Gravity in Higher Dimensions and Its Equation

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Gravity is Different

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- It is universal, links to all, including $m = 0$.
- Hence it can only be described by spacetime geometry — Curvature.
- No other force makes such a demand on spacetime.
- Its dynamics resides in and driven by the Riemann curvature, Can't be prescribed.
- Could it remain confined to four dimensions?

At the Beginning: "Free" State of Space and Time

- Space is homogeneous and isotropic: $x \leftrightarrow y$.
- Time is homogeneous
- Space and Time both are homogeneous: Why not $x \leftrightarrow t$?
- Dimensions don't match, make them match: $x \leftrightarrow ct$.
- c: Universal invariant velocity is the First constant of spacetime.
- Space and time $\rightarrow$ four dimensional spacetime.

Geometry of Homogeneous Spacetime

- Geometry must be homogeneous — Curvature tensor R_{abcd} should be homogeneous, covariantly constant, $\nabla_e R_{abcd} = 0$.
- The only solution being $R_{abcd} = \Lambda\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$.
- Constant curvature Λ is the Second constant of spacetime.
- "Free" spacetime is homogeneous with constant curvature Λ, not necessarily flat Minkowski.

Inhomogeneity $\rightarrow$ Gravity

- When Spacetime is inhomogeneous $\rightarrow$ presence of force.
- Its dynamics to follow from Spacetime Curvature.
- Bianchi Identity: $\nabla_{[e}R_{ab]cd} = 0$. By taking trace, we obtain, $\nabla_a G^a_b = 0$, where $G_{ab} = R_{ab} - (1/2)Rg_{ab}$.
- This implies $G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$, $\nabla_b T^{ab} = 0$.
- T_{ab} is the matter energy-momentum tensor.

- Thus we have Einstein Gravity!
- Homogeneity $\rightarrow$ constants, c and Λ, spacetime structure.
- Inhomogeneity $\rightarrow$ Einstein's Gravity.
- Homogeneity (Force free) $\rightarrow$ Inhomogeneity (Gravity).

Kinematicity

- Einstein gravity is kinematic in $d = 3$, i.e., $R_{abcd} = R_{abcd}(R_{ab})$ and vacuum is flat.
- In $d = 2$, $G_{ab} = 0$, so trivial.
- For $d \geq 4$, the spacetime is dynamic.
- Einstein is linear in curvature $\rightarrow$ order 1 Lovelock. Lovelock is homogeneous polynomial of Riemann of degree N. $N = 1$: Einstein, $N = 2$: Gauss-Bonnet, $\cdots$.
- Universalizing kinematic property in all critical odd $d = 2N + 1$.
- Note that for Lovelock: $d = 2N$: trivial, i.e., $G^{(N)}_{ab} = 0$; $d = 2N + 1$, kinematic, i.e., $R^{N}_{abcd} = R^{(N)}_{abcd}(R^{(N)}_{ab})$; for $d \geq 2N + 2$, it is dynamic.
- We have to define Lovelock Riemann R^{N}_{abcd}.
- Thus for $N > 1$, we have $d > 4$.

Higher Dimensions: Three Motivations

1. **Flat Space Embedding**
   - If gravity remains confined to four dimensions, spacetime should be embeddable in five dimensional flat spacetime.
   - Arbitrarily curved four dimensional spacetime requires ten dimensions for its flat space embedding.
   - Gravity may penetrate down to ten dimensions!
2. **High Energy Consideration**
   - Probing high energy regime, so one should consider R^2 or/and $R_{abcd}R^{abcd}$ terms in action.
   - Yet equation of motion should be second order.
   - This uniquely singles out Lovelock polynomial Lagrangian.
   - Higher order terms are non-trivial only in $d \geq 2N+1$ — quintessential higher dimensional!

Third Motivation

- For a classical field Total charge $= 0$, – electric force.
- So must be for gravity.
- Charge energy-momentum > 0, how to neutralize it?
- Gravitational field must have opposite negative polarity, and that is why it is attractive! negative charge is not localized, spread all over space. If we integrate over whole space it will exactly counter positive charge — the famous ADM calculation.
- Consider a mass point of mass M, and a local neighbourhood 3-ball of radius R. In this ball there would be overdominance of positive charge due to M, and hence field propagates off into extra dimension. As it propagates, its past lightcone would include region outside of R which has negative charge, and hence propagation is with diminishing field strength.
- Gravity propagates in higher dimensions.
- But not deep enough! Exactly similar to brane-world model.
- That is why extra dimensions are small or compactified.

Equation in Higher Dimensions

- What should be the correct equation? — (a) Second order to avoid ghosts. (b) Lovelock in general. (c) Einstein?
- Bound orbit should exist around a static object. For Einstein that happens in $d = 4$, none else. In higher dimensions, $\phi \sim r^{-(d-3)}$, while centrifugal potential $\sim r^{-2}$. Thus $d - 3 < 2$, i.e., $d < 5$.
- So we have Lovelock $\sum_N \alpha_N L^N$, where α_N are dimensionful.
- How to fix α_N? Measuring strength can fix only one α_N.
- This leaves only pure Lovelock with only one Nth order Lovelock.
- Also can we universalize kinematic property;i.e. vacuum trivial in all $d = 2N + 1$?
- Seeking equation that admits bound orbits as well as universalizes kinematic property.

Universalizing Kinematic Property

- Define Nth order Riemann

$$\mathcal{R}^{b_1 b_2 \ldots b_{2N}}_{a_1 a_2 \ldots a_{2N}} = \delta^{b_1 b_2 \ldots b_{2N}}_{c_1 c_2 \ldots c_{2N}} \delta^{d_1 d_2 \ldots d_{2N}}_{a_1 a_2 \ldots a_{2N}} R^{c_1 c_2}_{d_1 d_2} \cdots R^{c_{2N-1} c_{2N}}_{d_{2N-1} d_{2N}}$$

- The Lovelock Lagrangian becomes,

$$L^{(N)} = \frac{1}{2^N} \delta^{a_1 b_1 \cdots a_N b_N}_{c_1 d_1 \cdots c_N d_N} R^{c_1 d_1}_{a_1 b_1} \cdots R^{c_N d_N}_{a_N b_N}$$

- yielding Lovelock-Einstein tensor

$$P^{a\ (N)}_{bmn} R^{bmn}_{c} - \frac{1}{2} L^{(N)} \delta^a_c = \kappa T^a_c$$

- Here P^a_{bcd} is the derivative of the Lagrangian with respect to Riemann.
- It turns out that for $d = 2N + 1$, Nth order Riemann is entirely given in terms of Nth order Ricci.
- This verifies universalization of kinematic property — it uniquely singles out pure Lovelock and so does the existence of bound orbit property.

Pure Lovelock Equation

- How about bound orbits? In pure Lovelock theory of order N, the gravitational potential scales as $\phi \sim r^{-(d-2N-1)/N}$. Thus we must have $(d-2N-1)/N < 2$, i.e., $d < 4N+1$.
- Thus bound orbits always exist in $d < 4N+1$, and it singles out pure Lovelock equation.

- Both kinematic and bound orbit properties uniquely pick out pure Lovelock as the proper equation in higher dimension.
- Further pure Lovelock static black hole solution is given by

$$g_{tt} = g^{rr} = 1 - \left(\Lambda r^{2N} + \frac{M}{r^{d-2N-1}}\right)^{1/N} \sim 1 - \Lambda r^2 - \frac{M}{r^{d-3}}$$

- It asymptotically approximates to Schwarzschild-dS in d dimension though there was no Einstein term in the equation!
- Universality in $d = 2N+1, 2N+2$ dimensions. Gravitational dynamics is similar Thermodynamics is universal, Entropy always goes as $\sim r_{\rm h}^2$, horizon radius square in all $d = 2N+2$.

Pulling Up

- Using kinematic property as a guide — Pure Lovelock field equation in higher dimensions. Allows bound orbits to exist.
- Total charge $= 0$:
  Why gravity is attractive, propagates in higher dimension but not deep enough field? That is why higher dimensions are small/compact! attarctive fieldmust be vanishing — existence of higher dimensions, attractive nature and higher dimensions are small.
- In all $d = 2N + 1$, BTZ black hole solution also exist, for Einstein it exists only in $d = 3$.
- Universal gravitational dynamics in all $d = 2N + 1$ and $d = 2N + 2$ dimensions.
- Λ is a universal constant of spacetime structure on the same footing as the speed of light, c.
- Λ can have any value determined by measurement/observation.
- Nothing to do with vacuum energy.

Pulling Up Continued

- Pure Lovelock Riemann is defined through Riemann and hence it is not directly anchored on a connection and metric.
- Degrees of freedom, which follow from the metric, continue to be the same as in Einstein theory.
- For instance, in all odd $d = 2N + 1$, gravity is kinematic, and hence there should be no degrees of freedom!
- This is an open question!

Looking Ahead

arXiv:1206.0635

- How should vacuum energy gravitate — $\lambda g_{ab} = T_{ab}$?
- It is a secondary source produced by matter T_{ab} — Could mother and daughter sit alongside in the right hand side of the equation? No independent existence of its own — produced by matter fields.
- It is the same in character like gravitational field energy — self interaction.
- How does it gravitate **NOT through T_{ab} in GR, but through curvature of 3-space.**
- **Einstein is Newton with space curved (arxiv:1206.0635), enlarging framework.**
- **Something similar – framework enlargement should happen for vacuum energy?**

Vacuum Energy

- For vacuum energy $\rho + p = 0$, i.e., inertial density $= 0$.
- Recall $m = 0$ particles could not be incorporated in Newtonian mechanics, needed a new mechanics SR.

- Vanishing of inertial entity signals need for new theory.
- Thus to incorporate vacuum energy as well, we need a new theory, possibly quantum gravity — Leads to enlargement of framework.
- It is not possible to talk about the enlargement until the new theory arrives.
- In the new theory, gravitational interaction of vacuum energy will be automatically incorporated through enlargement of framework, won't know until new theory emerges!
- Any guesses about possible features — Higher dimensions; Micro structure; non-commutativity; perhaps all together.

Vacuum Energy Continued

- We have already come to higher dimensions from a general principle, Total charge must vanish.
- It is matter of principle that we need a new theory to incorporate vacuum energy in gravitational interaction. This should be taken as a guiding principle.
- Keep looking for new insights and directions.
- May be it gravitates via higher dimensions, while keeping GR intact in four dimensions.
- Exactly what GR did — kept Newton's law intact but curved the space.
- Once Λ is liberated from vacuum energy, it could have any value that 1997 supernova observation measure. It is for the first time Λ has been experimentally determined.
- Then the monster number $10^{2}0$ could be turned on its head to pronounce

  The Universe measures this much in the Planck area units !!!

Extremal Black Hole

- Consider the Black Hole mass Formula

$$M = \underbrace{\frac{\kappa}{4\pi}A}_{\text{Internal Energy}} + \underbrace{2\omega_{\mathrm{H}}J_{\mathrm{H}} + 2Q\phi_{\mathrm{H}}}_{\text{Work Done}}$$

- The surface gravity κ becomes,

$$\kappa = \frac{\sqrt{M^a - a^2 - Q^2}}{r_{\mathrm{h}}^2 + a^2}; \qquad r_{\mathrm{h}} = M + \sqrt{M^a - a^2 - Q^2}$$

- Extremality implies: $M^2 = a^2 + Q^2$.
- Thus extremal black hole has zero internal energy (rest mass) and the entire energy is due to rotation and electric interaction.
- Photon also has zero rest mass, while its energy is purely kinetic.
- Whenever $m = 0$, it signals for a new theory.
- Thus extremal black holes cannot be fully understood in GR. Quantum Gravity?

Finally

Looking at the existing theory deep and hard, one should generally be able to capture a guiding principle/direction that points to new theory.
Here we have reflected on two instances: vacuum energy and extremal black hole asking for new theory.
Road to Quantum Gravity should be illuminated by some general principle/considerations — geometric/physical!

Thank You