An exact derivation of Hawking effect in Canonical Formulation

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References

S. Barman, G. M. Hossain and C. Singha,
"Exact derivation of the Hawking effect in canonical formulation," Phys. Rev. **D97**, 025016 (2018), arXiv:1707.03614 [gr-qc].

Motivation

- Hawking effect is an interesting result in quantum field theory in curved spacetime which is usually realized using null coordinates [1].
- However, null coordinates do not lead to a true Hamiltonian as they do not describe dynamics.
- We have introduced a set of near-null coordinates to derive Hawking effect in Hamiltonian formulation.

[1] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975)

Schwarzschild spacetime and null rays

- Hawking effect [2] for Schwarzschild black hole is derived using metric

$$ds^2 = -\Omega dt^2 + \Omega^{-1} dr^2 + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right) \ ,$$

where $\Omega = (1 - r_s/r)$. For $dr_\star = \Omega^{-1} dr$ null coordinates are

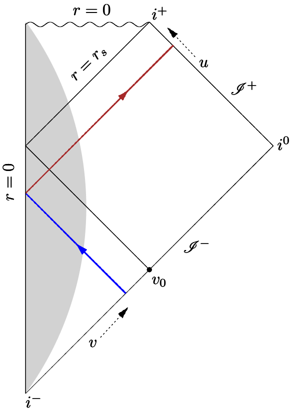
$$v = t + r_\star \ ; \quad u = t - r_\star \ .$$

- Using these null coordinates the metric is

$$ds^2 = -\Omega \, du \, dv + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right) \ .$$

[2] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975)

The Hawking effect



- Ingoing and outgoing modes are $f_{\omega'}(v) \sim e^{-i\omega' v}$ and $p_\omega(u) \sim e^{-i\omega u}$.
- The *Bogoliubov transformation* between these two modes give usual Hawking effect.

$$p_\omega(u) = \sum_{\omega'} \left[\alpha_{\omega\omega'} f_{\omega'}(v) + \beta_{\omega\omega'} f^*_{\omega'}(v)\right] \ .$$

- In the domain $|v| \ll 2r_s$, null coordinates are related as

$$v \approx -2r_s \ e^{-u/2r_s} \quad . \tag{1}$$

The temperature

- The number density for Hawking quanta

$$\langle N_\omega \rangle \equiv \langle 0_- | \hat{N}^+_\omega | 0_- \rangle = \sum_{\omega'} |\beta_{\omega\omega'}|^2 = \frac{1}{e^{2\pi\omega/\varkappa} - 1} \, .$$

The temperature associated with the Hawking effect

$$T_H = \varkappa/(2\pi k_B) = 1/(4\pi r_s k_B) \, .$$

- Therefore, an observer at $\mathscr{I}^+$ would perceive thermal quanta at temperature $T_H = \varkappa/(2\pi k_B) = 1/(4\pi r_s k_B)$.

Canonical formulation

- A timelike and a spacelike coordinates for $\mathbb{O}^-$ observer

$$\tau_- = t-(1-\epsilon)r_\star \ ; \ \xi_- = -t-(1+\epsilon)r_\star \ .$$

- For $\mathbb{O}^+$ observer

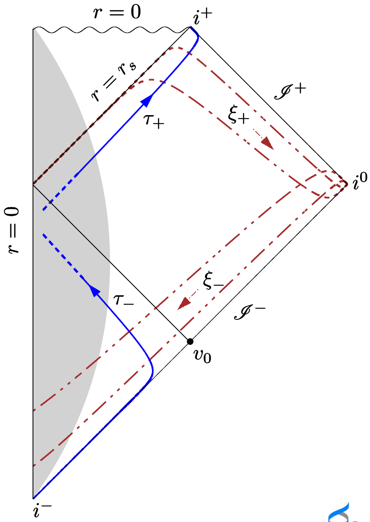
$$\tau_+ = t+(1-\epsilon)r_\star \ ; \ \xi_+ = -t+(1+\epsilon)r_\star \ .$$

- In the domain $|\xi_-| \ll 2r_s$

$$\xi_- \approx 2r_s \ e^{\xi_+/2r_s} \ .$$

- For both the observers $\mathbb{O}^-$ and $\mathbb{O}^+$

$$ds^2 = \frac{\epsilon\,\Omega}{2}\left[-d\tau_\pm^2 + \frac{2}{\epsilon}d\tau_\pm d\xi_\pm + d\xi_\pm^2\right] \ .$$



Scalar Field Hamiltonian

- The Hamiltonian for free scalar field

$$H^{\pm}_{\varphi} = \int d\xi_{\pm} \frac{1}{\epsilon} \left[\left\{ \frac{\Pi^2}{2} + \frac{1}{2}(\partial_{\xi_\pm}\varphi)^2 \right\} + \Pi\ \partial_{\xi_\pm}\varphi \right] .$$

- Fourier modes

$$\varphi = \frac{1}{\sqrt{V_\pm}} \sum_k \tilde{\phi}_k e^{ik\xi_\pm} \ ; \ \Pi = \frac{1}{\sqrt{V_\pm}} \sum_k \sqrt{q}\ \tilde{\pi}_k e^{ik\xi_\pm} .$$

- Hamiltonian in terms of the Fourier modes

$$H^{\pm}_{\varphi} = \textstyle\sum_k \frac{1}{\epsilon}(\mathcal{H}^{\pm}_k + \mathcal{D}^{\pm}_k).$$

- Hamiltonian density for k^{th} mode $\mathcal{H}^{\pm}_k = \frac{1}{2}\tilde{\pi}_k\tilde{\pi}_{-k} + \frac{1}{2}|k|^2\tilde{\phi}_k\tilde{\phi}_{-k}$.
- Diffeomorphism generator $\mathcal{D}^{\pm}_k = -\frac{ik}{2}\left(\tilde{\pi}_k\tilde{\phi}_{-k} - \tilde{\pi}_{-k}\tilde{\phi}_k\right)$.

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Number operator in Fock quantization

- $\frac{\langle \hat{\mathcal{H}}^{+}_{\kappa} \rangle}{\kappa} \equiv \frac{\langle 0_- | \hat{\mathcal{H}}^{+}_{\kappa} | 0_- \rangle}{\kappa} = \frac{e^{2\pi\kappa/\varkappa}+1}{e^{2\pi\kappa/\varkappa}-1} \left[\frac{1}{\zeta(1+2\delta)} \sum_{r=1}^{\infty} \frac{1}{r^{1+2\delta}} \; \frac{\langle \hat{\mathcal{H}}^{-}_{k_r} \rangle}{k_r} \right]$

- The number operator $\hat{N}_{\kappa} = \left[\hat{\mathcal{H}}^{+}_{\kappa} - \lim_{\varkappa \to 0} \hat{\mathcal{H}}^{+}_{\kappa} \right] |\kappa|^{-1}$.

- The vacuum expectation value of the number operator

$$\langle \hat{N}_{\kappa=\omega} \rangle = \frac{1}{e^{2\pi\kappa/\varkappa}-1} \left[\frac{2}{\zeta(1+2\delta)} \sum_{r=1}^{\infty} \frac{1}{r^{1+2\delta}} \; \frac{\langle \hat{\mathcal{H}}^{-}_{k_r} \rangle}{k_r} \right].$$

- In Fock quantization $\langle \hat{\mathcal{H}}^{-}_{k} \rangle = E^{0}_{k} = \frac{1}{2}|k|$ for all modes.

- $N_{\omega} \equiv \langle \hat{N}_{\kappa=\omega} \rangle = \frac{1}{e^{2\pi\omega/\varkappa}-1} = \frac{1}{e^{(4\pi r_s)\omega}-1}$.

- Therefore, an asymptotic future observer $\mathbb{O}^{+}$ would perceive thermal quanta at temperature $T_H = \varkappa/(2\pi k_B) = 1/(4\pi r_s k_B)$ in Fock quantization.

Conclusion

- Usually Hawking effect is realized through the use of null rays.
- We introduced near-null coordinates to derived hawking effect in Hamiltonian formulation.
- Using this formulation one can apply different canonical quantization methods such as polymer quantization.


Thank You !