Towards a Phase Space Description of Gravity

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A Phase space formulation

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- **Is there any natural way in the gauge theory to understand the diffeomorphism redundancy in gravity ?**

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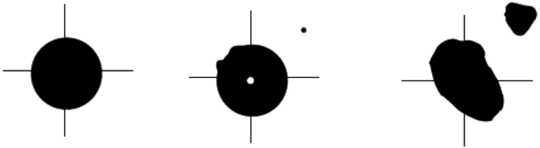
- Is there any natural way in the gauge theory to understand the diffeomorphism redundancy in gravity ?
- A hint came from the work of *Lin-Lunin-Mldacena* in 2004.
  - Geometry of a special class of solutions of the bulk theory is completely determined by specifying a "shape" in dimensions spanned by two of the bulk coordinates (x_1, x_2).



[LLM]

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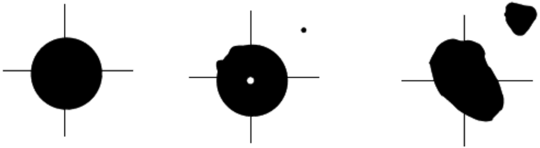

[LLM]

- This means the full bulk geometry is determined by the the shape of this 2D droplet.

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- Questions
  - Can phase space description of gauge theories be used to provide a dual quantum mechanical description of string degrees of freedom ?
  - In this talk :
    - How to construct a collective N particle description of a generic class of gauge theory and the corresponding phase space picture ?
    - What are the coordinate and momenta of these N particles ?
    - Hamiltonian ?
    - Constraints on large N representations for Chern-Simons theories.

Gauge theory as unitary matrix model

- Gauge theories on compact manifolds can be written as a unitary matrix model.
- For example, the thermal partition function of a gauge theory, coupled with matter and gauge group $SU(N)$ on a compact manifold $S^p \times S^1$ can be written as

$$Z = \int [DU] e^{S(U)}, \quad S(U) = \sum_{\vec{n}=0}^{\infty} \frac{a_{\vec{n}}(T, \lambda)}{N^k} \prod_{i=1}^{k} \mathrm{Tr}\, U^{n_i}, \quad \sum_{i}^{k} n_i = 0.$$

$$U = \exp[i \int_{S^1} \mathcal{A}_0], \quad \mathcal{A}_0 = \frac{1}{V_{S^p}} \int_{S^p} A_0$$

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- A particular class of unitary matrix model known as plaquette model is given by

$$\mathcal{Z} = \int \mathcal{D}U \exp\left[N \sum_{n=0}^{\infty} \frac{\beta_n}{n} (Tr[U^n] + Tr[U^{\dagger n}]) \right].$$

Unitary matrix model

- Unitary matrix model plays an important role in physics and mathematics.
- The field was initiated almost a century ago by statisticians and introduced in physics in the 50s-60s by Wigner and Dyson.
- Partition functions of different super-symmetric gauge theories, in particular Chern-Simons theories on certain manifolds boil down to UMMs.
- Partition functions of 2D gravity/string theory can also be written in terms of Hermitian matrix models.
- These models also have applications in a broad class of condensed matter systems.
- There exists a strong similarity between non-trivial zeros of $\zeta(s)$ function and eigenvalues of unitary matrix.

Unitary matrix model

- **An unitary matrix model is a statistical ensemble of unitary matrices, defined by the partition function**

$$\mathcal{Z} = \int [\mathcal{D}U] \exp[S(U)],$$

$$S(U) \quad : \quad \text{Action}, \quad \mathcal{D}U \rightarrow \text{measure}.$$

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$$S(U) \quad : \quad \text{Action}, \quad \mathcal{D}U \rightarrow \text{measure.}$$

- **Both $\mathcal{D}U$ and $S(U)$ are invariant under unitary transformation.**
- **One can go to a diagonal basis**

$$U \rightarrow e^{i\theta_i}\delta_{ij}, \quad \theta_i\text{'s are eigenvalues of } U.$$

- **In this basis the Haar measure and the action can be written as**

$$\int [DU] = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\theta_i \prod_{i<j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right), \quad S(U) = S(\{\theta_i\}).$$

Unitary matrix model

- **The partition function is given by (for plaquette model)**

$$\mathcal{Z} = \int \mathcal{D}U \exp\left[N\sum_{n=0}^{\infty}\frac{\beta_n}{n}\left(Tr[U^n] + Tr[U^{\dagger n}]\right)\right] = \int \prod_i [d\theta_i]\; e^{-NS_{eff}(\{\theta_i\})},$$

where

$$S_{eff}(\{\theta_i\}) = -\sum_{i=1}^{N}\sum_{n=1}^{\infty}\frac{2\beta_n}{n}\cos n\theta_i - \frac{2}{N}\sum_{i\neq j}\ln|\sin\frac{\theta_i-\theta_j}{2}|$$

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- **It describes a system of N particles interacting by the two-dimensional repulsive potential $\ln|\sin(\theta_i-\theta_j)/2|$ in the common potential**

$$V(\theta_i) = -\sum_{n=1}^{\infty}\frac{2\beta_n}{n}\cos n\theta_i, \quad \forall\ i.$$

Unitary matrix model

- **Because of the strong repulsive potential $\ln|\sin(\theta_i - \theta_j)/2|$ two particles do not come close to each other.**
- **If we neglected the repussive force, all eigenvalues would sit at the minima of the potential $V(\theta)$.**
- **Due to the Coulomb repulsion they are spread around these minima and fill some finite intervals.**

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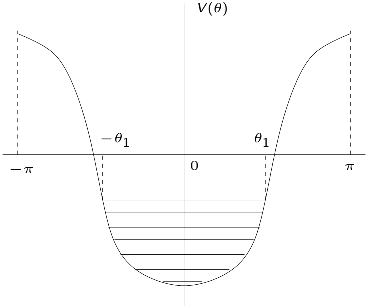
- **N Fermions moving in potential $V(\theta)$.**
- **In large N limit, partition function is dominated by minimum energy configuration.**



Eigenvalue distribution.

Unitary matrix model

- **The saddle point equation**
  Eigenvalue distribution is governed by the following equation

$$\frac{\partial S[\theta_i]}{\partial \theta_i} = 0,$$

which is given by,

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \cot\left(\frac{\theta_i - \theta_j}{2}\right) = S'[\theta_i].$$

- **In large N limit we define continuous variables**

$$\theta(x) = \theta_i, \quad x = \frac{i}{N}, \quad x \in [0,1].$$

- **The saddle point equation becomes**

$$\fint d\theta' \rho(\theta) \cot\left(\frac{\theta - \theta'}{2}\right) = S'(\theta), \quad \rho(\theta) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta \theta}.$$

Unitary matrix model - an example

- **Consider plaquette model with $\beta_1 \neq 0$ and $\beta_{n>1} = 0$. *Gross-Witten* potential**
- **Action is given by**

$$S(U) = N\left[\beta_1(\mathbf{Tr}U + \mathbf{Tr}U^\dagger)\right] = N\sum_{i=1}^{N} 2\beta_1 \cos\theta_i = -N^2 \int d\theta \rho(\theta) V(\theta).$$

- **The potential $V(\theta)$ is given by**

$$V(\theta) = -2\beta_1 \cos\theta.$$

Unitary matrix model - an example

- **We categorize different classes according to gaps in eigenvalue distributions.**

No-gap phase : Two possibilities

- **Potential** $V(\theta) = 0$, $\beta_1 = 0$.

- **All the eigenvalues are uniformly distributed on circle due to coulomb repulsion.**

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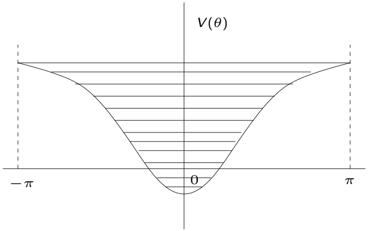
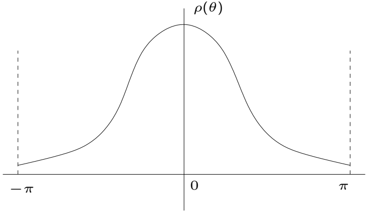


- **Potential $V(\theta) \neq 0$, but depth is less - can not accommodate N fermions.**

$$\rho(\theta) = \frac{1}{2\pi}(1 + 2\beta_1 \cos\theta) \text{ for } \theta \in [-\pi, \pi].$$
$$0 < \beta_1 < 1/2.$$



Unitary matrix model - an example

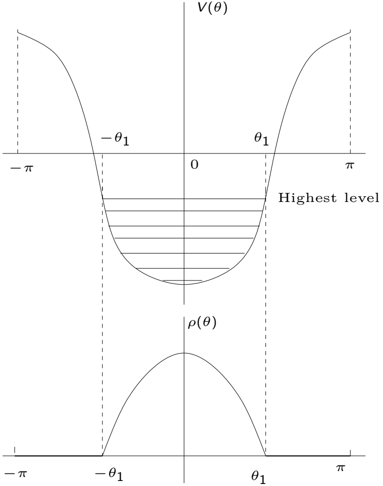
One-gap phase

- Potential depth is well enough to accommodate N states.
- Eigenvalue density is given by

$$\rho(\theta) = \frac{2\beta_1}{\pi}\sqrt{\sin^2\frac{\theta_1}{2} - \sin^2\frac{\theta}{2}}\cos\frac{\theta}{2}.$$

θ_1 is determined in terms of β_1

$$\sin^2\frac{\theta_1}{2} = \frac{1}{2\beta_1}.$$



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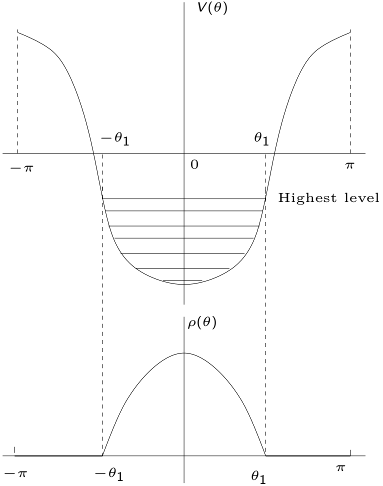


- $\beta_1 = 1/2$ is a phase transition point : Gross-Witten phase transition.

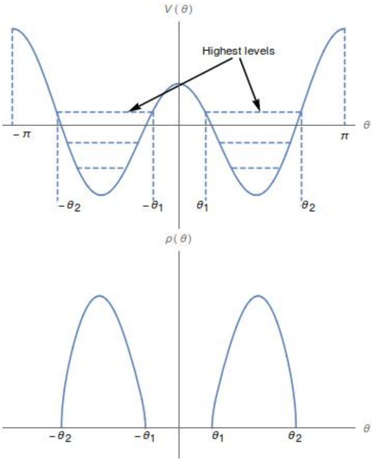
Unitary matrix model : other examples

Two-gap phase

- Potential looks like a double-well type.
- In large N limit, lowest energy configuration : lowest energy states are symmetrically filled in both the wells.
- Eigenvalue density is given by

$$\rho(\theta) = -\frac{\beta_2}{\pi}|\sin\theta| \sqrt{(\cos\theta_2 - \cos\theta)(\cos\theta - \cos\theta_1)}$$

$\beta_2 < 0$ and θ_1, θ_2 are determined in terms of β_1 and β_2.



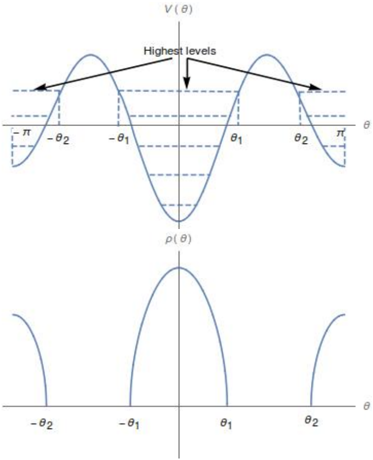
Unitary matrix model : other examples

Two-gap phase

- Potential has minima at $\theta = 0$ and $\theta = \pi$.
- In large N limit, lowest energy configuration : lowest energy states are symmetrically filled in both the wells.
- Eigenvalue density is given by

$$\rho(\theta) = \frac{1}{\pi}\sqrt{(\cos\theta_2 - \cos\theta)(\cos\theta_1 - \cos\theta)} \left[\beta_1 + \beta_2(2\cos\theta + \cos\theta_1 + \cos\theta_2)\right]$$

$\beta_2 > 0$ and θ_1, θ_2 are determined in terms of β_1 and β_2.



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- **Any representation of** $SU(N)$ **can be described in terms Young diagrams with maximum** N **number of rows.**

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$h_i = n_i + N - i, \quad n_i$ **number of boxes in i-th row of a Young diagram.**

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- **Later we shall see that** h_i**s play the role of momenta.**

Young diagram distribution for GWW case

- **For $\beta_1 < 1/2$, $u(h)$ saturates the maximum value in a finite range of h**

$$
\begin{aligned}
u(h) &= 1, && 0 \le h \le 1 - 2\beta_1. \\
&= \frac{1}{\pi}\cos^{-1}\left[\frac{h-1}{2\beta_1}\right], && 1 - 2\beta_1 \le h \le 1 + 2\beta_1
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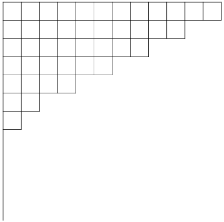

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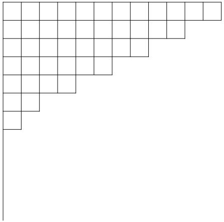

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u(h) = \frac{2}{\pi}\cos^{-1}\left[\frac{h+\beta_1-1/2}{2\sqrt{\beta_1 h}}\right]
$$

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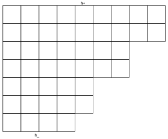

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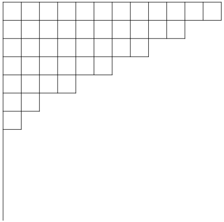

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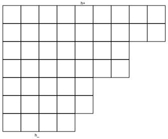

- **There exists a Douglas-Kazakov type phase transition between these saddle points as one varies the parameter β_1.**

Relation between Eigenvalue and Young diagram distribution

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- **We observed that there exists a surprising relation between eigenvalue distribution and Young tableaux distribution for different phases of the theory.**

$$\boxed{\rho(\theta) = \frac{h_+ - h_-}{2\pi} \quad \textbf{and} \quad u(h) = \theta/\pi} \quad \textbf{where} \quad h_\pm = \frac{1}{2} + \beta_1 \cos\theta \pm \pi\rho(\theta).$$

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- **Eigenvalue distribution and Young tableaux distribution can be expressed in a unified way in terms of a single constant distribution function $\omega(h,\theta)$ such that,**

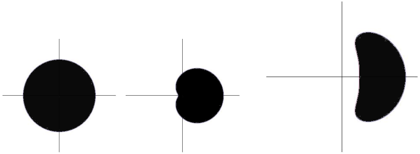
$$\rho(\theta) = \int_0^\infty \omega(h,\theta)dh, \qquad u(h) = \int_{-\pi}^{\pi} \omega(h,\theta)d\theta$$

where $\omega(h,\theta)$ is a distribution in a two dimensions.

$$\omega(h,\theta) = \frac{1}{2\pi}\Theta[(h_+ - h)(h - h_-)].$$

Relation between Eigenvalue and Young diagram distribution

- **2D droplets for GWW potential**



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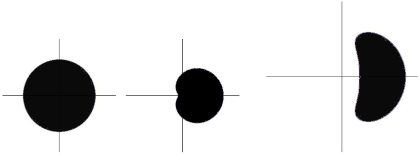

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Thermal AdS **Small black hole** **Big black hole**

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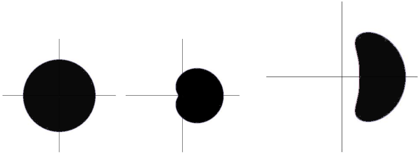

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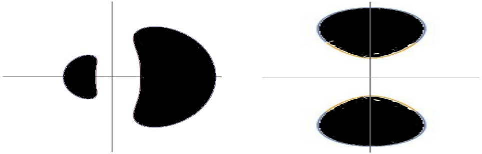
- **2D droplets in general (for two gap phase)**



Gravity dual ?

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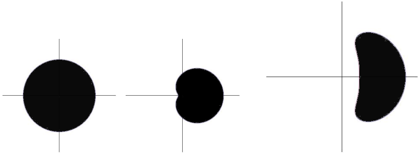

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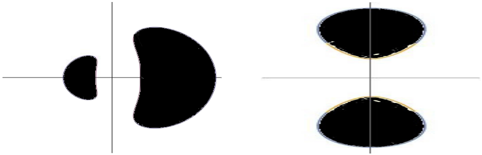

Gravity dual ?

- **These droplets are similar to Wigner distribution in Thomas-Fermi model.**
- **All the states inside/outside this region are filled/empty.**
- **Thus boundary defines fermi surface in (h, θ) plane.**
- **Topology/shape of fermi surface is different for different phases of the model.**
- **Phase transition corresponds to change of topology of fermi region.**

Collective field theory Hamiltonian

[A. Chattopadhyay, P. Dutta, SD]

- **From phase space distribution we define a Hamiltonian density**

$$\omega(h,\theta) = \frac{1}{2\pi}\Theta[(h_+ - h)(h - h_-)] = \frac{1}{2}\Theta(\mu - \mathfrak{h}(h,\theta))$$

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$$\mathfrak{h}(h,\theta) = \frac{h^2}{2} - S(\theta)h + \frac{g(\theta)}{2} + \mu, \quad \textit{where} \quad g(\theta) = h_+(\theta)h_-(\theta)$$

- **Momentum density $S(\theta)$ can be written from phase space distribution**

$$S(\theta) = \frac{1}{2\pi\rho}\int_0^\infty dh\ h\ \omega(h,\theta) = \frac{h_+(\theta) + h_-(\theta)}{2}.$$

- **The Hamiltonian can also be written as**

$$H_h = \int d\theta \left(\frac{S^2\rho}{2} + \frac{\pi^2\rho^3}{6}\right) + V_{eff}(\rho) + \mu$$

Collective field theory Hamiltonian

[A. Chattopadhyay, P. Dutta, SD]

- **From phase space distribution we define a Hamiltonian density**

$$\omega(h,\theta) = \frac{1}{2\pi}\Theta[(h_+ - h)(h - h_-)] = \frac{1}{2}\Theta(\mu - \mathfrak{h}(h,\theta))$$

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- **Collective field theory of $\rho(\theta,t)$. The Hamiltonian is given by,** [Jevicki-Sakita; Das-Jevicki]

$$H_B = \int d\theta \left(\frac{1}{2}\frac{\partial\pi(t,\theta)}{\partial\theta}\rho(t,\theta)\frac{\partial\pi(t,\theta)}{\partial\theta} + \frac{\pi^2\rho^3(t,\theta)}{6} + V_{eff}(\theta)\rho(t,\theta)\right), \quad S(\theta) = \partial_\theta\pi$$

Chern-Simons matter systems : The partition function

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- We consider CS theory on $S^2 \times S^1$ interacting with matter in fundamental representations. Partition function of this theory is given by

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$$\mathcal{Z} = \int [\mathcal{D}A][\mathcal{D}\mu] e^{i\frac{k}{4\pi}\mathbf{Tr}\int(AdA+\frac{2}{3}A^3)-S_{matter}}, \quad D\mu \text{ is the matter field measure}$$

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$$\mathcal{Z}=\int [\mathcal{D}U] \exp\left[S(U)\right], \quad S(U): \text{ Action, form depends on matter field}$$

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- However, eigenvalue density has an upper-cap

$$\boxed{0 \leq \rho(\theta) \leq \frac{1}{2\pi\lambda.}} \qquad \lambda = \frac{N}{k+N}$$

Chern-Simons matter systems : Gross-Witten-Wadia matter

Jain,Minwalla,Sharma,Takimi,Wadia, Yokoyama; Takimi

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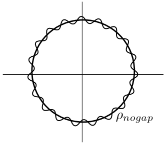

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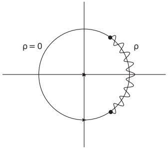

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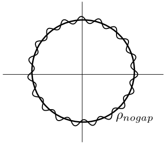

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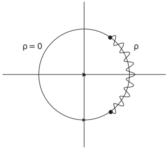

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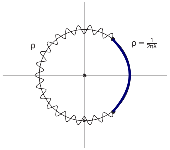
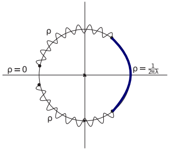


4. Cap-gap phase



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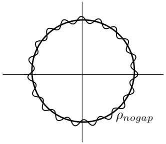

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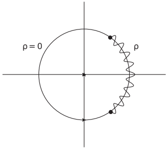

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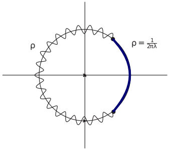

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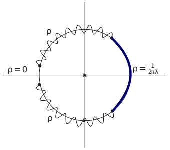

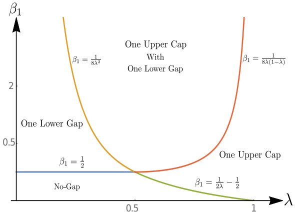

Representation for phases of CS-M theory

[A. Chattopadhyay, P.Dutta, SD 2018]

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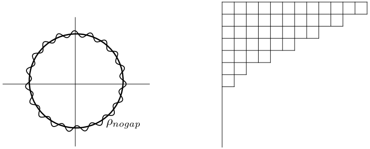

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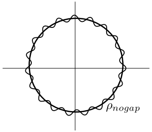

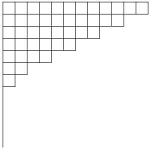

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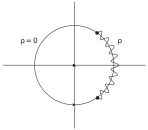

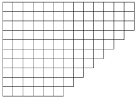

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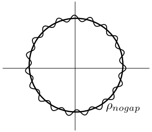

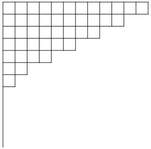

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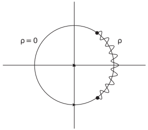

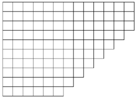

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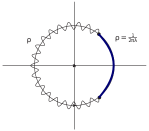

?

Constraint on Young diagrams for CS-M theory on $S^2 \times S^1$

[A. Chattopadhyay, P.Dutta, SD 2018]

$$\rho(\theta) = \frac{h_+ - h_-}{2\pi}$$

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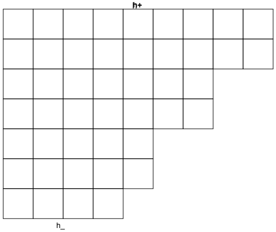

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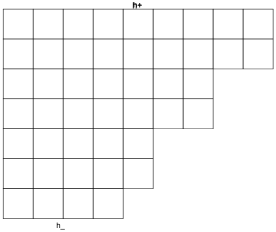

- **We see that the eigenvalue density is related to spread (support) of $u(h)$ i.e. width of Young distribution. Thus an upper cap on eigenvalue density puts restriction on spread of $u(h)$ or width of Young diagrams. Since, spread of Young distribution is identified with $2\pi\rho(\theta)$, we *claim* that**

for CS-matter theory on $S^2 \times S^1$ the dominant representations have a Young distribution function with maximum spread $1/\lambda$.

Constraint on Young diagrams for CS-M theory on $S^2 \times S^1$

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What does it mean ?

Constraint on Young diagrams for CS-M theory on $S^2 \times S^1$

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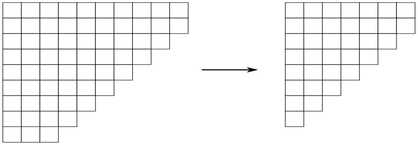
- **Reduced Young diagrams :** **The diagram on the right is obtained by taking conjugation of conjugation of the first diagram.**



- **Conjugation of conjugation of any representation is the representation itself.**
- **Hence both the diagrams describe the same representation.**

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$$h_1 < k \quad \textbf{which implies} \quad h(0) < \frac{1}{\lambda}, \quad (h(0) = h_1/N).$$

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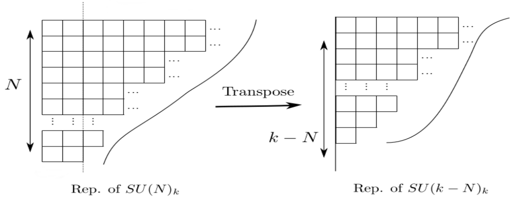
- **Eigenvalues are discrete with minimum spacing $2\pi/k$. This implies that corresponding reciprocal space (h space) is periodic and the size of first Brillouin zone is k ($h_{\mathbf{max}} = k$) - consistent with our claim.**

Representation for upper-cap phase for CS-M theory

[A. Chattopadhyay, P.Dutta, SD 2018]

- We use the level rank duality to find the dominant representation for the upper cap.
- The Young diagrams for these representations are related to each other by transpositions i.e. the interchange of rows and columns.
- We know that Lower gap is mapped to upper cap via level rank duality.
- Therefore, the dominant representations for upper-cap phase can be obtained by transposing the dominant representations for the lower-gap phase.



Representation for phases of CS-M theory

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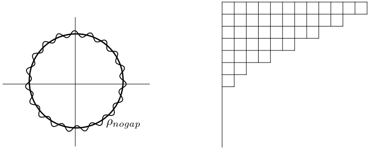

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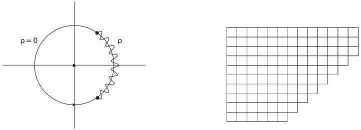

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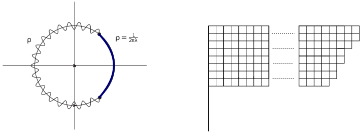

Outlook

- Constant phase space distribution is very similar to Thomas-Fermi (TF) model at zero temperature, where Wigner distribution is assumed to take constant value inside some region in phase space and zero elsewhere.
- We are yet to understand underlying many-body quantum mechanics associated with different large N phases of matrix model.
- It would also be interesting to understand correspondence between a low temperature fluctuation in a many body quantum system and fluctuations in interacting gauge theory.

$$\omega(h,\theta) = \frac{1}{e^{\beta(H(h,\theta)-\mu)} + 1}.$$

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- Chern-Simons gauge theory on S^3 is dual to topological closed string theory on resolved conifold (Gopakumar-Vafa). Does phase space description of CS theory will help us to understand a correspondence between topological string theory and quantum mechanics of many-body system ?
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Thank you