}_i$ are the dynamical variables with non-zero brackets.

Quick outline of derivation
$\{\rho, \rho\} = 0$ follows trivially since $\{X, X\} = 0$.

$$\{u_i(\mathbf{r}), \rho(\mathbf{r}')\} = \left[\frac{\int dx \dot{X}_i(x) \delta(X(x) - r)}{\int dx \delta(X(x) - r)}, \rho_0 \int \delta(X(y) - r') dy \right], \qquad (22)$$

$$\sim \frac{\rho_0}{\int dx \delta(X(x) - r)} \int dxdy \ \{\dot{X}_i(x), X_j(y)\} \ \frac{\partial \delta(X(y) - r')}{\partial_j X(y)} \delta(X(x) - r)$$

$$\sim \frac{\rho_0}{\int dx \delta(X(x) - r)} \int dxdy \ \frac{\delta_{ij}}{\rho_0} \delta(x - y) \ \frac{\partial \delta(X(y) - r')}{\partial_j X(y)} \delta(X(x) - r)$$

$$\sim \frac{1}{\int dx \delta(X(x) - r)} (\int dx \delta(X(x) - r)) \partial_i \delta(r - r')$$

$$\sim \partial_i \delta(r - r')$$

- We obtain,

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0, \tag{23}$$

$$\{u_i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i \delta(\mathbf{r} - \mathbf{r}'), \tag{24}$$

$$\{u_i(\mathbf{r}), u_j(\mathbf{r}')\} = -\frac{(\partial_i u_j - \partial_j u_i)}{\rho} \delta(\mathbf{r} - \mathbf{r}') \tag{25}$$

$\partial_i u_j - \partial_j u_i = \omega_{ij}$ is called vorticity. This is the canonical fluid algebra ((5-8) mentioned earlier).
Canonical Poisson brackets between Lagrangian D.O.F., and Lagrangian D.O.F. to fluid D.O.F. map→ Canonical Poisson brackets between fluid D.O.F.

- In the same way
NC Poisson brackets between Lagrangian D.O.F., and Lagrangian D.O.F. to fluid D.O.F. map→ NC Poisson brackets between fluid D.O.F.

- We follow same procedure now with NC brackets

$$\{\dot{X}^i(\mathbf{x}), X^j(\mathbf{x}')\} = \frac{1}{\rho_0}\delta^{ij}\delta(\mathbf{x}-\mathbf{x}'), \quad \{X^i(\mathbf{x}), X^j(\mathbf{x}')\} = \frac{1}{\rho_0}\theta^{ij}\delta(\mathbf{x}-\mathbf{x}'),$$
$$\{\dot{X}^i(\mathbf{x}), \dot{X}^j(\mathbf{x}')\} = 0. \quad (26)$$

(with $\theta^{ij} = -\theta^{ji}$) resulting in NC fluid algebra,

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = -\theta_{ij}\partial_i\rho(\mathbf{r})\partial_j\delta(\mathbf{r}-\mathbf{r}'), \quad (27)$$

$$\{u_i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i\delta(\mathbf{r}-\mathbf{r}') - \theta_{jk}\partial_j u_i(\mathbf{r})\partial_k\delta(\mathbf{r}-\mathbf{r}'), \quad (28)$$

$$\{u_i(\mathbf{r}), u_j(\mathbf{r}')\} = \frac{(\partial_j u_i - \partial_i u_j)}{\rho}\delta(\mathbf{r}-\mathbf{r}') + \theta_{kl}\frac{\partial_k u_i \partial_l u_j}{\rho}\delta(\mathbf{r}-\mathbf{r}'). \quad (29)$$

Noncommutativity induced vorticity: No (canonical) vorticity, $\omega_{ij} = \partial_i u_j - \partial_j u_i = 0$, a commonly used restriction. But note that due to NC effect effective θ-dependent vorticity is generated

$$\{u_i(\mathbf{r}), u_j(\mathbf{r}')\} = \theta_{kl}\frac{\partial_k u_i \partial_l u_j}{\rho}\delta(\mathbf{r}-\mathbf{r}'). \quad (30)$$

- **Jacobi identity considerations**: Jacobi identity plays a vital role in the internal consistency of the commutator structure in quantum mechanics or quantum field theory.
- In classical Hamiltonian mechanics or field theory, Jacobi identity defined in terms of Poisson brackets (or Dirac brackets in constrained systems) has to be satisfied

$$J(A(x), B(y), C(z)) \equiv \{\{A(x), B(y)\}, C(z)\} + \{\{B(y), C(z)\}, A(x)\} + \{\{C(z), A(x)\}, B(y)\} = 0. \quad (31)$$

Jacobi identity for NC coordinate algebra is trivially satisfied since the NC extension θ^{ij} is not operatorial.

- However, even the canonical fluid algebra has operators in RHS of brackets and checking its Jacobi is non-trivial. But $J(\rho, u^i, u^j) = 0$, true for other combinations also (Jackiw review).
- NC fluid algebra is still more complicated but we have checked Jacobi identity validity (Mitra, Banerjee and Ghosh, JCAP 2018).

Assumption: we use same form of Hamiltonian as the canonical one,

- Hamiltonian

$$H = \int dr \left(\frac{1}{2} \rho \mathbf{u}^2 + U(\rho) \right), \tag{32}$$

but NC brackets to compute the NC fluid equations of motion:

- NC Continuity equation remains unchanged,

$$\dot{\rho} = \{H, \rho\} = -\partial_i(\rho u_i), \tag{33}$$

Example of how antysymmetry of θ_{ij} affects results:

$$\{\rho(x), \int U(\rho(y))dy\} \sim \int U'(\rho(y))\theta_{ij}\partial_i\rho(x)\partial_j\delta(x-y) \sim \theta_{ij}\partial_i\rho(x)\partial_j U'(x)$$

$$\sim \theta_{ij}\partial_i\rho(x)\partial_j U'(\rho(x)) \sim \theta_{ij}\partial_i\rho(x)\partial_j\rho(x)U''(x) = 0$$

- NC Euler (force) equation

$$\dot{u}_i = \{H, u_i\} = -\partial_j \left((\frac{u^2}{2} + U')\delta^{ij} + \theta_{jk}(\frac{u^2}{2} + U')(\partial_k u_i) \right). \tag{34}$$

The continuity equation is unchanged but the Euler (force) equation has NC terms.

Properties of canonical fluid: Conservation laws and symmetries

Details in Jackiw et.al., review

- The energy density,

$$\mathcal{E} = \frac{1}{2}\rho\mathbf{u}^2 + U = T^{00}, \tag{35}$$

together with the energy flux

$$T^{j0} = \rho u^j(\frac{1}{2}\mathbf{u}^2 + U'), \tag{36}$$

satisfies the energy conservation law,

$$\dot{T}^{00} + \partial_j T^{j0} = 0. \tag{37}$$

- The momentum density, $\mathcal{P}$,

$$\mathcal{P}^i = \rho u^i = T^{oi}, \tag{38}$$

and the stress tensor T^{ij} with $P = \rho U' - U$ defined as the pressure,

$$T^{ij} = \delta^{ij}(\rho U' - U) + \rho u^i u^j = \delta^{ij} P + \rho u^i u^j, \tag{39}$$

satisfy

$$\dot{T}^{oi} + \partial_j T^{ji} = 0. \tag{40}$$

- The time translation and space translation are generated by the conserved quantities,

$$E = \int dx \mathcal{E} \qquad \text{(time-translation)}, \tag{41}$$

$$\mathbf{P} = \int dr\ \vec{\mathcal{P}} = \int dr\ \mathbf{j} \qquad \text{(space-translation)}. \tag{42}$$

Rotations are generated by the conserved angular momentum,

$$M^{ij} = \int dr\ (r^i \mathcal{P}^j - r^j \mathcal{P}^i) \qquad \text{(spatial rotation)}. \tag{43}$$

- The non-relativistic theory under Galilean transformation: Galilean boost constant of motion,

$$\mathbf{B} = t\ \mathbf{P} - \int dr\ \mathbf{r}\rho \qquad \text{(velocity boost)}. \tag{44}$$

- The continuity equation provides the total number or mass as the final conserved quantity

$$N = \int dr \rho \qquad \text{(number)}. \tag{45}$$

We will compare and contrast these important features of the ideal fluid with the NC generalized fluid model.

- **Conservation laws**
  From the NC fluid continuity equation derived earlier
  $\dot{\rho} = \{H, \rho\} = -\partial_i(\rho u_i)$ the Total number($\sim$ mass)
-
$$N = \int dr \rho$$

  is a conserved quantity.
  Using both continuity and Euler equation

$$\dot{u}_i = \{H, u_i\} = -\partial_j \left((\frac{u^2}{2} + U')\delta^{ij} + \theta_{jk}(\frac{u^2}{2} + U')(\partial_k u_i) \right)$$

  and energy density $\mathcal{E} = \frac{\rho u^2}{2} + U(\rho)$ we obtain
-
$$\dot{\mathcal{E}} = -\partial_i \left[\rho u_i(\frac{u^2}{2} + U') + \theta_{ij}\{\frac{u^2}{2}\partial_j P\} \right], \quad (46)$$

  where $P = \rho U' - U$ is the pressure.

- Thus, as per our assumption, canonical form of energy density is preserved but the energy flux T^{j0} receives an NC correction,

$$T^{j0} = \rho u_i(\frac{u^2}{2} + U') + \theta_{ij}\{\frac{u^2}{2}\partial_j(\rho U' - U)\}. \quad (47)$$

- We still have total energy conservation in NC fluid since the θ-term is also a total derivative.
  $H = \int dr\mathcal{E}$ is conserved quantity.
- **Space-time symmetries**
  Translation invariance: for momentum π^i defined as
  $\Pi^i = \int dr\pi^i = \int dr\rho u_i$

$$\{\Pi^i, \rho(\mathbf{r})\} = -\partial_i\rho, \quad (48)$$

$$\{\Pi^i, , u_j(\mathbf{r})\} = -\partial_i u_j + \theta_{kl}\partial_k u_j \partial_l u_i. \quad (49)$$

- $\mathbf{\Pi}$ translates ρ correctly but fails to do so for $\mathbf{u}$.
  Time derivative of π_i:

$$\pi^i = \rho u_i, \quad \dot{\pi}^i = \partial_j[-\{\rho u_i u_j + \delta_{ij}P\} + \theta_{jk}u_i\partial_k P] + \frac{1}{2}\theta_{jk}\rho(\partial_j u_i)\partial_k u^2. \quad (50)$$

- The last term in RHS is not a total derivative but $\sim$ small for small $u_i \rightarrow$ "weaker" momentum conservation principle.

- Thus,
$$\dot{\pi}^i = \partial_j \left(-(\rho u_i u_j + \delta_{ij} P) + \theta_{jk} u_i \partial_k P\right), \quad (51)$$
leading to a modified T^{ij},
$$T^{ij} = -(\rho u_i u_j + \delta_{ij} P) + \theta_{jk} u_i \partial_k P. \quad (52)$$

- θ-contribution in T^{ij} is not symmetric under interchange of $i, j \rightarrow$ the total angular momentum will not be conserved.

$$\begin{aligned} \dot{M}_{ij} &= \int dr (r^i \dot{\pi}^j - r^j \dot{\pi}^i) \\ &= \int dr \theta_{kl} [r^i \partial_j u_l - r^j \partial_i u_l](\partial_k u^2)\rho. \quad (53) \end{aligned}$$

- This result is also expected since the constant set parameters θ^{ij} does not transform under rotations. However once again RHS is of higher powers in $u_i \rightarrow$ "weaker" angular momentum conservation principle.

Action formulation for canonical fluid

Subtleties involved in action formalism (Review by Jackiw et.al., J.Ph.A)

- Fluid Lagrangian in Lagrangian variables $\rightarrow$ Fluid Hamiltonian in Lagrangian variables $\rightarrow$ Fluid Hamiltonian in Euler variables (using the map) $\equiv$ Euler Hamiltonian (used earlier)

$$L = \int dx(\frac{1}{2}m\dot{X}(x)^2 - U(X(x))) \rightarrow H = \int dx(\frac{P(x)^2}{2m}m + U(X(x))); P = m\dot{X}$$

$$\rightarrow H(Euler) = \int dr(\frac{1}{2}\rho u^2 + V(\rho))$$

  The above is straightforward.

- Deriving Euler Hamiltonian directly from a Lagrangian (the latter written in terms of Euler variables) is problematic.

  Reasons: (i) Fluid Euler equations are first order in time.
  (ii) The above has to be reproduced from the Hamiltonian using fluid variable algebra which should be derivable from the time derivative (kinetic) part of Lagrangian.
  (iii) Directing using the map

$$L = \int dx(\frac{1}{2}m\dot{X}(x)^2 - U(X(x))) \rightarrow \int dr(\frac{1}{2}\rho u^2 - V(\rho)) \neq L(Euler)$$

  Not the correct Euler L $\rightarrow$ this will not generate any non-trivial brackets since there are no time derivatives.

Deeper reason: Presence of a Casimir, Chern-Simons term (in three space dim.) $C_3 = \int d r \vec{u}.(\nabla \times \vec{u}) = \int d r \vec{u}.\vec{\omega}$ with $\vec{\omega} \equiv$ vorticity. For non-zero $\vec{\omega}$ this creates an obstruction.

- Represent velocity $u_i = \partial_i \theta + \alpha \partial_i \beta$ in terms of three scalar variables $\rightarrow (\theta, \alpha, \beta)$ the Clebsch variables.
  Now $\vec{\omega} = \nabla \alpha \times \nabla \beta \rightarrow C_3 = \int d r \partial_i (\theta \epsilon^{ijk} \partial_j \alpha \partial_k \beta) = \int d\vec{S}.(\theta \vec{\omega})$. Since C_3 is a surface integral it has no bulk contribution and brackets for θ, α, β can be derived.
- Lagrangian due to Eckart (1938) and Lin (1963)

$$L = \int dr \, (\frac{1}{2} \rho \vec{u}^2 - V(\rho) + \theta(\dot{\rho} + \nabla.(\vec{u}\rho)) - \rho\alpha(\dot{\beta} + \vec{u}.\nabla\beta) \qquad (54)$$

  u_i is now an auxiliary variable and its variation $\rightarrow u_i = \partial_i \theta + \alpha \partial_i \beta$. Generalization to interacting gauge-fluid models are studied in Mitra, Banerjee and SG, EPJC 2015.
- First order Lagrangian L has Second Class Constraints $\rightarrow$ Dirac analysis yields the (Dirac) brackets:

$$\{\rho(x), \rho(y)\} = \{\rho(x), \alpha(y)\} = \{\rho(x), \beta(y)\} = 0,$$

$$\{\rho(x), \theta(y)\} = \delta(x-y), \ \{\alpha(x), \theta(y)\} = -\frac{\alpha}{\rho}\delta(x-y),$$
$$\{\alpha(x), \beta(y)\} = \frac{1}{\rho}\delta(x-y).$$

The action for canonical fluid with brackets correctly reproduces Euler Hamiltonian and $\rho, u_i = \partial_i\theta + \alpha\partial_i\beta$ brackets.

- We posit a NC corrected fluid action that reproduces the NC density bracket
$$\{\rho(x), \rho(y)\} = -\theta_{ij}\partial_i\rho(x)\partial_j\delta(x-y).$$
- Proposed form of NC fluid Lagrangian for irrotational (zero vorticity) fluid, (α, β do not appear),
$$L = -\dot{\theta}(\rho - \frac{1}{2}\theta^{ij}\partial_i\rho\partial_j\theta) - (\frac{1}{2}\rho(\partial_i\theta)^2 + U(\rho)). \quad (55)$$

Lagrangian (variational) equations of motion for ρ and θ:

$$\dot{\rho} = -\partial_i\left[(\rho\partial_i\theta) + \frac{\theta^{ij}}{2}[\partial_j\theta\partial_k(\rho\partial_k\theta) + \rho\partial_j(\partial_k\theta)^2]\right], \quad (56)$$

$$\dot{\theta} = -\left[(\frac{(\partial\theta)^2}{2} + u') - \frac{\theta^{jk}}{2}\partial_k\theta\partial_j(\frac{(\partial\theta)^2}{2} + U')\right]. \quad (57)$$

- Noether prescription yields the canonical energy momentum tensor:

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu \theta)}\partial^\nu \theta + \frac{\partial L}{\partial(\partial_\mu \rho)}\partial^\nu \rho - \eta^{\mu\nu} L \quad (58)$$

where $\eta^{\mu\nu} = diag(1, -1, -1, -1)$ is the flat metric.

- Explicit expressions for energy and momentum densities are,

$$T^{00} = \frac{1}{2}\rho(\partial_i \theta)^2 + U(\rho), \quad T^{0i} = \rho\partial_i \theta - \frac{1}{2}\theta^{jk}\partial_j \rho \partial_k \theta \partial_i \theta. \quad (59)$$

Consistent with our earlier assumption, T^{00} does not receive any NC correction.
$T^{ij} \neq T^{ji}, T^{0i} \neq T^{i0}$ indicates that rotational and Lorentz symmetries are lost due to constant θ^{ij} parameter.

- NC Dirac brackets to $O(\theta^{ij})$,

$$\{\theta(x), \theta(y)\} = 0 \ , \quad \{\rho(x), \rho(y)\} = -\theta^{ij}\partial_i \rho(x)\partial^x_j \delta(x - y),$$
$$\{\rho(x), \theta(y)\} = \delta(x - y) + \frac{1}{2}\theta^{ij}\partial_j \theta(x)\partial^x_i \delta(x - y). \quad (60)$$

These brackets reproduce same equations of motion (55,56).

T^{00}, T^{0i} correctly generate correct time and space translations respectively.

- With the Hamiltonian $H = \int d^3x T^{00}$ from (59) and the NC algebra (60), we compute

$$\dot{\rho} = \{\rho, H\}, \quad \dot{u^i} = \{u^i, H\}. \tag{61}$$

RHS same as Lagrangian dynamical equations derived earlier.

- Momentum $P^i = \int d^3x \; T^{0i}$ yields

$$\{\theta(x), P^i\} = -\partial_i\theta, \quad \{\rho(x), P^i\} = -\partial_i\rho. \tag{62}$$

$\rightarrow$ spatial translations for ρ and θ.

- Total mass operator $M = \int d^3x \; \rho(x)$ satisfies

$$\{M, \rho(x)\} = \{M, \partial_i\theta(x)\} = 0 \tag{63}$$

indicating that M will lie at the centre of the Galilean algebra and will act as the central extension.

- **Two parameter central extension in Galilean algebra for Noncommutative fluid**

  Total mass $M = \int d^3x\ \rho$ is conserved,

$$\{M, H\} = 0. \quad (64)$$

  Energy-momentum conservation,

$$\partial_\mu T^{\mu\nu} = 0 \quad (65)$$

  or component form energy and momentum conservation laws are satisfied,

$$\partial_0 T^{00} + \partial_i T^{i0} = 0, \quad \partial_0 T^{0i} + \partial_j T^{ji} = 0. \quad (66)$$

  But the total momentum $P^i = \int d^3x\ T^{0i}$ is conserved but the local conservation law receives NC corrections.

- **Galilean boost algebra:** Define Galileo boost generator,

$$B^i = tP^i - \int d^3x\ \rho x^i \quad (67)$$

  Non-canonical θ^{ij}-dependent transformation of θ and ρ under boost,

$$\{\theta(x), B^i\} = -t\partial_i\theta + x^i - \frac{1}{2}\theta^{ij}\partial_j\theta, \quad \{\rho(x), B^i\} = -t\partial_i\rho - \theta^{ij}\partial_j\rho. \quad (68)$$

  Both θ and ρ behave in a non-canonical way.

- Compute

$$\{B^i, P^j\} = -\delta^{ij} \int d^3x \ \rho = -\delta^{ij} M, \tag{69}$$

$$\{B^i, B^j\} = \theta^{ij} \int d^3x \ \rho = \theta^{ij} M. \tag{70}$$

$\rightarrow$ two parameter central extension in Galilean algebra (Das and S.G., PRD (rap.) 2018).

- The first one, M, is the well known Bargman central extension.
- A structure, similar to the second one, depending on NC parameter θ^{ij} was discovered *only* in $2+1$-dimensional planar models having Exotic symmetry (for review see C.Duval, P. A. Horvarthy, J. Phys A: Math. Gen. 2001). In NC fluid second extension can occur in higher (three) dimensions.

- Rest of NC Galilean algebra in three space dimensions: (angular momentum $\mathbf{J} = \int d^3x\ (\mathbf{x} \times \mathbf{T})$ and NC parameter $\Theta^k = (1/2)\epsilon^{kij}\theta^{ij}$)

$$\{J^i, J^j\} = \epsilon^{ijk} J^k, \quad \{J^i, P^j\} = \epsilon^{ijk} P^k \tag{71}$$

$$\{J^i, B^j\} = \epsilon^{ijk} B^k + \frac{1}{2}(\mathbf{\Theta}.\mathbf{P}\delta^{ij} - \Theta^j P^i), \tag{72}$$

$$\{\mathbf{B}, H\} = -\mathbf{P} + \int d^3x\ [\frac{1}{2}\mathbf{\Theta}.(\nabla\frac{1}{\rho} \times \mathbf{T})\mathbf{T} + \frac{1}{4}(\mathbf{\Theta} \times \nabla(\frac{1}{\rho}))], \tag{73}$$

$$\{\mathbf{J}, H\} = \frac{1}{4}\int d^3x\ \mathbf{T}^2[(\mathbf{\Theta}.\mathbf{T})\nabla\frac{1}{\rho^2} - (\mathbf{\Theta}.\nabla\frac{1}{\rho^2})\mathbf{T}]. \tag{74}$$

- P^i transforms canonically which is expected since (as shown before) it correctly translates both θ, ρ.
- $\mathbf{J} - \mathbf{J}$ angular momentum algebra is also canonical – somewhat unexpected.
- Rest of the algebra receive NC corrections.Thus NC generalization leads to non-conservation of boost and angular momentum which is expected and agrees with earlier results.
- In our (low energy) approximation some of the NC corrections can be neglected but the second central extension survives.

Darboux map, noncommutativity induced vorticity and non-isentropy

- Darboux's theorem, a fundamental property of symplectic geometry, states that any symplectic manifold is locally isomorphic to some R^{2n} with its standard symplectic form.
  In physics language the NC variables ρ, θ can be expressed (at least locally) in terms of a canonical set ρ_c, θ_c obeying canonical algebra

$$\{\rho_c(x), \rho_c(y)\} = \{\theta_c(x), \theta_c(y)\} = 0; \ \{\rho_c(x), \theta_c(y)\} = \delta(x - y).$$

  The explicit form of Darboux map to $O(\theta)$:

$$\rho = \rho_c - \frac{1}{2}\theta^{ij}\partial_j\rho_c\partial_i\theta_c; \quad \theta = \theta_c.$$

- From now on we will work with ρ_c, θ_c but keep the original notation ρ, θ. The Hamiltonian, to $O(\theta)$ and to $O(u^2)$ is,

$$H \approx \int dr[T_c - \frac{1}{2}\theta^{ij}\frac{\partial_j \rho u_i}{\rho}(T_c + P_c)] \qquad (75)$$

  where, $u_i = \partial_i\theta$ and $T_c = \frac{1}{2}\rho u^2 + U(\rho)$, $P_c = \rho U' - U$ are canonical energy density and pressure (Das and S.G., PRD 2018).

- The continuity equation to $O(u^2)$,

$$\dot{\rho} \approx \{\rho, H\}$$
$$= \partial_l[-\rho(u^l - \frac{1}{2}\theta^{lj}\partial_j\rho\frac{1}{\rho}(\frac{1}{2}u^2 + U') - \frac{1}{2}\theta^{ij}\frac{(\partial_j\rho)}{\rho}u^i u^l)] \quad (76)$$

defines an effective velocity $\dot{\rho} = -\partial_l(\rho\bar{u}^l)$ where, to $O(u^2)$,

$$\bar{u}^l \approx u^l - \frac{1}{2}\theta^{lj}\partial_j\rho\frac{1}{\rho}(\frac{1}{2}u^2 + U') - \frac{1}{2}\theta^{ij}\frac{(\partial_j\rho)}{\rho}u^i u^l.$$

Clearly $\bar{u}^l$ is no longer irrotational $\rightarrow$ induced vorticity to $O(u^2)$:

$$\{\bar{u}^l(x), \bar{u}^k(y)\} \approx \frac{1}{2}[\theta^{lm}\partial_k^y(\frac{1}{\rho(y)}U'(y)\partial_m^y\delta(x-y))$$
$$-\theta^{km}\partial_l^x(\frac{1}{\rho(x)}U'(x)\partial_m^x\delta(x-y))]. \quad (77)$$

- NC induced vorticity is structurally totally different from the conventional form of vorticity ($\sim \nabla \times u$). The leading term (written here) is independent of $\bar{u}$ and will survive the low energy limit.

- Explicit example: $U(\rho) = K\rho^{\lambda}$ with K, λ numerical constants, for which $P_c = (\lambda - 1)U$.
  For the special case of pressureless dust, $(\lambda = 1, P_c = 0)$, induced NC vorticity is

$$\{\bar{u}^l(x), \bar{u}^k(y)\} \approx \frac{K}{2}[\theta^{lm}\partial_m(\frac{1}{\rho}\partial_k\delta(x-y)) - \theta^{km}\partial_l(\frac{1}{\rho}\partial_m\delta(x-y))].$$

  (All arguments of fields and derivatives are on x.)
  Non-abelian like feature, reminiscent of NC field theories since $\{u^k(x), \bar{u}^k(y)\}$ even for *same* k is non-zero:

$$\{\bar{u}^k(x), \bar{u}^k(y)\} = \frac{\theta^{km}}{\rho^2}(\partial_k\rho\partial_m\delta(x-y) - \partial_m\rho\partial_k\delta(x-y))$$

  (no sum on k). (Das and S.G., PRD 2018)

Effective pressure :

- Euler equation for $\bar{u}^i$,

$$\dot{\bar{u}}^m \approx -\partial_m(\frac{\bar{u}^2}{2}) - \frac{1}{\rho}\partial_m P_c + \frac{1}{2\rho}\theta^{ij}\partial_m(\bar{u}^i\partial_j U) - \frac{1}{2}\theta^{ij}U'\partial_m(\frac{1}{\rho}\bar{u}^i\partial_j\rho) + \frac{1}{2}\theta^{mj}[U'\partial_j(\frac{\partial_k(\rho\bar{u}^k)}{\rho}) + \frac{\bar{u}^k\partial_j\rho\partial_k U'}{\rho}]. \quad (78)$$

- Effective pressure depends explicitly on $\bar{u}^i$ (apart from ρ) $\rightarrow$ non-barotropy in the fluid $\rightarrow$ may lead to non-isentropic dynamics?
  NC corrected effective pressure for pressureless fluid ($P = 0, U = K\rho$):

$$\dot{\bar{u}}^l \approx -\partial_l(\frac{\bar{u}^2}{2}) + \frac{K}{2}(\frac{1}{\rho^2}\theta^{kj}\bar{u}^k\partial_j\rho\partial_l\rho + \theta^{lj}\partial_j(\frac{\partial_k(\rho\bar{u}_k)}{\rho})).$$

- Signature of the NC pressure can be both positive or negative (depending on θ and the fields) $\rightarrow$ might be interesting in cosmological scenario ? (Das and S.G., PRD 2018)

- Noncommutative fluid action for generic fluid (with non-zero vorticity)
  Eckart - Lin form of first order fluid action with NC correction terms,

$$L = -\partial_t\theta(\rho - \frac{1}{2}\theta^{ij}\partial_i\rho\partial_j\theta) - (\frac{1}{2}\rho u^2 + U(\rho)) - \rho\alpha\partial_t\beta, \quad (79)$$

  where $u_i = \partial_i\theta + \alpha\partial_i\beta$. For $u_i = \partial_i\theta(x) \rightarrow$ no vorticity condition.
- The explicit form of Darboux map to $O(\theta)$, is given by

$$\rho = \rho_c + \frac{1}{2}\theta^{ij}\partial_i\rho_c\partial_j\theta_c; \quad \theta = \theta_c; \quad \beta = \beta_c; \quad \alpha = \alpha_c - \frac{\theta^{ij}}{2}\alpha\frac{\partial_i\rho\partial_j\theta}{\rho} \quad (80)$$

  such that the NC algebra in is reproduced. For simplicity we will just keep the notation $\rho, \theta, \alpha, \beta$ instead of $\rho_c, \theta_c, \alpha_c, \beta_c$. The Hamiltonian is now written in terms of canonical variables, (to order of θ^{ij}),

$$H = \int dr[T - \frac{1}{2}\theta^{ij}\frac{\partial_j\rho\partial_i\theta}{\rho}(\frac{1}{2}\rho(\partial_i\theta)^2 - \frac{1}{2}\alpha^2(\partial_i\beta)^2 + U + P_c)]. \quad (81)$$

  where, $u_i = \partial_i\theta + \alpha\partial_i\beta$, $T = \frac{1}{2}\rho u^2 + U(\rho)$ is the canonical energy density and $P = \rho U' - U$ is the pressure (P.Das and S.G. EPJC).

- Further modification in Non Commutative algebra

  Further extensions of the θ_{ij}-NC structure with a new set of NC parameters σ_{ij} ,

$$\{X_i(\mathbf{x}), X_j(\mathbf{y})\} = \frac{\theta_{ij}}{\rho_0}\delta(\mathbf{x}-\mathbf{y}),\ \{\dot{X}_i(\mathbf{x}), X_j(\mathbf{y})\} = \frac{1}{\rho_0}(\delta_{ij}+\sigma_{ij})\delta(\mathbf{x}-\mathbf{y}),$$
$$\{X_i(\mathbf{x}), \dot{X}_j(\mathbf{y})\} = 0. \quad (82)$$

  This can have non-trivial consequence in cosmology. We noticed that some of the NC contributions, (that could have been relevant in cosmology), vanished due to $\theta^{ij} = -\theta^{ji}$ but σ^{ij} has no such symmetry and contributes. (Mitra, Banerjee and SG, JCAP 2018).
- Extended NC brackets

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = -\ \partial_i \rho \theta_{ij} \partial_j \delta(\mathbf{r}-\mathbf{r}'), \quad (83)$$

$$\{u^i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i \delta(\mathbf{r}-\mathbf{r}') + \sigma_{ij}\partial_j\delta(\mathbf{r}-\mathbf{r}') + \theta_{kj}\partial_k\delta(\mathbf{r}-\mathbf{r}')\partial_j u_i(\mathbf{r}') \quad (84)$$

$$\{u_i(\mathbf{r}), u_j(\mathbf{r}')\} = \frac{\partial_j u_i - \partial_i u_j}{\rho}\delta(\mathbf{r}-\mathbf{r}') + \theta^{lm}\frac{\partial_l v_i \partial_m u_j}{\rho}\delta(\mathbf{r}-\mathbf{r}') + \frac{1}{\rho}(\sigma_{kj}\partial_k u_i - \sigma_{ik}\partial_k u_j)\delta(\mathbf{r}-\mathbf{r}'). \quad (85)$$

Density (ρ) algebra remains unaltered but the rest receives σ_{ij}-contribution.

- Keeping the form of Hamiltonian unaltered,

$$H = \int dV\ \mathcal{H} = \int (\frac{1}{2}\rho u^2 + U(\rho))$$

the NC-generalized Euler dynamics follows,

$$\dot{\rho} = \{\rho, H\} = -\partial_i(\rho u_i) - \sigma_{ij}\partial_j(\rho u_i) = -\partial_i(\rho u_i + \sigma_{ji}\rho u_j), \quad (86)$$

$$\dot{u}_k = \{u_k, H\} = -u_i\partial_i u_k - \partial_k U'(\rho) + \theta_{ji}\partial_i U'(\rho)$$
$$-\sigma_{ij}u_i\partial_j u_k - \sigma_{ij}\partial_j U'(\rho)\partial_i u_k. \quad (87)$$

- Continuity equation now gets a NC σ-term. Euler equation has both σ-terms and a θ-term.

- Fluid dynamics mapped to comoving coordinates
  As is customary in cosmology we now work in a comoving frame $(a(t), \mathbf{x})$:
- Map between laboratory and comoving coordinates ($\mathbf{r}$ and $a(t), \mathbf{x}$ respectively) is given by,

$$\mathbf{r}(t) = a(t)\mathbf{x} \tag{88}$$

  with $a(t)$ the scale factor and $\mathbf{x}$, the time independent comoving distance.
- The canonical continuity and Euler equations in Friedmann-Robertson-Walker (FRW) cosmology are given by.

$$\dot{\rho} = -3H(\rho + P) = -3\frac{\dot{a}}{a}(\rho + P), \tag{89}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}, \tag{90}$$

  with pressure P, cosmological constant Λ and Newton's constant G. $H(t) = \dot{a}/a$ is the Hubble parameter.
- The Friedmann equation follows:

$$\frac{\dot{a}^2}{a^2} = H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{91}$$

- Rewrite the conventional fluid dynamical equations in comoving frame $\rightarrow$ FRW equations
- NC fluid equations mapped to comoving coordinates $\rightarrow$ NC FRW dynamics.
  Consider fluid without vorticity as canonical vorticity does not play any major role in cosmology.
- We consider comoving coordinates

$$\mathbf{r} = a(t)\mathbf{x}(t), \tag{92}$$

  where $\mathbf{r}(t), \mathbf{x}(t)$ and $a(t)$ denote the lab. coordinates, comoving coordinates and the scale factor respectively. $\mathbf{x}(t)$ now depends on t.
- Recast the dynamics in the comoving coordinates $\mathbf{x}, t$.

- The laboratory velocity $\mathbf{u} = \dot{\mathbf{r}}$

$$\dot{\mathbf{r}} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}}(t) \ \rightarrow \mathbf{u} = \dot{a}\mathbf{x} + \mathbf{v}, \tag{93}$$

with $\mathbf{v}$ defined as the peculiar velocity (*Cosmological Inflation and Large-Scale Structure*, Liddle and Lyth). In standard FRW $\mathbf{v}$ is taken as zero and now it is considered as a perturbation.

- The space derivatives are related by

$$\partial/\partial\mathbf{r} = (1/a)\partial/\partial\mathbf{x}.$$

- Time derivatives at constant $\mathbf{r}$ and constant $\mathbf{x}$ are related by,

$$\frac{\partial}{\partial t} \mid_{\mathbf{r}} = \frac{\partial}{\partial t} \mid_{\mathbf{x}} - \frac{\dot{a}}{a}(\mathbf{x}.\partial_{\mathbf{x}}).$$

This can be seen as $\partial_t \mid_r = \partial_t \mid_x + (\frac{\partial x}{\partial t})\partial_x$ and replace $(\frac{\partial x}{\partial t})$ at constant r using $\dot{r} = 0 = \dot{a}x + a\dot{x}$.

Noncommutative FRW from noncommutative fluid:: Using the above identities the NC fluid dynamics (85,86)

$$\dot{\rho} = \{\rho, H\} = -\partial_i(\rho u_i) - \sigma_{ij}\partial_j(\rho u_i) = -\partial_i(\rho u_i + \sigma_{ji}\rho u_j), \quad (94)$$

$$\dot{u}_k = \{u_k, H\} = -u_i\partial_i u_k - \partial_k U'(\rho) + \theta_{ji}\partial_i U'(\rho)$$
$$-\sigma_{ij}u_i\partial_j u_k - \sigma_{ij}\partial_j U'(\rho)\partial_i u_k - \partial_k\Phi_G. \quad (95)$$

- The term $-\partial_k\Phi_G$ is put in by hand. Φ_G is the gravitational potential that satisfies $\nabla^2\Phi_G = 4\pi G\rho$: Poisson equation.
  $\Phi_G = \frac{2}{3}\pi G\rho_0 r^2 + \phi$: the first term satisfies $\nabla^2\Phi_G = 4\pi G\rho_0$ for homogeneous background. ϕ is a perturbation also called peculiar potential.
  In comoving frames (Mitra, Banerjee and Ghosh, JCAP 2018) and using $P = \rho U' - U \rightarrow \partial_i U' = \frac{1}{\rho}\partial_i P$ for pressure P,
- Continuity equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\partial_i(\rho v_i) + \frac{\sigma_{ij}}{a}\partial_j(\rho\dot{a}x_i + \rho v_i) = 0 \quad (96)$$

- Euler equation,

$$\ddot{a}x_k + \frac{\partial v_k}{\partial t} + \frac{\dot{a}}{a}v_k + \frac{1}{a}v_i\partial_i v_k + \frac{\dot{a}}{a}\sigma_{ik}(\dot{a}x_i + v_i) + \frac{1}{a}\sigma_{ij}(\dot{a}x_i + v_i)\partial_j v_k$$
$$= -\frac{1}{a}[\frac{\partial_k P}{\rho} + \sigma_{kj}\frac{\partial_j P}{\rho} + \frac{\dot{a}}{a\rho}\theta_{ik}\partial_i P + \frac{1}{a\rho}\theta_{ij}\partial_i P\partial_j v_k + \frac{4\pi}{3}a^2 G\rho x_k + \partial_k\phi]. \quad (97)$$

Recover conventional FRW with

$$P = \Lambda = 0;\ \phi = v_i = 0;\ \theta^{ij} = \sigma^{ij} = 0;\ \rho = \rho_0,$$

- Continuity equation

$$\dot{\rho_0} + 3\frac{\dot{a}}{a}\rho_0 = 0;$$

- *Euler equation*
  Isolate structurally similar terms in (97) and requiring that the combinations vanish separately. In the present case the x_i-dependent terms read (with ρ replaced by its homogeneous background value ρ_0):

$$(\ddot{a} + \frac{4\pi}{3}aG\rho_0)x_i = 0 \ \rightarrow \ddot{a} + \frac{4\pi}{3}aG\rho_0 = 0.$$

NC correction to the background

$P = \Lambda = 0;\ \ \rho = \rho_0(t),\ \ v_i = 0$

- Continuity equation

$$\dot{\rho_0} + 3\frac{\dot{a}}{a}\rho_0 + \frac{1}{a}\sigma_{ij}\partial_j(\rho_0 \dot{a} x_i) = 0 \quad \rightarrow \dot{\rho_0} + \frac{\dot{a}}{a}\rho_0(3+\sigma) = 0 \qquad (98)$$

where $Tr(\sigma_{ij}) = \sigma$.
NC correction to Euler equation $\phi = v_i = 0$

$$[(\ddot{a} + \frac{4\pi}{3}G\rho_0 a)\delta_{ik} + \dot{a}H\sigma_{ik}]x_i = 0. \qquad (99)$$

To satisfy the above for arbitrary x_i we require determinant of the coefficient matrix of x_i to vanish,

$$\begin{vmatrix} (\lambda + \dot{a}H\sigma_{11}) & \dot{a}H\sigma_{12} & \dot{a}H\sigma_{13} \\ \dot{a}H\sigma_{21} & (\lambda + \dot{a}H\sigma_{22}) & \sigma_{23}\dot{a}H \\ \dot{a}H\sigma_{31} & \dot{a}H\sigma_{32} & (\lambda + \dot{a}H\sigma_{33}) \end{vmatrix} = 0, \qquad (100)$$

where,

$$\lambda = \ddot{a} + \frac{4\pi}{3}aG\rho_0.$$

- Expand the determinant

$$(\lambda + \dot{a}H\sigma_{11})[(\lambda + \dot{a}H\sigma_{22})(\lambda + \dot{a}H\sigma_{33}) - (\dot{a}H)^2\sigma_{23}\sigma_{32}]$$

$$+(\dot{a}H)\sigma_{12}[(\dot{a}H)^2\sigma_{23}\sigma_{31} - \dot{a}H\sigma_{21}(\lambda + \dot{a}H\sigma_{33})] \quad (101)$$

$$+\dot{a}H\sigma_{13}[(\dot{a}H)^2\sigma_{21}\sigma_{32} - \dot{a}H\sigma_{31}(\lambda + \dot{a}H\sigma_{22})] = 0. \quad (102)$$

To $O(\sigma)$ the above equation reduces to,

$$(\lambda)^3 + \lambda^2\dot{a}H(\sigma_{11} + \sigma_{22} + \sigma_{33}) \approx 0, \quad (103)$$

leading to ($Trace(\sigma_{ij}) = \sigma$)

$$\lambda + \dot{a}H\sigma = 0 \quad (104)$$

→ NC corrected Euler equation in cosmology

$$\ddot{a} + \frac{4\pi}{3}aG\rho_0 + \dot{a}H\sigma = 0. \quad (105)$$

After a little more algebra we find that (98) and (105) together yield,

$$\frac{1}{2}\frac{d}{dt}(\dot{a}^2) = \frac{4\pi G\rho_0}{3}[\frac{1}{\rho_0}(\frac{d}{dt}(\rho_0 a^2) + \frac{a}{\rho_0}\sigma\partial_j(\rho_0\dot{a}))] - \dot{a}^2H\sigma \quad (106)$$

- Friedmann equation with NC correction:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{8\pi G}{3a^2}\int dt\ a(\rho_0\dot{a})\sigma - \frac{2}{a^2}\int dt\ \dot{a}^2 H\sigma.$$

$$= \frac{8\pi G\rho}{3} - \frac{k_{eff}}{a^2}, \quad (107)$$

where

$$k_{eff} = k - \sigma(\frac{8\pi G}{3}\int dt\ a\dot{a}\rho_0 - 2\int dt\ \dot{a}^2 H). \quad (108)$$

In NC space, flatness condition will be dictated by the effective curvature k_{eff}. For instance for a flat universe in NC cosmology $k_{eff} = 0$ will lead to a relation,

$$k(t) = \sigma(\frac{8\pi G}{3}\int dt\ a\dot{a}\rho_0 - 2\int dt\ \dot{a}^2 H). \quad (109)$$

Cosmological perturbations::

The aim of of introducing perturbations in the FRW "Standard Model" of cosmology is to explain how large scale structures were formed in the expanding Universe.

In particular, this means that starting from an isotropic and homogeneous universe with an average background density ρ_0, how does the fluctuation $\delta\rho = \rho - \rho_0$ grow so that the density contrast $\delta = \delta\rho/\rho_0$ can reach unity. Once δ reaches values of the order of unity, their growth becomes non-linear. From then onwards, they rapidly evolve towards bound structures such as star formation and other astrophysical process.

Perturbation theory: it is customary to split the fields into a flat FRW background part and a perturbation part that can be analyzed order by order. However, we have already found out that the background FRW equations are modified by NC contribution. Also the flatness condition $k_{eff} = 0$ means that $k \rightarrow k(t)$ due to NC correction.

Introduction of perturbations

- Customary to absorb the small parameter in perturbations of the respective quantities and treat these as small with respect to the background

$$
\begin{aligned}
\rho(\mathbf{x},t) &= \rho_0(t) + \delta\rho((\mathbf{x},t)) = \rho_0 + \rho_1 + \rho_2 + \ldots. \\
P(\mathbf{x},t) &= P_0(t) + \delta P(\mathbf{x},t) = P_0 + P_1 + P_2 + \ldots \\
H(\mathbf{x},t) &= H_0(t) + \delta H(\mathbf{x},t) = H_0 + H_1 + H_2 \ldots \\
\phi(\mathbf{x},t) &= \phi_0(t) + \delta\phi(\mathbf{x},t) = \phi_0 + \phi_1 + \phi_2 + .. \\
\mathbf{u} &= \dot{a}\mathbf{x} + \mathbf{v} = \dot{a}\mathbf{x} + \mathbf{v_1} + \mathbf{v_2} \ldots. \qquad (110)
\end{aligned}
$$

- The peculiar velocity $\mathbf{v}$ in (110) is considered to be the perturbation in the velocity field.
- Density contrast (of order n), a useful parameter : $\delta_n = \frac{\rho_n}{\rho_0}$.

- Now, we would like to write the perturbation equation corresponding to the Euler equation (97) (without the background terms in (105) that has already been taken in to account). The perturbed equation is,

$$\frac{\partial v_k}{\partial t} + (H_0 + \delta H)v_k + \frac{1}{a} v_i \partial_i v_k + (H_0 + \delta H)\sigma_{ik} v_i + \frac{1}{a}\sigma_{ij} v_i \partial_j v_k$$
$$= -\frac{1}{a}[c_s^2 \frac{\partial_k(\rho_0 + \delta\rho)}{\rho_0 + \delta\rho} + \sigma_{kj}\frac{\partial_j(P_0 + \delta P)}{\rho_0 + \delta\rho} + (H_0 + \delta H)\theta_{ik}\frac{\partial_i(P_0 + \delta P)}{\rho_0 + \delta\rho}$$
$$+ \frac{1}{a}\theta_{ij}\frac{\partial_i(P_0 + \delta P)}{\rho_0 + \delta\rho}\partial_j v_k + \partial_k \delta\phi]. \quad (111)$$

- 1st order in perturbation so that terms of the form $\frac{\partial_k(\rho_0+\delta\rho)}{\rho_0+\delta\rho} \approx \frac{\partial_k \delta\rho}{\rho_0}$ (homogeneous background).
  Thus we find

$$\dot{v}_k{}^1 + H_0(v_k^1 + \sigma_{ik} v_i^1) = -[\frac{1}{a} c_s^2 \frac{\partial_k \delta\rho}{\rho_0} + \frac{1}{a}\partial_k \phi_1 + \frac{1}{a\rho_0} H_0 \theta_{ik} \partial_i P_1 + \frac{1}{a\rho_0}\sigma_{kj}\partial_j P_1] \quad (112)$$

- To define perturbations in H:

$$H = \frac{1}{3}\nabla.\vec{u} = \frac{1}{3a}\partial_i^x(\dot{a}x_i + v_i) = \frac{\dot{a}}{a} + \frac{1}{3a}\partial_i v_i.$$

  So that $H = H_0 + \delta H = \frac{\dot{a}}{a} + \frac{1}{3a}\partial_i v_i^1 \rightarrow H_1 = \frac{1}{3a}\partial_i v_i^1$.
- From perturbed Poisson equation: $\frac{1}{a^2}\partial_k^2\Phi^1 = 4\pi G\delta\rho^1$
- From perturbed continuity equation:

$$(\dot{\rho^1}) = -\rho^0 H^1(3+\sigma) - H^0\rho^1(3+\sigma). \tag{113}$$

- Using background NC continuity equation $\dot{\rho_0} + \frac{\dot{a}}{a}\rho_0(3+\sigma) = 0$ and (113) we find

$$(\dot{\delta^1}) = (\frac{\rho^1}{\rho^0})^{\cdot} = -H^1(3+\sigma). \tag{114}$$

- Noncommutative corrections in density perturbation equation:
  Divergence of the perturbation equation (112):

$$\partial_k \dot{v_k}^1 + H_0 \partial_k (v_k^1 + \sigma_{ik} v_i^1) = -\frac{1}{a}[c_s^2 \frac{\partial_k^2 \rho^1}{\rho_0} + \partial_k^2 \phi^1 + \sigma_{kj} \frac{\partial_k \partial_j P^1}{\rho_0}]. \quad (115)$$

- Note that $\theta^{ik}\partial_i\partial_k P_1 = 0 \to \theta^{ij}$-contribution vanishes.
  Use relation connecting the divergence of the peculiar velocity and the Hubble parameter (113), $H^1 = \frac{1}{3a}\partial_i v_i^1 \quad \to$

$$\dot{H}^1 = -2H^0 H^1 - \frac{1}{3a}[c_s^2 \frac{\partial_k^2 \rho^1}{a\rho_0} + \frac{\partial_k^2 \phi^1}{a} + H_0 \sigma_{ik} \partial_k v_i^1 + \sigma_{kj} \frac{\partial_k \partial_j P^1}{a\rho_0}]. \quad (116)$$

- Wave Equation for Growth of Density Perturbations::
  Use (114) $(\dot{\delta}^1) = -H^1(3+\sigma) \rightarrow$ NC corrected density perturbation equation:

$$\ddot{\delta}^1 = -2H_0\dot{\delta}^1 + \frac{(3+\sigma)}{3a}[H_0\sigma_{ik}\partial_k v_i^1 + c_s^2\frac{\partial_k^2\delta^1}{a} + \sigma_{kj}\frac{\partial_k\partial_j P^1}{a\rho_0} + \frac{\partial_k^2\phi^1}{a}]. \quad (117)$$

- Only $O(\sigma_{ij})$ are considered.
- Consider solutions of the form $\delta^1(x,t) \sim \delta^1(t)exp\ (i\mathbf{k_c}.\mathbf{x})$ so that $\partial_k^2\delta^1 = -k_c^2\delta^1 = -k^2a^2\delta^1$,
  $\mathbf{k_c}$ and $\mathbf{k}$ are respectively the comoving and proper wave vector,

$$\ddot{\delta}^1 = -2H_0\dot{\delta}^1 + \frac{\partial_k^2\phi^1}{a^2} + c_s^2\frac{\partial_k^2\delta^1}{a^2} + \frac{\sigma}{3}\frac{\partial_k^2\phi^1}{a^2} + \frac{1}{a}H_0\sigma_{ik}\partial_k v_i^1 + \sigma_{kj}\frac{\partial_k\partial_j P^1}{a^2\rho_0}$$

$$= -2H_0\dot{\delta}^1 + (4\pi G\rho_0 - c_s^2k^2)\delta^1 + \frac{4\pi G\rho_0}{3}\sigma\delta^1 + \frac{1}{a}\sigma_{ik}(H_0\partial_k v_i^1 + \frac{\partial_i\partial_k P^1}{a\rho_0}).(118)$$

  We used perturbed Poisson equation: $(1/a^2)\partial^2\phi^1 = 4\pi G\delta\rho = 4\pi G\rho_0\delta^1$.
  This is the density perturbation equation. This equation governs the dynamics of small density fluctuations in a noncommutative fluid for an expanding background cosmology without cosmological constant.

- We rewrite the above equation in the convenient form,

$$\ddot{\delta}^1 = -2H_0\dot{\delta}^1 + 4\pi G\rho_0(1+\frac{\sigma}{3})\delta^1 - c_s^2 k^2\delta^1 + \Sigma, \tag{119}$$

where $\Sigma = \frac{1}{a}\sigma_{ik}(H_0\partial_k v_i^1 + \frac{\partial_i\partial_k P^1}{a\rho_0})$. σ and Σ are both NC contributions.

- In Σ both the terms are $\sim O(\sigma\partial v^1), O(\sigma\partial P^1) \approx 0$. We neglect Σ. Hence in the long wavelength limit ($\lambda\rangle\rangle\lambda_J = c_s\sqrt{\frac{\pi}{G\rho_0}}$, λ_j is the Jeans' wavelength in conventional cosmology), (119) $\rightarrow$

$$\ddot{\delta}^1 = -2H_0\dot{\delta}^1 + 4\pi G\rho_0(1+\frac{\sigma}{3})\delta^1. \tag{120}$$

In the linear regime, density fluctuations on different scales evolve independently.
In the Fourier space as (114) and (119) are rewritten:

$$H_k^1 = -\frac{\dot{\delta}_k}{3+\sigma},$$

$$\ddot{\delta}_k^1 + 2H_0\dot{\delta}_k^1 = 4\pi G\rho_0(1+\frac{\sigma}{3})\delta_k^1 \tag{121}$$

- We will try to find solution of the NC-modified equation (121) in a flat space which implies at critical density $(\rho = \rho_c)$,

$$\ddot{\delta}_k^1 + 2H_0\dot{\delta}_k^1 = 4\pi G\rho_0(1 + \frac{\sigma}{3})\delta_k^1. \quad (122)$$

- First find out the time dependence of H_0 and ρ_0.
- Note that the background, is no longer the conventional one since it has already received a NC correction, as is seen from NC-continuity equation (98), $\dot{\rho_0} + \frac{\dot{a}}{a}\rho_0(3+\sigma) = 0$ with solution

$$\rho_0 = \bar{\rho}a^{-(3+\sigma)}. \quad (123)$$

- To $O(\sigma)$ we use the conventional (matter dominated) time dependence of $a(= A_0 t^{\frac{2}{3}})$.
$\rightarrow$ We calculate $k(t)$ under flat space condition from (109),

$$k(t) = \sigma(\frac{8\pi G}{3}\bar{\rho}\int dt\ a\dot{a}a^{-(3+\sigma)} - 2\int dt\ \dot{a}^2 H).$$

$$k(t) = \frac{8}{3}\sigma t^{-2/3}(-\pi G\bar{\rho}A_0^{-(1+\sigma)}t^{-2\sigma/3} + \frac{A_0^2}{3}). \tag{124}$$

In the conventional (flat space) case, for $\sigma = 0 \rightarrow k = 0$.
Use this k in the Friedmann equation (91) (with $\Lambda = 0$, no cosmological constant),

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0}{3} - \frac{\frac{8}{3}\sigma t^{-2/3}(-\pi G\bar{\rho}A_0^{-(1+\sigma)}t^{-2\sigma/3} + \frac{A_0^2}{3})}{a^2}. \tag{125}$$

Try a solution of a as a polynomial in t. In the RHS of (125) substitute

$$\rho_0 = \bar{\rho}a^{-(3+\sigma)}, \quad a = A_0 t^{2/3}, \tag{126}$$

$\rightarrow$ σ-corrected background and conventional form of $a(t)$ so that (125) will yield the $O(\sigma)$ corrected $a(t)$.
Here A_0 and $\bar{\rho}$ are two constants that take care of the dimensions.

- Solution ,

$$t = Aa^{\frac{3+\sigma}{2}} + Ba^{3(\frac{1+\sigma}{2})} \tag{127}$$

where A and B are constants,

$$A = \frac{2(1-\sigma)}{3+\sigma}\sqrt{\frac{3}{8\pi G\bar{\rho}}}, \quad B = \frac{8\sigma A_0^3}{27(1+\sigma)}(\frac{3}{8\pi G\bar{\rho}})^{\frac{3}{2}}.$$

- Invert (127) to express a as a function of t in the familiar form,

$$a = (\frac{t}{A})^{\frac{2}{3+\sigma}}[1 - \frac{BA_0^{2\sigma/3}}{A} t^{\frac{2\sigma}{3}}]^{\frac{2}{3+\sigma}} \quad (128)$$

where, $\frac{B}{A} = \frac{2A_0^3\sigma}{3\pi G\bar{\rho}}$.
For $\sigma = 0 \to B = 0$ and the familiar form, $a(t) \sim t^{2/3}$ is recovered.
For convenience we further approximate $a(t) \sim t^{2/(3+\sigma)}$ in subsequent analysis.
Putting everything together in (122), the NC evolution equation for δ_k^1:

$$\ddot{\delta}_k^1 + \frac{4(1-\sigma/3)}{3t}\dot{\delta}_k^1 - \frac{2}{3t^2}(1+\frac{\sigma}{6})\delta_k^1 = 0. \quad (129)$$

- By inspection a power law solution $\delta_k^1 \sim t^n$ gives

$$n = \frac{1}{6}[-1 + \frac{4\sigma}{3} \pm 5\sqrt{1 - \frac{11}{75}\sigma}] \approx \frac{1}{6}[-1 + \frac{4\sigma}{3} \pm 5(1 - \frac{11}{150}\sigma)]. \quad (130)$$

The NC corrected values of n are

$$n = \frac{2}{3} + \frac{29}{180}\sigma, \quad n = -1 + \frac{51}{180}\sigma. \quad (131)$$

Note that σ can be either positive or negative. Positive and negative values of n signify growing or decaying modes. Obviously allowed values of σ have to be such that the original nature of the mode (growing or decaying) is not altered.

Noncommutative effect on Hubble parameter and linear growth of structure ::

- To what extent can NC affect the curvature and related evolutionary history of the universe in a quantitative way:
  Generically numerical upper bounds of NC parameters, obtained from areas in quantum mechanics or particle physics are in fact extremely small. From a theoretical perspective NC effects are expected to become relevant at approximately around Planck scale when the spacetime continuum tends to get replaced by discreteness with noncommutativity manifesting itself by inducing an inherent length scale.
- Strictly speaking, we are considering a *non-canonical* Poisson bracket structure in classical physics, rather than a noncommutative structure in the quantum commutators.
  Thus smallness of the quantum NC parameter θ, σ may not directly apply to our study. Still our NC parameters are small since we have used them as perturbations.

NC effect on Hubble parameter $H(t)$:
From $a(t) \sim t^{2/(3+\sigma)}$ derived earlier, we find

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{(3+\sigma)t}. \qquad (132)$$

Thus larger negative values of σ (pink and red line in Figure 1) tend to stay more and more above the $\sigma = 0$ line whereas larger positive values of σ (blue and green line in Figure 1) stay below the $\sigma = 0(blackline)$ line.

- **Figure 1**: Using the explicit form of NC-modified scale factor $a(t)$ we compute $H(t)$ and plot it against t for two values of $\sigma = 0.1, \ +0.5$ and $\sigma = -0.1, \ -0.5$ (since σ can take positive or negative values). These can be compared with the conventional case, $\sigma = 0$, the middle black line.
- Comparing with a conventional matter dominated universe $H \sim 2/(3t)$, one might conclude that NC correction for positive σ reduces H indicating that the rate of expansion of universe slows down $\rightarrow$ simulating maybe a dark matter?
  Negative σ seems to behave in a way that opposes the conventional matter contribution.

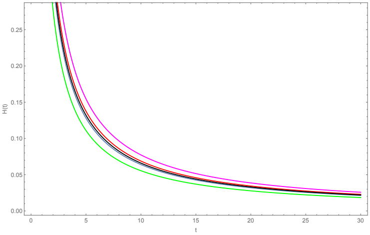

Figure : $H(t)$ is plotted against t for $\sigma = 0$ black line (conventional case), $\sigma = -0.1, \ -0.5$ red and majenta lines respectively and $\sigma = +0.1, \ +0.5$ blue and green lines respectively.

NC-correction in the evolution of the density contrast modes $\delta_k^1 \sim t^n$:

- For $\sigma = 0$ the conventional values $n = -1$ and $n = +2/3 \approx 0.66$. The latter increasing mode is of interest.
  From (131) we get for $\sigma = +0.1, \ +0.5$, n changes to $0.68, \ 0.74$ respectively and for $\sigma = -0.1, \ -0.5$, n changes to $0.63, \ 0.58$ respectively for the increasing mode.

  In Figure 2 we have plotted δ_k^1 against t for the above four values of n along with $n = +2/3$ (for $\sigma = 0$) for comparison.
- Positive values of σ enhances the growing modes so that structure formation is favored. In this sense our model of generalized fluid dynamics in the cosmological perspective becomes interesting since it might lead to a dark matter model, (that is essential for explaining the observed large-scale structure in the Universe), in classical framework.

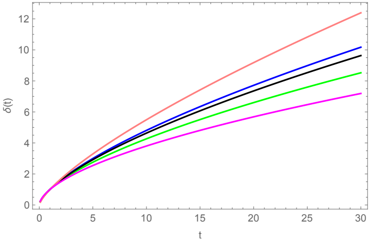

Figure : $\delta_k^1(t) \sim t^n$ is plotted against t for $n = 2/3,\ \sigma = 0$ black line (conventional case), $n = 0.68,\ 0.74$; $(\sigma = +0.1, +0.5)$ for blue and red lines respectively and $n =\ 0.58, 0.63$; $(\sigma = -0.1, -0.5)$ for green and magenta lines respectively.

Conclusion::

- A generalized fluid model has been developed that lives in a noncommutative ($\sim$ noncanonical) space.
- Noncommutative fluid brackets are derived that generate a generalized fluid dynamics.
- Symmetries of the extended model have been discussed.
- Action for the noncommutative fluid has been constructed.
- Possible consequences of this extended fluid dynamics in cosmology (non-relativistic Newtonian framework) are pointed out.

Future directions::

- We have considered the simplest form of approximation and a more detailed analysis of the model is needed. Specifically one of our future projects is to find solutions of the scale factor directly computed from the noncommutativity extended equations derived here.
- It would be interesting to exploit the rigorous cosmological averaging principles developed by Buchert and coworkers in the present context where the modifications stem from the fact that the evolution and averaging of dynamical variables do not commute.

THANK YOU