Canonical Transformations, Duality & Disorder Operator
Ising Model $\rightarrow$ SU(N) LGT

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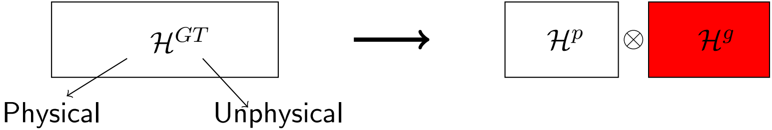
The Goal

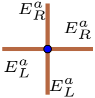
Gauge Theory Hilbert Space $\mathcal{H}$



Gauge Transformations: $U_{\alpha\beta} \to (\Lambda_L)_{\alpha\bar{\alpha}} (U)_{\bar{\alpha}\bar{\beta}} \left(\Lambda_R^{\dagger}\right)_{\bar{\beta}\beta}$

Two ways to construct $\mathcal{H}^{p}$:

Break Holonomies (E, A)

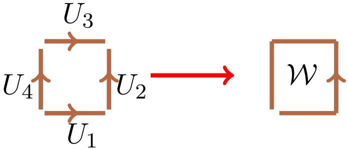


$U_{\alpha\beta} \equiv a_{\alpha}^{\dagger} b_{\beta}^{\dagger}$:

Prepotential Formulation.

Quantum Simulations, U(1) Link Gauge Invariance

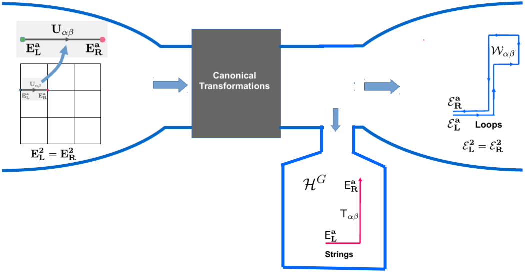
Glue Holonomies $(B, \tilde{A})$



Canonical Transformation $\Rightarrow$ Duality

Canonical Transformations: Gluing Links to Loops

SU(N) Lattice Gauge Theory, $\mathcal{H} = \mathcal{H}^{Loops} \otimes \mathcal{H}^{Gauge}$



- Loop Space $\rightarrow$ No Gauss Law Constraints
- Canonical transformations $\rightarrow$ Loop Space without Notorious Mandelstam Constraints.
- Can. Trans. $\rightarrow$ Duality $\rightarrow$ Disorder Operator For FREE.
- New Order-Disorder Algebra for SU(N) LGT.

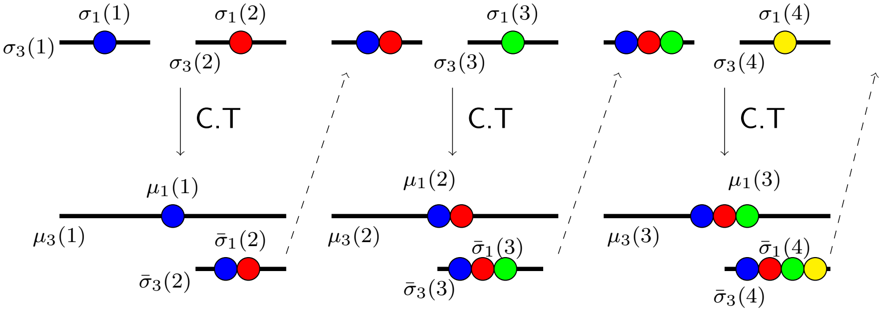
Flow Chart of the talk:

- Ising Model (1+1)
  - Kramers- Wannier Duality (1931)
  - Order-Disorder Operators & Algebra
- Ising Gauge Theory $(2+1)$
  - Z_2 Loops & Strings
  - Wegner Z_2 Gauge-Ising Spin Duality
  - Z_2 Order-Disorder Operators & Alegebra
- SU(N) Lattice Gauge Theory $(2+1), (3+1)$
  - SU(N) Loops & Strings
  - SU(N) Gauge-Spin Duality
  - SU(N) Order-Disorder Operators & Algebra

Ising-Model: Canonical Trans. $\rightarrow$ Kram.-Wan. Duality $_{(1+1)\text{ d}}$

$$H=\sum_{m=0}[-\sigma_1(m)-\lambda\sigma_3(m)\sigma_3(m+1)]$$

The conjugate pairs (σ_1, σ_3) satisfy: $\{\sigma_1, \sigma_3\} = 0$.
For convenience σ_3 is denoted by — and σ_1 denoted by •.



Canonical Transfromation

$$\mu_3(m)=\sigma_3(m)\sigma_3(m+1)$$

$$\mu_1(m)=\prod_{\hookleftarrow}\sigma_1(m)$$

Inverse Canonical Transfromation

$$\sigma_3(m)=\prod_{\hookrightarrow}\mu_3(m)$$

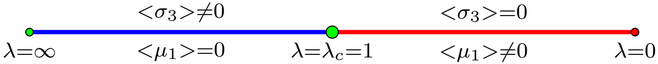
$$\sigma_1(m)=\mu_1(m-1)\mu_1(m)$$

Self-Dual Ising Hamiltonian

$$H = \sum_{m=0} [-\sigma_1(m) - \lambda \sigma_3(m) \sigma_3(m+1)]$$
$$= \sum_{m=0} [-\mu_1(m)\mu_1(m+1) - \lambda \mu_3(m)]$$

Kramers-Wannier Duality and Order Disorder

Ordered Phase   Disordered Phase

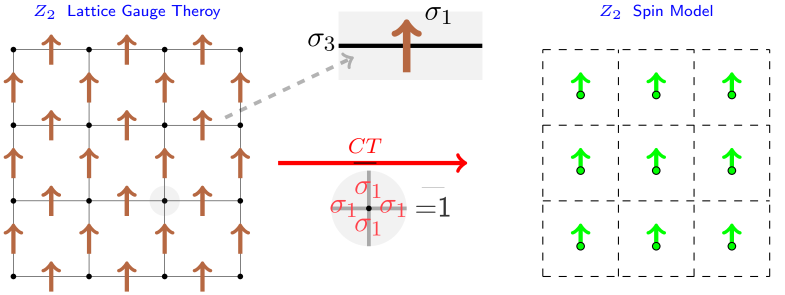


What Are These Order - Disorder Operators in Gauge Theories ??

Can you obtain GT Duality from CT ??

- Z_2 LGT
- SU(N) LGT

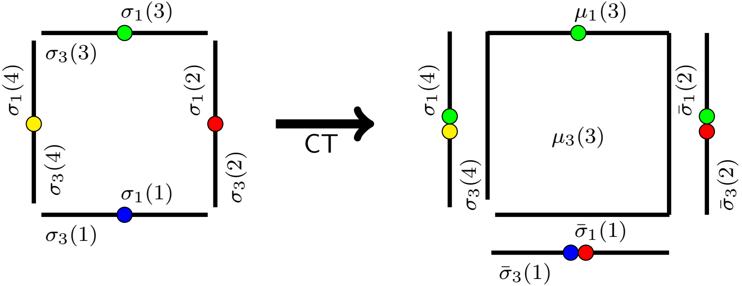
Z_2 Gauge Theory & Wegner Duality (2+1)d

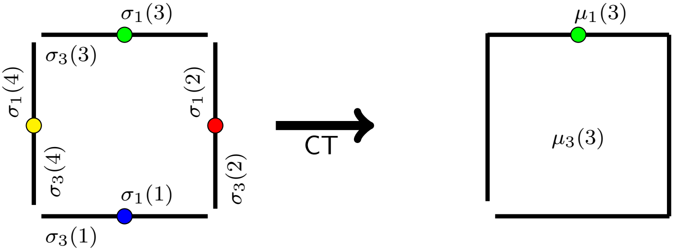


$$H_{Gauge} = -\sum_{links} \sigma_1 - \lambda \sum_{plaquettes} \sigma_3 \sigma_3 \sigma_3 \sigma_3$$

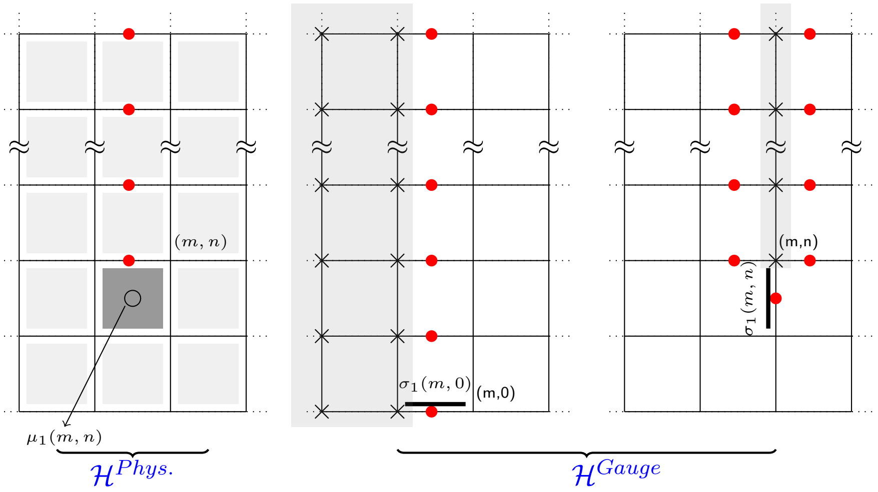
$$H_{Ising} = -\lambda \sum_{sites} \sigma_1 - \sum_{sites} \sigma_3 \sigma_3, \qquad H_{Gauge} \sim H_{Ising}$$

Loop & Stings(Glueing Of Links)

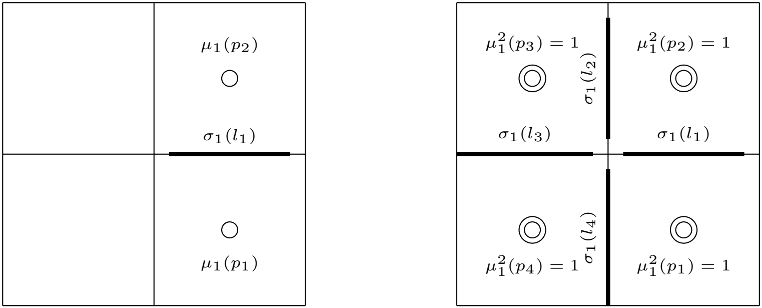




Canonical Transformations: From Links to Loops & Strings



Gauss Law becomes Identity



Z_2 Order-Disorder Operators

Z_2 **Wilson loop Operator**

$$\mathcal{W}_{\mathcal{C}}(p) = \prod_{l \in \mathcal{C}} \sigma_3(l) = \{\mu_{\mathbf{3}}\mu_{\mathbf{3}}.......\mu_{\mathbf{3}}\}$$

Z_2 **Disoder Operator**

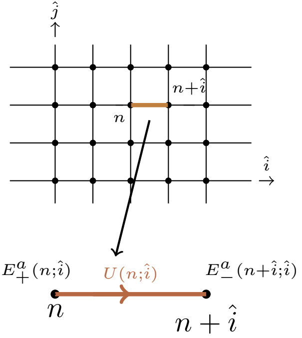
$$\Sigma(m,n) = \prod_{n'=n}^{N} \sigma_1(m,n') = \mu_{\mathbf{1}}(\mathbf{m},\mathbf{n})$$

The Algebra:

$$\Sigma(m,n)\mathcal{W}_{\mathcal{C}}(p)\Sigma^{\dagger}(m,n) = -\eta\mathcal{W}_{\mathcal{C}}(p)$$

Hamiltonian formulation of LGT

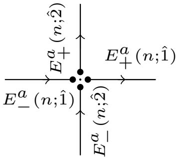
Link Formulation



$$[E^a_L, U_{\alpha\beta}] = -(\lambda^a U)_{\alpha\beta}$$

$$[E^a_R, U_{\alpha\beta}] = (U\lambda^a)_{\alpha\beta}$$

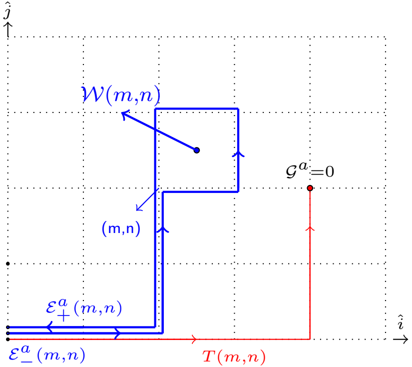
Gauss-Law(Mandelstam Constraints)



$$\mathcal{G}^a(n) = \sum_{\hat{i}=1}^{d=2} [E^a_+(n,\hat{i}) + E^a_-(n,\hat{i})] = 0$$

$$H = g^2 \sum_{links} E^2 + \frac{1}{g^2} \sum_{plaq} [2N - (Tr(UUUU) + h.c)]$$

SU(N) Loops & SU(N) String



$$\mathcal{G}^a_{global} = \sum_{\Box} \left[\mathcal{E}^a_+(\Box) + \mathcal{E}^a_-(\Box)\right] = 0$$

From SU(N) LGT to SU(N) spin model

$SU(N)$ Lattice gauge theory

SU(N) spin model.

Local Gauss Law :

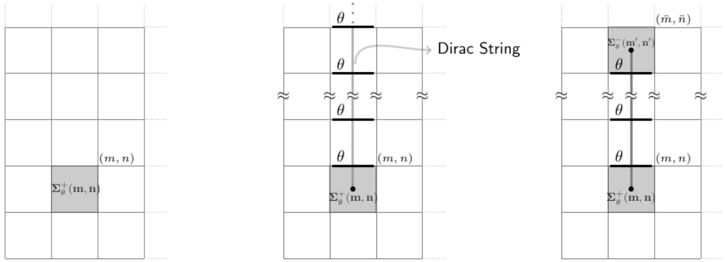
$$\mathcal{G}^a(n) = \sum_{i=1,2} \left[E^a_+(n,\hat{i}) + E^a_-(n,\hat{i})\right] = 0$$

Global Gauss Law :

$$\mathcal{G}^a_{global} = \sum_{n^*} \left[\mathcal{E}^a_+(n^*) + \mathcal{E}^a_-(n^*)\right] = 0$$

Order & Disorder Operator

SU(2) Magnetic Eigenstates $|\omega, \hat{n}\rangle$: $Tr\mathcal{W}\ |\omega, \hat{n}\rangle = cos\omega\ |\omega, \hat{n}\rangle, (\omega, \hat{n}) \in S^3$



SU(2) Disorder-Operators (Generalized t'Hooft Operators)

$$\Sigma_\theta^{\pm}(m,n) \equiv \exp i\left(\hat{w}(m,n).\mathcal{E}_{\pm}(m,n)\theta\right)$$
$$\Sigma_\theta^{+}(m,n)|\omega,\hat{n}\rangle = |\omega+\theta,\hat{n}\rangle$$

SU(2) Order-Disorder Algebra

$$\Sigma_\theta^{+}(m,n)\mathcal{W}^{[j]}_{\alpha\beta}\Sigma_\theta^{-}(m,n) = D^{[j]}_{\alpha\gamma}(\hat{n},\theta)\mathcal{W}^{p}_{\gamma\beta}$$

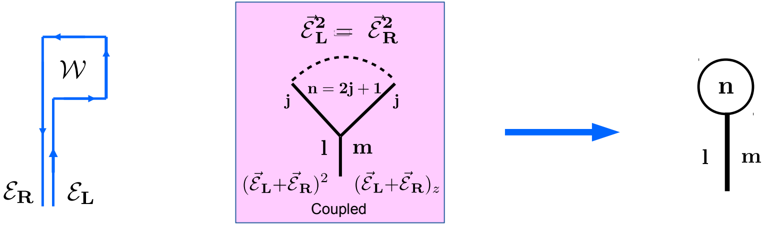
$\eta = \pm 1 \rightarrow D^{j}_{mn}(\hat{n},\omega)$: Wigner rotation matrices in spin j

Summary

- Canonical Transformations $\rightarrow$ Duality Transformations. The new formulation is in terms of conjugate pairs of Magnetic fields, Electric vector potentials.
- Duality Transformations $\rightarrow$ New SU(N) Order-Disorder Algebra.
- All non-abelian Gauss laws are explicitly solved.
- Loop formulation without Mandelstam constraints.
- Interactions now are via Electric vector potentials.

SU(2) LGT and Hydrogen atom

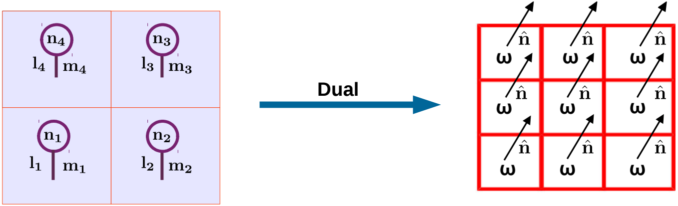
$$J_L \sim \mathcal{E}_L \qquad J_R \sim \mathcal{E}_R$$



Therefore, there is a H atom corresponding to each plaquette.

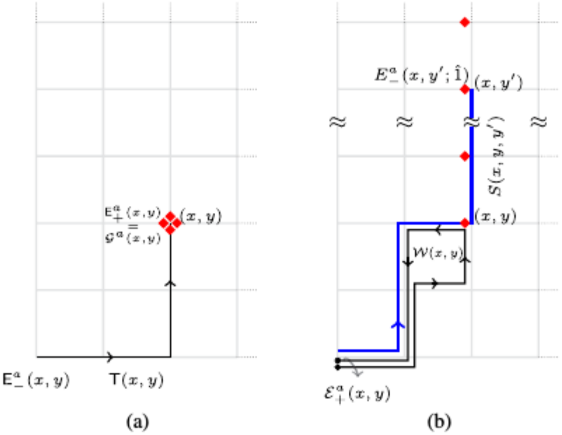
Diagonalizing all loops

$$|\omega, \hat{n}\rangle = \sum_{n,l,m} Y_{nlm}\,(\omega, \hat{n})\ \ |n\ l\ m\ \rangle$$



$\mathcal{W}(p)$ are Diagonal in this Basis.

From Links to Loops



Inverse Canonical Transfromation

From Loops to Links