The curious case of torsion

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What and why?

1 Torsion is set to zero in GR

2 Palatini variation (metric and connection independent) confirms torsion-free condition

3 vierbein-Einstein-Palatini (VEP)/Einstein-Cartan-Sciama-Kibble (ECSK) theory

4 Fermions, torsion and conformal transformation

- Gravity is described in terms of independent variables (Palatini type) tetrads e^I_μ and spin connection A^{IJ}_μ [Kibble (1961), Sciama (1964), Hehl and Datta (1971)]

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}, \Gamma^\alpha_{\mu\nu} = e^\alpha_I \partial_\mu e^I_\nu + A^I_{\mu J} e^J_\nu e^\alpha_I \tag{1}$$

- Torsion

$$C^\alpha{}_{\alpha\mu} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} \tag{2}$$

- Action

$$S = \frac{1}{2\kappa} \int |e| d^4x F^{IJ}_{\mu\nu} + S_M \tag{3}$$

- Equations

$$\delta e : F^{IJ}_{\alpha\mu} e^\alpha_I - \frac{1}{2} e^J_\mu F^{KL}_{\alpha\beta} e^\alpha_K e^\beta_L = \kappa \Theta^J_\mu \tag{4}$$

$$\delta A : A^{IJ}_\mu = \omega^{IJ}_\mu(e) + \Lambda^{IJ}_\mu \tag{5}$$

- Torsion

$$C^\alpha{}_{\mu\nu} = e^\alpha_I \Lambda^{IJ}_{[\mu} e_{\nu]J} \tag{6}$$

- Fermion (ψ) [Hehl and Datta (1971)]

  Spinor covariant derivative:

$$ {}^{\psi}D_{\mu}\psi = \partial_{\mu}\psi - \frac{i}{4} A_{\mu}^{IJ}\sigma_{IJ}\psi \, . \tag{7} $$

  Action:

$$ S[e, A, \phi, \psi] = S[e, A] + \int |e| \, d^4x \, \frac{i}{2} \Big[\Big(\bar{\psi}\gamma^K e_K^{\mu} \; {}^{\psi}D_{\mu}\psi - (\bar{\psi}\gamma^K e_K^{\mu} \; {}^{\psi}D_{\mu}\psi)^{\dagger} \Big) \Big] \tag{8} $$

Fermion

- On-shell spin connection

$$A_\mu^{IJ} = \omega_\mu^{IJ} + \frac{\kappa}{8}\bar{\psi}\{\gamma_K, \sigma^{IJ}\}\psi e_\mu^K \tag{9}$$

- On-shell torsion

$$^{OS}C^\alpha{}_{\mu\nu} = \frac{\kappa}{2}\epsilon^{IJKL}\bar{\psi}\gamma_L\gamma_5\psi e_I^\alpha e_{J\mu} e_{K\nu}\,. \tag{10}$$

- Non-linear Dirac equation (NLD)

$$\gamma^K e_K^\mu \partial_\mu \psi + \frac{1}{2}C^\alpha{}_{\mu\alpha} e_K^\mu \gamma^K \psi - \frac{i}{4}A_\mu^{IJ} e^{\mu K}\gamma_K \sigma_{IJ}\psi = 0. \tag{11}$$

If C is identified with ^{OS}C:

$$\gamma^K e_K^\mu \partial_\mu \psi - \frac{i}{4}\omega_\mu^{IJ} e^{\mu K}\gamma_K \sigma_{IJ}\psi - \frac{i\kappa}{64}\bar{\psi}\{\gamma_K, \sigma_{IJ}\}\psi\{\gamma^K, \sigma^{IJ}\}\psi = 0\,. \tag{12}$$

- With *a priori* torsion-free condition

$$\gamma^K e_K^\mu \partial_\mu \psi - \frac{i}{4}\omega_\mu^{IJ} e^{\mu K}\gamma_K \sigma_{IJ}\psi = 0\,. \tag{13}$$

- Einstein's equation

$$\widehat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{R} = \kappa \widehat{T}_{\mu\nu}(\psi, \bar{\psi}) - \frac{3\kappa^2}{16} g_{\mu\nu} \bar{\psi}\gamma_I \gamma_5 \psi \bar{\psi} \gamma^I \gamma_5 \psi \,. \tag{14}$$

- Effective Lagrangian in torsion-free theory

$$\mathcal{L}_F = \frac{i}{2} \Big(\bar{\psi}\gamma^K e^\mu_K \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^K e^\mu_K \psi - \frac{i}{2} \epsilon_{IJKL} \omega^{IJ}_\mu e^{\mu K} \bar{\psi} \gamma^L \gamma_5 \psi \\ - \frac{3i\kappa}{8} \left((\bar{\psi}\gamma_5 \psi)^2 - (\bar{\psi}\psi)^2 \right) \Big) \,. \tag{15}$$

Conformal transformation

- Torsion-free Dirac equation is invariant; NLD is not under conformal transformation $(g_{\mu\nu} \to \Omega^2 g_{\mu\nu},\ \tilde{e}^I_\mu = \Omega e^I_\mu, \psi \to \Omega^{-\frac{3}{2}}\psi)$ [Chakrabarty and Lahiri (2018)]

$$\Omega^{-\frac{5}{2}} \left(\gamma^K e^\mu_K \partial_\mu \psi - \frac{i}{4} \omega^{IJ}_\mu e^{\mu K} \gamma_K \sigma_{IJ} \psi \right) - \frac{i\kappa}{64} \Omega^{-\frac{9}{2}} \bar{\psi} \{\gamma_K, \sigma_{IJ}\} \psi \{\gamma^K, \sigma^{IJ}\} \psi = 0\,. \tag{16}$$

- How should torsion (spin connection) behave under conformal transformation?
  - Off-shell:
    - Nieh-Yan theory [Nieh and Yan 1982] $A^{IJ}_\mu \to A^{IJ}_\mu$
    - Invariant torsion $A^{IJ}_\mu \to A^{IJ}_\mu + (e^I_\mu e^{J\alpha} - e^J_\mu e^{I\alpha})\partial_\alpha \ln \Omega$
  - On-shell: Dynamically generated torsion → NLD

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Conformal properties of torsion

- General off-shell transformation

$$A_\mu^{IJ} \to A_\mu^{IJ} + \xi \left(e_\mu^I e^{J\nu} - e_\mu^J e^{I\nu} \right) \partial_\nu \ln \Omega \tag{17}$$

$$\Lambda_\mu^{IJ} \to \Lambda_\mu^{IJ} - (1-\xi) \left(e_\mu^I e^{J\nu} - e_\mu^J e^{I\nu} \right) \partial_\nu \ln \Omega \,, \tag{18}$$

$\xi = 0$: Nieh-Yan theory
$\xi = 1$: Invariant torsion

- General conformal transformation of torsion

$$C^\alpha{}_{\mu\nu} \to C^\alpha{}_{\mu\nu} + (\xi - 1)\delta^\alpha_{[\mu}\partial_{\nu]} \ln \Omega \tag{19}$$

- Dirac equation is invariant with the general transformations

- Gravity is different for different ξ!

$$R \to \Omega^{-2}\{R - \xi\{6g^{\mu\nu}\nabla_\mu\nabla_\nu \ln \Omega - 6(\nabla_\mu \ln \Omega)(\nabla^\mu \ln \Omega)\}\} \,. \tag{20}$$

Summary

- On-shell torsion breaks conformal invariance
- Off-shell transformations correspond to different theories of gravity
- Multiple fermions

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Thank You!