Exact Dynamics for the Entanglement Entropy

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Overview of the talk

- Introduction
- Entanglement dynamics of an interacting many-body system
- Interacting oscillators and the Bose-Hubbard model
- Solution of time dependent SHO
- Entanglement dynamics
- Concluding remarks

Motivation

- Understanding the out of equilibrium quantum many-body systems is one of the central problems for both theoretical and experimental physics.
- Thermalization of isolated quantum systems, many-body localization and related concepts are at the forefront of research today.
- Entanglement entropy and entanglement dynamics play a key role in many of these ideas.
- Entanglement entropy for the bosonic systems was realized experimentally by R. Islam, *et al.*, Nature, (2015) and A. M. Kaufman *et al.*, Science, (2016).
- With recent advances in cold atoms and optical lattices it is plausible that detailed predictions of entanglement dynamics may be amenable to experimental studies.

Motivation

Measuring entanglement entropy in a quantum many-body system

Rajibul Islam[1], Ruichao Ma[1], Philipp M. Preiss[1], M. Eric Tai[1], Alexander Lukin[1], Matthew Rispoli[1] & Markus Greiner[1]

Entanglement is one of the most intriguing features of quantum mechanics. It describes non-local correlations between quantum objects, and is at the heart of quantum information sciences. Entanglement is now being studied in diverse fields ranging from condensed matter to quantum gravity. However, measuring entanglement remains a challenge. This is especially so in systems of interacting delocalized particles, for which a direct experimental measurement of spatial entanglement has been elusive. Here, we measure entanglement in such a system of itinerant particles using quantum interference of many-body twins. Making use of our single-site-resolved control of ultracold bosonic atoms in optical lattices, we prepare two identical copies of a many-body state and interfere them. This enables us to directly measure quantum purity, Rényi entanglement entropy, and mutual information. These experiments pave the way for using entanglement to characterize quantum phases and dynamics of strongly correlated many-body systems.

Ref: R. Islam, *et al.*, Nature, (2015)

Motivation

Quantum thermalization through entanglement in an isolated many-body system

Adam M. Kaufman, M. Eric Tai, Alexander Lukin, Matthew Rispoli, Robert Schittko, Philipp M. Preiss, Markus Greiner*

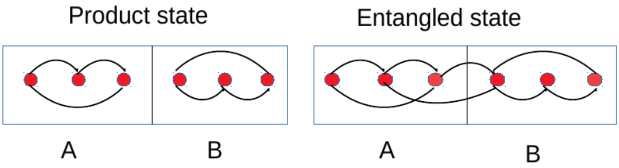
Statistical mechanics relies on the maximization of entropy in a system at thermal equilibrium. However, an isolated quantum many-body system initialized in a pure state remains pure during Schrödinger evolution, and in this sense it has static, zero entropy. We experimentally studied the emergence of statistical mechanics in a quantum state and observed the fundamental role of quantum entanglement in facilitating this emergence. Microscopy of an evolving quantum system indicates that the full quantum state remains pure, whereas thermalization occurs on a local scale. We directly measured entanglement entropy, which assumes the role of the thermal entropy in thermalization. The entanglement creates local entropy that validates the use of statistical physics for local observables. Our measurements are consistent with the eigenstate thermalization hypothesis.

Ref: A. M. Kaufman *et al.*, Science, (2016)

Motivation

- Another motivation of this study arises from black hole physics.
- The black hole horizon provides a bipartition of space-time into exterior and interior region.
- With respect to the exterior region, the black hole behaves as a thermodynamic object.
- Bekenstein-Hawking entropy law gives $S \propto A$.
- This can be interpreted as entanglement entropy [Sorkin (1986), Srednicki (1993)].
- It is interesting to ask how the system would behave in a non-equilibrium situation.

Introduction



- State of system is described by

$$\psi_{AB}$$

- Product state if

$$\psi_{AB} = \psi_A \otimes \psi_B$$

- Entangled state if

$$\psi_{AB} \neq \psi_A \otimes \psi_B$$

Introduction

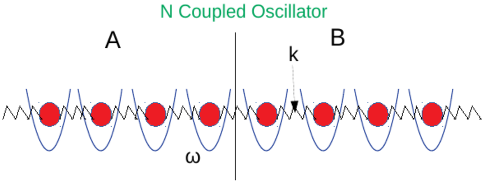
- Density matrix of a system is defined as $\rho = |\psi >< \psi|$
- The reduced density matrix of any subsystem is

$$\rho_{A/B} = Tr_{B/A}(\rho)$$

- The von Neumann entropy is a measure of quantum entanglement

$$S_1 \equiv -Tr(\rho_A \log \rho_A) = -\sum_i (p_i \log p_i)$$

Model



- The Hamiltonian of the many-body system is given by

$$H^N(t) = \frac{1}{2}\left[\sum_{j=1}^{N}(p_j^2 + \omega^2(t)x_j^2) + k(t)\sum_{j=1}^{N-1}(x_j - x_{j+1})^2\right]$$

$$= \frac{1}{2}\left[\sum_{j=1}^{N} p_j^2 + X^T.K(t).X\right]$$

- where ω is the trapping frequency and k is the interaction strength.

$N = 2$

- We will first discuss the case for $N = 2$

$$H(t) = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2(t)(x_1^2 + x_2^2) + k(t)(x_1 - x_2)^2\right],$$

which could be mapped to the two site Bose-Hubbard model

$$H = \omega_{BH}(a_1^\dagger a_1 + a_2^\dagger a_2) - J(a_1^\dagger a_2 + a_2^\dagger a_1).$$

for $\omega = (\omega_{BH} - J)$ and $k = 2\omega_{BH} J$.

- The above Hamiltonian could be written as two independent harmonic oscillator with time dependent frequencies

$$H(t) = \frac{1}{2}\left[p_+^2 + p_-^2 + \omega_+^2(t)x_+^2 + \omega_-^2(t)x_-^2\right],$$

where $p_\pm = \frac{p_1 \pm p_2}{\sqrt{2}}$ and $x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}}$ and with $\omega_+ = \omega$ and $\omega_- = \sqrt{\omega^2 + 2k}$

TDSE

- A time dependent Hamiltonian satisfies the time dependent Schrödinger equation (TDSE)

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle$$

- In order to find the solutions, it is necessary to look for an invariant Hermetian operator I such that

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0$$

- The solutions of the TDSE can be constructed from the eigenstates of the operator I [Lewis and Riesenfeld (1969)].
- For a given Hamiltonian there is no general procedure to find the invariant. If H is an element of a dynamical algebra, then I can be expanded in terms of its elements with time dependent factors.

Time dependent SHO

- The Hamiltonian for a single time dependent oscillator is

$$H(t) = \frac{p^2}{2m} + \frac{m\omega^2(t)}{2}x^2$$

- The corresponding time dependent Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle$$

- There exists a time invariant Hermitian operator $I(t)$ satisfying

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0$$

- By using above two equations we obtain

$$i\hbar\frac{\partial}{\partial t}(I|\psi\rangle) = H(t)(I|\psi\rangle)$$

Time dependent SHO

- Solution of the Schrödinger equation can be written as,

$$\psi_n(x,t) = e^{i\kappa(t)}\phi_n^I(x,t)$$

where $\phi^I(x,t)$ is an eigenstate of I and $\kappa(t)$ is a real function of time which satisfies the equation given by

$$\frac{d\kappa}{dt} = \langle\phi|\left(i\hbar\frac{\partial}{\partial t} - H\right)|\phi\rangle$$

- Let us use following ansatz for invariant operator in the quadratic form as

$$I(t) = \frac{1}{2}[\alpha(t)x^2 + \beta(t)p^2 + \gamma(t)(px + xp)]$$

where α, β, γ are the dimensionful factors

Time dependent SHO

- The equations satisfied by α, β, γ given by

$$\dot{\alpha} = 2m\omega^2\gamma, \;\; \dot{\beta} = -\frac{2}{m}\gamma, \;\; \dot{\gamma} = -\frac{1}{m}\alpha + m\omega^2\beta$$

- If we now introduce a new real function $b(t)$ as

$$\beta(t) = b^2(t)$$

- Using the above equations we get

$$b\frac{d}{dt}(m^2\ddot{b} + m^2\omega^2 b) + 3\dot{b}(m^2\ddot{b} + m^2\omega^2 b) = 0$$

- The integration of the above equation will produce a nonlinear differential equation

$$\ddot{b} + \omega^2(t)b = \frac{\omega^2(0)}{b^3}$$

Time dependent SHO

- The invariant operator now can be written as

$$I = \frac{1}{2}\left[\frac{m^2\omega(0)^2}{b^2}x^2 + (bp - m\dot{b}x)^2\right]$$

- In terms of the operators

$$\hat{a} = 2^{-\frac{1}{2}}\left[\frac{\sqrt{m\omega(0)}}{b}x + i\frac{(bp - m\dot{b}x)}{\sqrt{m\omega(0)}}\right],$$

$$\hat{a}^\dagger = 2^{-\frac{1}{2}}\left[\frac{\sqrt{m\omega(0)}}{b}x - i\frac{(bp - m\dot{b}x)}{\sqrt{m\omega(0)}}\right].$$

the invariant I has the form

$$I = (\hat{a}^\dagger\hat{a} + \frac{1}{2})m\omega_0.$$

Time dependent SHO

- If $\phi_0^I(x,t)$ denotes the ground state of $\hat{a}$ in position basis, then

$$\hat{a}\phi_0^I(x,t) = 0,$$

- Thus the ground state of I is

$$\phi_0^I(x,t) = \left(\frac{m\omega(0)}{\pi b^2}\right)^{\frac{1}{4}} e^{-\left(\frac{m\omega(0)}{2b^2} - im\frac{\dot{b}}{2b}\right)x^2}$$

- All the excited states are accessed by acting the creation operator on the ground state, the n^{th} order eigenstate is

$$\phi_n^I(x,t) = \left(\frac{m\omega(0)}{2^{2n}n!^2\pi b^2}\right)^{\frac{1}{4}} H_n\left(x\sqrt{\frac{m\omega(0)}{b^2}}\right) e^{\frac{im\dot{b}}{2b}x^2} e^{-\frac{m\omega(0)x^2}{2b^2}}$$

Time dependent SHO

- Now the phase factor $\kappa(t)$ using Schrödinger equation given by

$$\kappa(t) = -\left(n+\frac{1}{2}\right)\omega(0)\int_0^t \frac{1}{b^2(t')}dt',$$
$$= -\left(n+\frac{1}{2}\right)\omega(0)\tau.$$

- Hence the ground state solution at time t is

$$\psi_{gs}(x,t) = e^{i\left(\frac{\dot{b}}{2b}x^2 - E_0\tau\right)}\psi_{gs}\left(\frac{x}{b},0\right),$$

where $b(t)$ is a scaling parameter satisfies Ermakov equation and $\tau = \int_0^t \frac{dt'}{b^2(t')}$ and E_0 is the ground state energy of the oscillator at time $t=0$.

Detailed steps for $N = 2$

- The TDSE of the two oscillator Hamiltonian is given by,

$$i\frac{\partial\psi(x_1,x_2,t)}{\partial t} = H(t)\psi(x_1,x_2,t).$$

- The full time dependent wave function can be found as

$$\psi(x_1,x_2,t) = \tilde{A}(t)\exp\Big[i\Big(a_1x_1^2 + a_1x_2^2 + 2a_2x_1x_2\Big)\Big] \times \exp\Big[-i\Big(E_+\tau_+ + E_-\tau_-\Big)\Big]$$
$$\times \exp\Big[-\frac{1}{4b_1^2(t)}\omega_+(0)(x_1+x_2)^2\Big] \times \exp\Big[-\frac{1}{4b_2^2(t)}\omega_-(0)(x_1-x_2)^2\Big]$$

where $\tilde{A}(t) = \frac{(\omega_+(0)\omega_-(0))^{1/4}}{\sqrt{\pi b_1(t)b_2(t)}}$, $a_1(t) = (\frac{\dot{b}_1}{4b_1} + \frac{\dot{b}_2}{4b_2})$, $a_2(t) = (\frac{\dot{b}_1}{4b_1} - \frac{\dot{b}_2}{4b_2})$ and $\omega_+ = \omega$ and $\omega_- = \sqrt{\omega^2 + 2k}$.

Solution of Ermakov equation

- Here $b_1(t)$, $b_2(t)$ satisfy the nonlinear Ermakov equations given by

$$\ddot{b}_j + \omega_{\pm}^2(t) b_j = \frac{\omega_{\pm}^2(0)}{b_j^3}$$

- Boundary condition $b_j(0) = 1$, $\dot{b}_j(0) = 0$.
- Then the general solution of $b_j(t)$ can be written by a nonlinear superposition principle *(Ref: E. Pinney, (1950))*,

$$b_j(t) = \sqrt{Au(t)^2 + Cv(t)^2 + 2Bu(t)v(t)}$$

where $u(t), v(t)$ are the linearly independent solutions of the Hill's equations, A, B, C are constants, related by $AC - B^2 = \frac{\omega^2(0)}{W^2}$ and $W = \dot{u}v - \dot{v}u$ is the Wronskian.

Solution of Ermakov equation

- For example we consider a particular form of $\omega(t) = \omega_i\theta(-t) + \omega_f\theta(t)$.
- Now the corresponding Hill's equation is

$$\ddot{x} + \omega_f^2 x = 0.$$

The two linearly independent solutions of the above equation are

$$u(t) = e^{i\omega_f t},$$
$$v(t) = e^{-i\omega_f t},$$

The Wronskian is, $W = u\dot{v} - v\dot{u} = -2i\omega_f$

Solution of Ermakov equation

- Therefore the general solution can be written as

$$b(t) = \sqrt{Ae^{2i\omega_f t} + Ce^{-2i\omega_f t} + 2\sqrt{AC + \frac{\omega_i^2}{4\omega_f^2}}}$$

- Applying the boundary conditions $b_j(0) = 1$, $\dot{b}_j(0) = 0$, we get

$$b(t) = \sqrt{n_1 \cos(2\omega_f t) + n_2},$$

where $n_1 = \frac{\omega_f^2 - \omega_i^2}{2\omega_f^2}$, $n_2 = \frac{\omega_f^2 + \omega_i^2}{2\omega_f^2}$

Detailed steps for $N = 2$

- The reduced density matrix is defined as

$$\rho_{red}(x_1, x_1', t) = \int dx_2 \rho(x_1, x_2, x_1', x_2, t)$$

- The reduced density matrix is

$$\rho_{red}(x_1, x_1', t) = \pi^{-1/2}(\gamma - \beta)^{1/2} \exp\left[i(x_1^2 - x_1'^2) z(t)\right] \exp\left[-\frac{\gamma}{2}(x_1^2 + x_1'^2) + \beta x_1 x_1'\right]$$

where

$$\gamma(t) = \frac{\left(\frac{\omega_+(0)}{b_1^2(t)} + \frac{\omega_-(0)}{b_2^2(t)}\right)}{2} - \frac{\left(\frac{\omega_+(0)}{b_1^2(t)} - \frac{\omega_-(0)}{b_2^2(t)}\right)^2 - \left(\frac{\dot{b}_1}{b_1} - \frac{\dot{b}_2}{b_2}\right)^2}{4\left(\frac{\omega_+(0)}{b_1^2(t)} + \frac{\omega_-(0)}{b_2^2(t)}\right)},$$

$$\beta(t) = \frac{\left(\frac{\omega_+(0)}{b_1^2(t)} - \frac{\omega_-(0)}{b_2^2(t)}\right)^2 + \left(\frac{\dot{b}_1}{b_1} - \frac{\dot{b}_2}{b_2}\right)^2}{4\left(\frac{\omega_+(0)}{b_1^2(t)} + \frac{\omega_-(0)}{b_2^2(t)}\right)},$$

$$z(t) = \left(\frac{\dot{b}_1}{4b_1} + \frac{\dot{b}_2}{4b_2}\right) - \frac{\frac{\omega_+(0)}{b_1^2(t)} - \frac{\omega_-(0)}{b_2^2(t)}}{\frac{\omega_+(0)}{b_1^2(t)} + \frac{\omega_-(0)}{b_2^2(t)}}\left(\frac{\dot{b}_1}{4b_1} - \frac{\dot{b}_2}{4b_2}\right).$$

Detailed steps for $N = 2$

- The eigenvalue equation satisfied by ρ_{red} is

$$\int_{-\infty}^{\infty} dx_1' \rho_{red}(x_1, x_1', t) f_n(x_1', t) = p_n(t) f_n(x_1, t)$$

- The eigenvalue has the following form

$$p_n(t) = (1 - \xi(t))\xi(t)^n$$

- The time-dependence of $\xi(t)$ is given by,

$$\xi(t) = \frac{\beta}{\gamma + \epsilon} = \frac{\frac{\beta}{\gamma}}{1 + \sqrt{1 - \frac{\beta^2}{\gamma^2}}} < 1,$$

- The von Neumann entropy can be now written as

$$S_1(t) = -\log(1 - \xi(t)) - \frac{\xi(t)}{1 - \xi(t)} \log \xi(t).$$

Detailed steps for $N = 2$

- This expression of von Neumann entropy is true for arbitrary time dependence in the system
- For the time independent frequencies, the entropy reduces to that derived by Sorkin et al (1986) and Srednicki (1993) in the context of black holes.
- Next we will consider two different cases of quench.

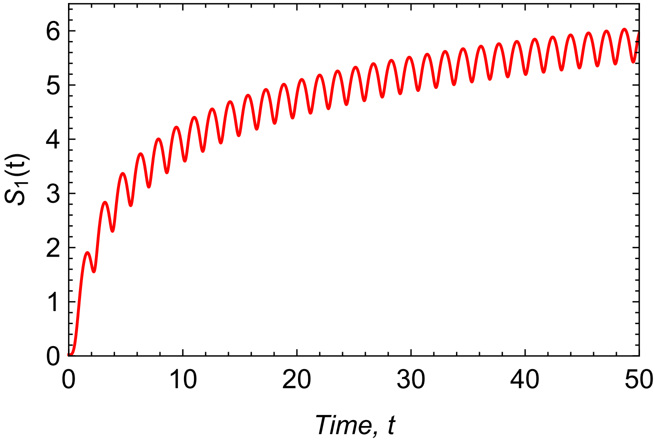
Quench in $N = 2$

- The time dependence is given by sudden change in the parameters.
- The solutions of Ermakov equations for a sudden quench in ω from $\omega(i) = \omega \to \omega(f) = 0$ are

$$b_1(t) = \sqrt{1 + \omega^2 t^2},$$
$$b_2(t) = \sqrt{n_2 \cos(2kt) + m_2},$$

where $n_2 = \frac{2k-(\omega^2+2k)}{4k}$ and $m_2 = \frac{2k+(\omega^2+2k)}{4k}$.

Von Neumann Entropy for N=2



- We start with $\omega(i) = 3$ and $k = 2$ then quench $\omega(i)$ to $\omega(f) = 0$.

Quench in $N = 2$

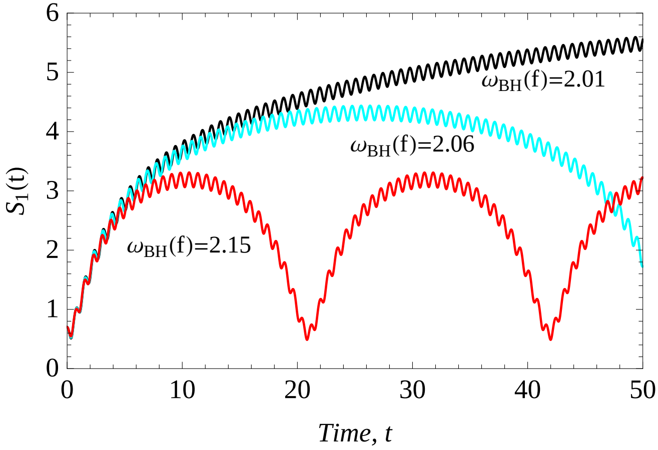
- Here we do the quench in the Bose-Hubbard parameters.
- The solutions of Ermakov equations for a sudden quench in ω_{BH} $(\omega_{BH}(i) \to \omega_{BH}(f))$ is

$$b_1(t) = \sqrt{n_1 \cos(2(\omega_{BH}(f) - J)t) + m_1},$$
$$b_2(t) = \sqrt{n_2 \cos(2(\omega_{BH}(f) + J)t) + m_2},$$

where $n_1 = \frac{(\omega_{BH}(f)-J)^2-(\omega_{BH}(i)-J)^2}{2(\omega_{BH}(f)-J)^2}$, $m_1 = \frac{(\omega_{BH}(f)-J)^2+(\omega_{BH}(i)-J)^2}{2(\omega_{BH}(f)-J)^2}$

$n_2 = \frac{(\omega_{BH}(f)+J)^2-(\omega_{BH}(i)+J)^2}{2(\omega_{BH}(f)+J)^2}$ and $m_2 = \frac{(\omega_{BH}(f)+J)^2+(\omega_{BH}(i)+J)^2}{2(\omega_{BH}(f)+J)^2}$.

Von Neumann Entropy for N=2



- We start with $\omega_{BH}(i) = 3$ and $J = 2$ then quench $\omega_{BH}(i)$ to $\omega_{BH}(f) = 2.15, 2.06$ and 2.01.

Time-dependent entangled system

- The Hamiltonian for N coupled oscillators with time dependent parameters is given by

$$H^N(t) = \frac{1}{2}\left[\sum_{j=1}^{N} p_j^2 + X^T.K(t).X\right]$$

where $X{=}(x_1, x_2 ... x_N)^T$ and K is a real symmetric $N \times N$ matrix with real eigenvalues.

- The time dependent density matrix of the whole system has the form

$$\rho(X, X', t) = \left(\det\frac{\Omega}{\pi}\right)^{\frac{1}{2}} \exp\left[i\left(X^T\tilde{b}X - X'^T\tilde{b}X'\right)\right]$$

$$\exp\left[-\frac{X^T\Omega X}{2} - \frac{X'^T\Omega X'}{2}\right]$$

here $\Omega = U^T\sqrt{K'^D}U$, $K'^D_{jj} = \frac{K^D_{jj}(0)}{b_j^4(t)}$, $\tilde{b} = U^T\tilde{b}^D U$

Reduced system and entropy

- Tracing over the subsystem X^α, the reduced density matrix of X^a is given by

$$\rho_{red}(X^a, X'^a, t) = \int \prod_{\alpha=1}^{n} dX^\alpha \rho(X^a, X^\alpha, X'^a, X^\alpha)$$

- We write the matrices Ω and $\tilde{b}$ respectively as

$$\Omega = \begin{pmatrix} \Omega_{n\times n} & \Omega_{n\times N-n} \\ \Omega^T_{n\times N-n} & \Omega_{N-n\times N-n} \end{pmatrix}$$

$$\tilde{b} = \begin{pmatrix} \tilde{b}_{n\times n} & \tilde{b}_{n\times N-n} \\ \tilde{b}^T_{n\times N-n} & \tilde{b}_{N-n\times N-n} \end{pmatrix}$$

Reduced system and entropy

- Using these and after some algebra, we get

$$\rho_{red}(X^a, X'^a, t)$$
$$= \left(\frac{\det\frac{\Omega}{\pi}}{\det\frac{\Omega_{n\times n}}{\pi}}\right)^{\frac{1}{2}} \exp\left[i\left(X^{aT} Z X^a - X'^{aT} Z X'^a\right)\right]$$
$$\times \exp\left[-\frac{1}{2}\left(X^{aT}\gamma X^a + X'^{aT}\gamma X'^a\right) + X^{aT}\beta X'^a\right]$$

  where X^a, X'^a has $N-n$ components

- $Z(t)$, γ, β are $(N-n)\times(N-n)$ matrices given by

Reduced system and entropy

$$Z(t) = \tilde{b}_{N-n\times N-n} - \tilde{b}^T_{n\times N-n}\Omega^{-1}_{n\times n}\Omega_{n\times N-n},$$

$$\gamma(t) = \Omega_{N-n\times N-n} - \frac{1}{2}\Omega^T_{n\times N-n}\Omega^{-1}_{n\times n}\Omega_{n\times N-n} + 2\tilde{b}^T_{n\times N-n}\Omega^{-1}_{n\times n}\tilde{b}_{n\times N-n},$$

$$\beta(t) = \frac{1}{2}\Omega^T_{n\times N-n}\Omega^{-1}_{n\times n}\Omega_{n\times N-n} + 2\tilde{b}^T_{n\times N-n}\Omega^{-1}_{n\times n}\tilde{b}_{n\times N-n}$$

- The reduced density matrix in new coordinates takes the form

$$\rho_{red}(R^a, R'^a, t) = \exp\left[iR^{aT}Z''R^a - iR'^{aT}Z''R'^a\right] \times \prod_{j=n+1}^{N} \frac{\left(1-\tilde{\beta}_j\right)^{\frac{1}{2}}}{\pi^{N-n}} \exp\left[-\frac{1}{2}(r_j^2 + r_j'^2) + \tilde{\beta}_j r_j r_j'\right]$$

Time-dependent many-body system

- The eigenvalues are given by

$$p_l(t) = \prod_{j=n+1}^{N} (1 - \xi_j)\xi_j^l$$

- where

$$\xi_j(t) = \frac{\tilde{\beta}_j}{1 + \sqrt{1 - \tilde{\beta}_j^{\,2}}}$$

- In this case there will be N number of $b_j(t)$ satisfying Ermakov equations given by

$$\ddot{b}_j + \lambda_j(t) b_j = \frac{\lambda_j(0)}{b_j^3}$$

where λ_j are eigenvalues of K which dependence on ω and k.

Time-dependent many-body system

- The von Neumann entropy takes the form

$$S_1(t) = \sum_{j=1}^{N-n} \left[-\log(1-\xi_j(t)) - \frac{\xi_j(t)}{1-\xi_j(t)} \log \xi_j(t) \right]$$

after partitioning the system to n versus $N-n$ degrees of freedom.

Entropy plots for N=4

- As a specific example, we now consider a chain of $N = 4$ oscillators.
- The Hamiltonian is given by

$$H(t) = \frac{1}{2}\left[\sum_{j=1}^{4}(p_j^2 + \omega^2(t)x_j^2) + k(t)\sum_{j=1}^{4}(x_j - x_{j+1})^2\right]$$

- We consider the periodic boundary condition given by $x_5 = x_1$. The matrix K has the form

$$K = \begin{pmatrix} \omega^2 + 2k & -k & 0 & -k \\ -k & \omega^2 + 2k & -k & 0 \\ 0 & -k & \omega^2 + 2k & -k \\ -k & 0 & -k & \omega^2 + 2k \end{pmatrix}$$

Entropy plots for N=4

- Eigenvalues are given by $\lambda_j = (\omega^2 + 2k) - 2k\cos\left(\frac{2\pi j}{4}\right)$.
- The corresponding eigenfunctions are

$$\hat{e}_j = N^{-1/2} \begin{pmatrix} 1 \\ \exp(2\pi ij/4) \\ \exp(4\pi ij/4) \\ \exp(6\pi ij/4) \end{pmatrix}$$

where $j = 1, 2, 3, 4$.

- Note that there are only three distinct eigenvalues $(\lambda_1 = \lambda_3)$.
- Hence there will be only three distinct Ermakov equations $(b_1(t) = b_3(t))$.

Entropy plots for N=4

- The matrices U, $\tilde{b}^D$ and $(K'^D)^{\frac{1}{2}}$ are given by

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -1 & -i & 1 \\ -1 & 1 & -1 & 1 \\ -i & -1 & i & 1 \end{pmatrix}$$

$$\tilde{b}^D = \begin{pmatrix} \frac{\dot{b}_1(t)}{2b_1(t)} & 0 & 0 & 0 \\ 0 & \frac{\dot{b}_2(t)}{2b_2(t)} & 0 & 0 \\ 0 & 0 & \frac{\dot{b}_1(t)}{2b_1(t)} & 0 \\ 0 & 0 & 0 & \frac{\dot{b}_4(t)}{2b_4(t)} \end{pmatrix}$$

$$(K'^D)^{1/2} = \begin{pmatrix} \frac{\sqrt{\omega^2+2k}}{b_1^2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\omega^2+4k}}{b_2^2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\omega^2+2k}}{b_1^2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\omega^2}}{b_4^2} \end{pmatrix}$$

Entropy plots for N=4

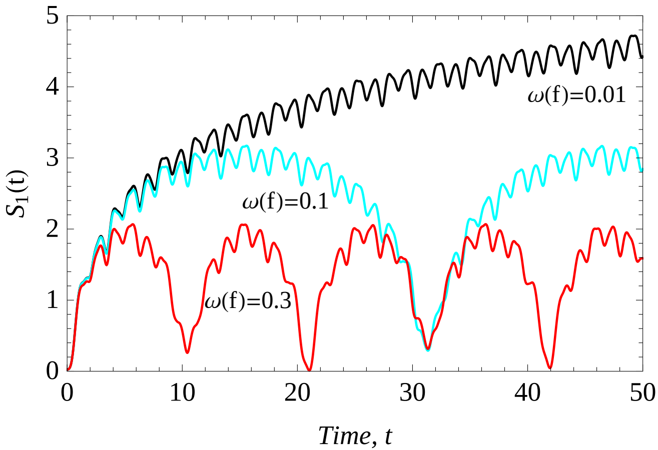
- We now perform a sudden quench at time $t = 0$, when ω, k change from a constant values $(\omega(i), k(i))$ to $(\omega(f), k(f))$.
- The reduced system is defined by tracing out the last two oscillators in the chain.
- In this case there are four Ermakov equations whose solutions (with $b_j(t=0) = 1$ and $\dot{b}_j(t=0) = 0$) are given by

$$b_j(t) = \sqrt{n_j \cos(2\sqrt{\lambda_j(f)}t) + m_j}$$

here $n_j = \frac{\lambda_j(f) - \lambda_j(i)}{2\lambda_j(f)}$, $m_j = \frac{\lambda_j(f) + \lambda_j(i)}{2\lambda_j(f)}$

- $\lambda_j(i)$, $\lambda_j(f)$ are the eigenvalues of K before and after the quench.

Entropy plots for N=4

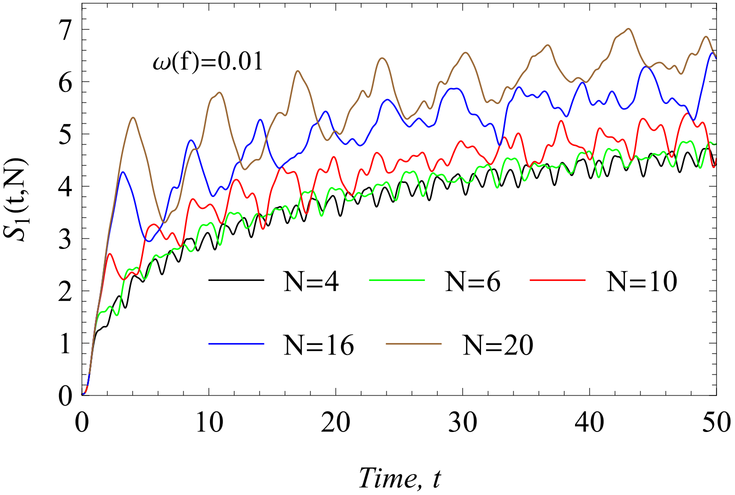


- Initial values $\omega(i) = 3$ and $k(i) = 2$.

Scaling of entropy

- where $n_j = \frac{\lambda_j(f)-\lambda_j(i)}{2\lambda_j(f)}$, $m_j = \frac{\lambda_j(f)+\lambda_j(i)}{2\lambda_j(f)}$
- Three independent time scales contribute to the entanglement dynamics.
- This plot shows entanglement revival whose time period increases with decreasing $\omega(f)$.
- Each revival period contains several quasi-revivals on shorter time scales due to the effect of the Ermakov solutions
- At large times, the profile of the entanglement dynamics is dominated by the smallest frequency, which being independent of interaction k.
- Existence of multiple time scales within each revival period is a new feature due to the solutions of the Ermakov equations.

Entropy plots for different N

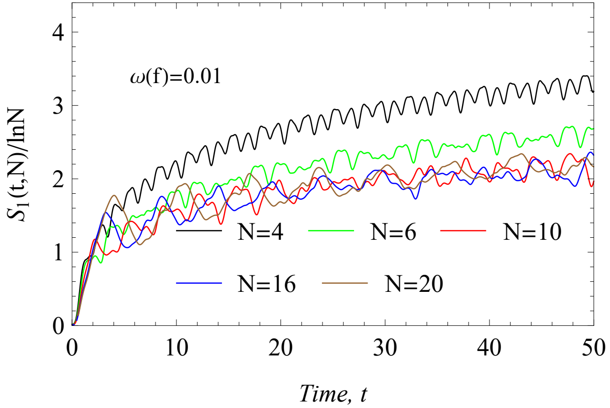


- Plot of von Neumann entropy with different N and same quenched parameters $\omega(f) = 0.01$

Entropy plots for different N

- The time evolution of $S_1(t, N)$ shows the effect of multiple time scales whose number increases with N.
- In addition, the von Neumann entropy itself increases as a function of N.
- In order to extract the N dependence of the entropy, in next section we have plotted the ratio $\frac{S_1(t,N)}{\ln N}$ as a function of time t for several N.

Scaling of entropy



- Plot of von Neumann entropy with different N and scaled by $log(N)$

Scaling of entropy

- Now we plot the ratio $\frac{S_1(t,N)}{\ln N}$ as a function of time t for several N.
- It is clear that for $N \geq 10$, the nature of this plot is consistent with the scaling relation

$$S_1(t, N) = c(t) \ln N + O(1)$$

- where $c(t)$ is a time dependent function that encodes the cumulative effect of the dynamically generated multiple time scales.

Connection to criticality

- The Hamiltonian

$$H^N(t) = \frac{1}{2}\left[\sum_{j=1}^{N}(p_j^2 + \omega^2(t)x_j^2) + k(t)\sum_{j=1}^{N-1}(x_j - x_{j+1})^2\right]$$

- Under the following canonical transformation

$$(x_j, p_j) \longrightarrow (k^{1/4}x_j, p_j/k^{1/4}),$$

- Introducing $a = \sqrt{m/k}$ $(m = 1)$, Hamiltonian takes the form

$$H^N(t) = \frac{1}{2}\left[\sum_{j=1}^{N}(\frac{p_j^2}{a} + a\omega^2(t)x_j^2) + \frac{1}{a}\sum_{j=1}^{N-1}(x_j - x_{j+1})^2\right]$$

- Lattice discretization of a free boson with lattice spacing a and mass ω.

Connection to criticality

- In the limit $a \to 0$, $N \to \infty$ we can replace

$$x_j \to \phi(x), \quad \frac{p_j}{a} \to \pi(x) = \dot{\phi}(x), \quad \text{with } x = na$$

- Hamiltonian reduces to the two-dimension Euclidean action given by

$$Action, \quad S = \frac{1}{2} \int dx \int d\tau \left[(\partial_\mu \phi)^2 + \omega^2 \phi^2 \right].$$

- In the limit $\omega \longrightarrow 0$ it is conformally invariant
- Therefore when $\omega \to 0$ and large N above Hamiltonian becomes critical.

[1]

[1] P. Calabrese *et al.*, (2013).

Lieb Robinson bound

- The wavefunction for this N coupled oscillator model with time dependent coefficients is

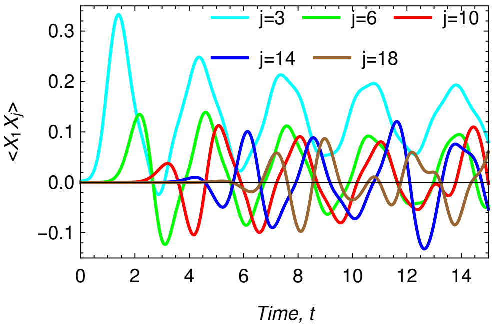
$$\psi(x_1, ..., x_N, t) = \left(\det\frac{\Omega}{\pi}\right)^{\frac{1}{4}} \exp\left[i\left(X^T\tilde{b}X - \sum_{j=1}^{N} E_j\tau_j\right)\right] \times \exp\left[-\frac{X^T\Omega X}{2}\right]. \qquad (1)$$

here $\Omega = U^T\sqrt{K'^D}U,\ K'^D_{jj} = \frac{K^D_{jj}(0)}{b^4_j(t)},\ \tilde{b} = U^T\tilde{b}^D U$

- The equal time correlation has the form

$$< x_i(t)x_j(t) >= \frac{1}{2}\sum_{m=1}^{N} U_{mi}\left(\frac{1}{\sqrt{K'^D(t)}}\right)U_{mj}$$

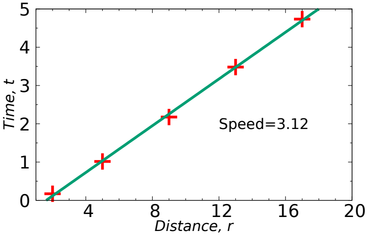
Lieb Robinson bound



- Here we have quenched from $\omega(i) = 3$ and $k(i) = 2$ to $\omega(f) = 2$ and $k(f) = 2.5$
- It takes finite time to propagate correlation from site i to site $(i + r)$

Lieb Robinson bound

- The plot of this distance versus time of propagation gives us propagation velocity with a finite bound
- The bound in the entanglement propagation speed for harmonic chain system is given by (for quench from $\omega(i) = 3$, $k(i) = 2$ to $\omega(f) = 2$, $k(f) = 2.5$)



This shows a finite speed of entanglement propagation

Summary

- We have obtained exact analytical expressions of von Neumann entropy for any arbitrary time-dependence.
- The entanglement dynamics is characterized by a multi-oscillatory behaviour and the number of time scales appearing in the entanglement dynamics increases with N.
- We saw that the entropy for this system violates the area law.
- In the critical limit there is a logarithmic scaling of the entanglement entropy.
- This method can be used to obtain exact solutions to a variety of quenches, which are under investigation.

Thank You