$\mathcal{PT}$-symmetric systems:Integrability, Symmetry & Related Aspects

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1. Introduction:$\mathcal{PT}$-Symmetric System
2. Integrable Many-body System With Balanced Loss & Gain

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   - Hamiltonian Formulation: Many-body System
   - Exactly Solvable Calogero-type Model
   - Exact Correlation Functions via Matrix Model

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4. Epilogue

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- 1998-2018: Emerging viewpoint $\Rightarrow$ $\mathcal{PT}$-symmetric QS

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- H with unbroken $\mathcal{PT}$ symmetry admits entirely real spectra.

$$\mathcal{PT}H\psi = \mathcal{PT}E\psi \Rightarrow H(\mathcal{PT}\psi) = E^*(\mathcal{PT}\psi) \Rightarrow E = E^*$$

$\mathcal{PT}$-unbroken phase: Orthogonality, Unitarity etc.

- Standard Norm: Non-orthonormal, incomplete set of states
- $\mathcal{PT}$-normalized states are not necessarily positive-definite

$$\langle \phi_m | \phi_n \rangle_{\mathcal{PT}} = \int_C dx \, [\mathcal{PT} \phi_m(x)] \, \phi_n(x) = (-1)^n \delta_{mn}$$

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- $\mathcal{C}$ shares the properties of charge-conjugation operator

$$\mathcal{C}(x,y) = \sum_n \phi_n(x)\phi_n(y), \ [H, \mathcal{PT}] = 0 \Rightarrow [H, \mathcal{C}] = 0$$

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- Orthonormality & completeness of states with $\mathcal{CPT}$-norm

$$\langle \phi_m | \phi_n \rangle_{\mathcal{CPT}} = \int_C dx \left[\mathcal{CPT} \phi_m(x) \right] \phi_n(x) = \delta_{mn}$$

Example:Potential $V(x) = x^2(ix)^{\mathcal{E}}, \mathcal{E} \in \Re$

Numerical Results: Bender & Boettcher, PRL 80, 5243(1998)

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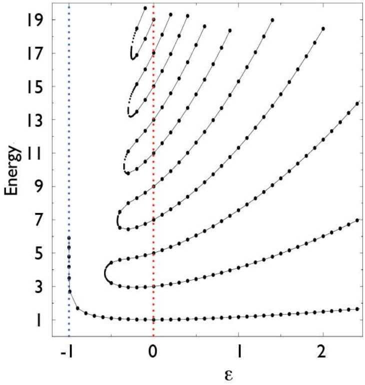

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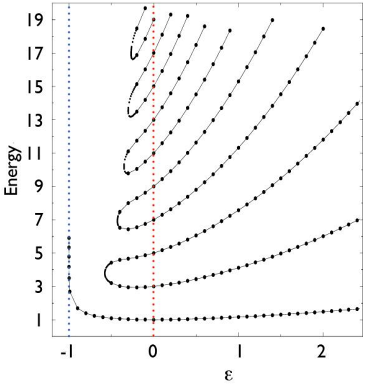

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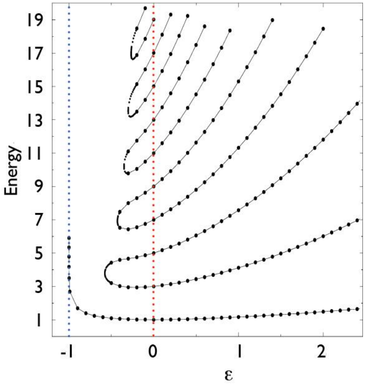

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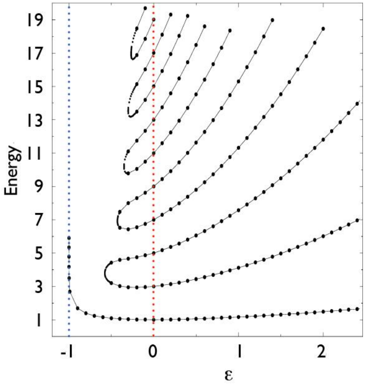

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Rigorous proof on real spectra

Dorey, Dunning & Tateo,
JPA **40**, R205 (2007)

Bateman Oscillator: Hamiltonian for a dissipative oscillator

System: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \Rightarrow$ Dissipative Oscillator

Bath: $\ddot{y} - 2\gamma\dot{y} + \omega_0^2 y = 0 \Rightarrow$ Auxiliary Oscillator

DO & AO together form a Hamiltonian system:

$$H_B = P_x P_y + \gamma(yP_y - xP_x) + (\omega_0^2 - \gamma^2)xy$$
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- No equilibrium state

Equilibrium state via System-bath coupling: an example

$$V(x,y) = \frac{\epsilon}{2}\left(x^2+y^2\right) + \frac{g}{2(x-y)^2}$$
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- Condition for equilibrium state (Unbroken $\mathcal{PT}$-phase)

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- Classical H: Periodic solutions in unbroken $\mathcal{PT}$-phase
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- Quantum H: Real, discrete, positive spectra, unitarity

General Constructions

- Definitions, Notations etc.

$$X^T = (x_1, x_2, \ldots, x_N), \ P^T = (p_1, p_2, \ldots, p_N),$$
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- Equations of Motion

$$\ddot{X} - 2D\dot{X} + 2M\frac{\partial V}{\partial X} = 0$$
$$[J]_{ij} \equiv \frac{\partial F_i}{\partial x_j}, \quad R \equiv AJ - (AJ)^T, \ D := MR$$

Generic features

- Hamiltonian $\Rightarrow$ Balanced loss-gain $[Tr(D) = 0]$

$$M^T = M, \; R^T = -R, \; D^T = D$$
$$\{M, R\} = 0, \; \{M, D\} = 0, \; \{R, D\} = 0$$

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- H in the background of a Pseudo-Euclidean metric

$$\begin{aligned}
M_d &= \hat{O} M \hat{O}^T \quad \left(O^T O = I_{2m}\right)\\
&= diagonal(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_m, -\lambda_m)\\
\tilde{X} &= \hat{O} X,\ \tilde{P} = \hat{O} P,\ \tilde{\Pi} = \hat{O}\Pi\\
H &= \tilde{\Pi}^T M_d \tilde{\Pi} + V(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)
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- Landau Hamiltonian with balanced loss/gain
  (i) Particle moves in an elliptic orbit with reduced cyclotron frequency
  (ii) Hall current is not necessarily in the perpendicular direction to the applied electric field

Representation of Matrices

- A particular choice for $N = 2m$

$$M = I_m \otimes \sigma_x, A = \frac{-i\gamma}{2} I_m \otimes \sigma_y, \ \ D = \gamma \chi_m \otimes \sigma_z$$

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- J has the expression: $J = \sum_{i=1}^{m} U_i^{(m)} \otimes V_i^{(2)}$

$$\left[U_a^{(m)} \right]_{ij} \equiv \delta_{ia} \delta_{ja}, \quad V_a^{(2)} \equiv \begin{pmatrix} \frac{\partial F_{2a-1}}{\partial x_{2a-1}} & \frac{\partial F_{2a-1}}{\partial x_{2a}} \\ \frac{\partial F_{2a}}{\partial x_{2a-1}} & \frac{\partial F_{2a}}{\partial x_{2a}} \end{pmatrix}$$

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- $Q_a(x_{2a-1}, x_{2a}) = Trace(V_a^{(2)})$

An interpretation

- $\hat{O} = \frac{1}{\sqrt{2}} \left[I_m \otimes (\sigma_x + \sigma_z) \right]$ diagonalizes M and generates the Co-ordinate transformation

$$z_i^{\pm} = \frac{1}{\sqrt{2}}(x_{2i-1} \pm x_{2i}), \ P_{z_i^{\pm}} = \pm\frac{1}{2}\left(\dot{z}_i^{\pm} - \gamma F_i^{\mp}\right)$$

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- H describes a system of m particles on a Pseudo-Euclidean plane interacting with each other through V

$$H = \sum_{i=1}^{m} \left[\left(P_{z_i^+} + \frac{\gamma}{2} F_i^- \right)^2 - \left(P_{z_i^-} - \frac{\gamma}{2} F_i^+ \right)^2 \right] + V$$

An interpretation

- $\hat{O} = \frac{1}{\sqrt{2}} \left[I_m \otimes (\sigma_x + \sigma_z) \right]$ diagonalizes M and generates the Co-ordinate transformation

$$z_i^{\pm} = \frac{1}{\sqrt{2}}(x_{2i-1} \pm x_{2i}), \ \ P_{z_i^{\pm}} = \pm \frac{1}{2} \left(\dot{z}_i^{\pm} - \gamma F_i^{\mp} \right)$$

$$F_i^{\pm} = \frac{1}{\sqrt{2}} \left(F_{2i-1} \pm F_{2i} \right), \ \ F_i^{\pm} \equiv F_i^{\pm}(z_i^{+}, z_i^{-})$$

- H describes a system of m particles on a Pseudo-Euclidean plane interacting with each other through V

$$H = \sum_{i=1}^{m} \left[\left(P_{z_i^+} + \frac{\gamma}{2} F_i^{-} \right)^2 - \left(P_{z_i^-} - \frac{\gamma}{2} F_i^{+} \right)^2 \right] + V$$

- The i'th particle is subjected to magnetic field Q_i

$$Q_i = \frac{\partial F_i^{+}}{\partial z_i^{+}} + \frac{\partial F_i^{-}}{\partial z_i^{-}}$$

Quantization

- $z_i^{\pm}$ and $P_{z_i}^{\pm} := -i\partial_{z_i^{\pm}}$ are treated as operators with the non-vanishing commutation relations($\hbar = 1$):

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- In general, $\hat{H}$ is non-hermitian for standard B.C.

$$\hat{H} = \sum_{i=1}^{m}\left[\left(\hat{\Pi}_{z_i^{+}}\right)^2 - \left(\hat{\Pi}_{z_i^{-}}\right)^2\right] + V(z_1^{\pm}, \ldots, z_m^{\pm})$$

Normalizable wf only in appropriate Stoke wedges

Integrability

- Translational invariant system(TIS)

$$V \equiv V(z_1^-, z_2^-, \dots, z_m^-), \quad Q_i \equiv Q_i(z_1^-, z_2^-, \dots, z_m^-)$$

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Classical Hamiltonian

$$V_C(z_i^-) = -\sum_{i=1}^{m} 2\omega_0^2 (z_i^-)^2 - \sum_{\substack{i,j=1 \\ i<j}}^{m} \frac{g^2}{2(z_i^- - z_j^-)^2},$$

$$\ddot{z}_i^- + \omega^2 z_i^- - \sum_{j,(j \neq i)}^{m} \frac{g^2}{(z_i^- - z_j^-)^3} = 0$$

$$z_i^+(t) = 2\gamma \int z_i^-(t) dt + C_i, \quad i = 1, 2, \ldots m.$$

Unlike RCM, V_{II} (second term of V) is not invariant under permutation symmetry S_{2m}. If each pair (x_{2i-1}, x_{2i}) is considered as an element, then, V_{II} is invariant under S_m

Exactly solvable with periodic solutions for $-\frac{\omega_0}{\sqrt{2}} < \gamma < \frac{\omega_0}{\sqrt{2}}$

Quantum Hamiltonian in translational invariant gauge

$$\hat{H}_L = \sum_{i=1}^{m} \left[\left(-i\partial_{z_i^+} + \gamma z_i^- \right)^2 - P_{z_i^-}^2 \right] + V_C$$

Energy eigenvalues:

$$E = -2\Omega[2n + l + \frac{1}{2}m + \frac{\lambda}{2}m(m-1)] + \frac{mk^2\omega^2}{2\Omega^2},$$
$$\Omega^2 = \frac{1}{2}(\omega_0^2 - 2\gamma^2), \ -\frac{\omega_0}{\sqrt{2}} < \gamma < \frac{\omega_0}{\sqrt{2}}$$

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- E consists of discrete as well as continuous spectra
- Box normalization: $0 \leq z_i^+ \leq L, \ \forall i$

$$E = 2|\ \Omega\ |[2n + l + \frac{1}{2}m + \frac{\lambda}{2}m(m-1)] + \frac{2m\pi^2\omega^2\hat{k}^2}{L^2\Omega^2}$$

Normalization of wave-functions

- Asymptotic form of the wave function χ

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- Possible solution: $\theta_i = \theta \ \forall \ i$, a pair of Stoke wedges with opening angle $\frac{\pi}{2}$ and centered about the positive and negative imaginary axes

Correlation functions

$$R_n(x_1, x_2, ...x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=n+1}^{N} dx_i \times \mid \chi(x_1, x_2, ..., x_N) \mid^2, \; n < N$$

Define $y_i = \sqrt{\frac{\Omega}{\lambda}} z_i$. Results from RMT & RCM may be used

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- Integrations over z_i^- in proper Stoke Wedges
- Mapping to integrals of RCM only for even n $(y = y_1)$

$$R_2 = \begin{cases} \frac{N(N-1)}{m\pi L}(2m - y^2)^{\frac{1}{2}}, & y^2 < 2m \\ 0, & y^2 > 2m. \end{cases}$$

Differs from RCM by a constant multiplicative factor

Non-local Nonlinear Schrödinger Equation

Ablowitz & Musslimani, PRL **110**, 064105(2013)
Sinha & Ghosh, PRE **91**, (2015) 042908; PLA **381**, (2017) 124

$$i\psi_t(x,t) = -\frac{1}{2}\psi_{xx}(x,t) + g\ \underbrace{\psi^*(-x,t)\psi(x,t)}_{V(x,t)}\psi(x,t), \ \ g \in \Re.$$

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- In contrast to SNLSE, both bright & dark solitons for $g < 0$.
- Vector Nonlocal NLSE is integrable & share all the properties of scalar Nonlocal NLSE

Lagrangian formulation of non-local NLSE

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- Lagrangian density of a $d+1$ dimensional NLSE

$$\begin{aligned}\mathcal{L} &= i\psi^*(\mathcal{P}\mathbf{x}, t)\partial_t\psi(x, t) - \frac{1}{2}\nabla\psi^*(\mathcal{P}\mathbf{x}, t)\cdot\nabla\psi(\mathbf{x}, t) \\ &- \frac{g}{p+1}\left\{\psi^*(\mathcal{P}\mathbf{x}, t)\psi(\mathbf{x}, t)\right\}^{p+1},\end{aligned}$$

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Real-valued Charges

- Density $\rho = \psi^*(\mathcal{P}\mathbf{x}, t)\psi(\mathbf{x}, \mathbf{t})$ is complex-valued.

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$$N = \int d^d x \left(|\psi_e(\mathbf{x}, \mathbf{t})|^2 - |\psi_o(\mathbf{x}, t)|^2 \right)$$

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- Hamiltonian $\mathcal{H}$ is real-valued and non-positive-definite

$$\mathcal{H} = \frac{1}{2}\int d^d\mathbf{x} \left[\underbrace{|\nabla\psi_e(\mathbf{x}, t)|^2 - |\nabla\psi_o(\mathbf{x}, t)|^2}_{\text{Non-positive definite Kinetic Energy}} \right] + \frac{g}{p+1}\int d^d\mathbf{x} \sum_{k=0}^{[\frac{p+1}{2}]} (-1)^k \ {}^{p+1}C_{2k} |\rho_c|^{2k} \rho_r^{p+1-2k}$$

$\rho = \rho_r + \rho_c$, $\rho_r(\rho_c)$ =real(complex) part of ρ

Complex-valued Charges

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \bigtriangledown \cdot \mathbf{J} = 0,$$

$$\mathbf{J} = \frac{i}{2}[\psi(\mathbf{x}, t)\nabla\psi^*(\mathcal{P}\mathbf{x}, t) - \psi^*(\mathcal{P}\mathbf{x}, t)\nabla\psi(\mathbf{x}, t)],$$

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- Momentum **P**, center of mass **X** and boost **B** are complex

$$\mathbf{P} = \int \mathbf{J}\, d^d\mathbf{x},\ \ \mathbf{X} = \frac{1}{Nd}\int \mathbf{x}\rho(\mathbf{x}, t) d^d\mathbf{x},\ \ \mathbf{B} = t\ \mathbf{P} - \mathbf{X}$$

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- Angular momenta $L_{ij}, \forall\ i,j$ are real for odd d only

$$L_{ij} \quad = \quad \int (x_i J_j - x_j J_i)\, d^d\mathbf{x}, \quad i,j = 1,2,\ldots d$$

Conformal Symmetry for $pd = 2$

$$\tau(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \ \alpha\delta - \beta\gamma = 1,$$

$$\mathbf{x} \rightarrow \mathbf{x}_h = \dot{\tau}^{-\frac{1}{2}}(t)\mathbf{x}, \quad t \rightarrow \tau = \tau(t)$$

$$\psi(\mathbf{x}, t) \rightarrow \psi_h(\mathbf{x}_h, \tau) = \dot{\tau}^{\frac{d}{4}} exp(-i\frac{\ddot{\tau}}{4\dot{\tau}}x_h^2)\psi(\mathbf{x}, t)$$

$$\psi^*(\mathcal{P}\mathbf{x}, t) \rightarrow \psi_h^*(\mathcal{P}\mathbf{x}_h, \tau) = \dot{\tau}^{\frac{d}{4}} exp(i\frac{\ddot{\tau}}{4\dot{\tau}}x_h^2)\psi^*(\mathcal{P}\mathbf{x}, t),$$

- Time-translation:$\tau(t) = t + \beta$

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$$\psi^*(\mathcal{P}\mathbf{x}, t) \rightarrow \psi_h^*(\mathcal{P}\mathbf{x}_h, \tau) = \dot{\tau}^{\frac{d}{4}} exp(i\frac{\ddot{\tau}}{4\dot{\tau}} x_h^2)\psi^*(\mathcal{P}\mathbf{x}, t),$$

- Time-translation:$\tau(t) = t + \beta$
- Dilation: $\tau(t) = \alpha^2 t$

Conformal Symmetry for $pd = 2$

$$\tau(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \ \alpha\delta - \beta\gamma = 1,$$

$$\mathbf{x} \rightarrow \mathbf{x}_h = \dot{\tau}^{-\frac{1}{2}}(t)\mathbf{x}, \quad t \rightarrow \tau = \tau(t)$$

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- Time-translation:$\tau(t) = t + \beta$
- Dilation: $\tau(t) = \alpha^2 t$
- Special conformal transformation: $\tau(t) = \frac{t}{1+\gamma t}$

Symmetry: Schrödinger Invariance

$$I_1(t) = \frac{1}{2}\int d^d\mathbf{x}\; x^2\; \rho(\mathbf{x},t), \quad I_2(t) = \frac{1}{2}\int d^d\mathbf{x}\;\mathbf{x}\cdot\mathbf{J},$$
$$D = tH - I_2, \quad K = -t^2H + 2tD + I_1$$

- D and K are real

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- D and K are real
- $H, D, K, \mathbf{P}, L_{ij}, \mathbf{B}$ form $d+1$ dimensional Schrödinger algebra
- Complex charges have no physical significance. It is to be seen, whether the corresponding quantum charges could be hermitian wrt some modified norm or not.

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- Hamiltonian formulation of generic many-particle systems with space-dependent balanced loss & gain is presented along with general features

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- Constructed partial & completely integrable systems related to underlying translation and rotational symmetry
- A Calogero-type model with balanced loss/gain is introduced and solved at the classical as well as quantum level including exact $2n$-particle correlation functions for the ground-state

Ongoing & Future Works

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- QFT formulations

Graffiti

MURRAY GELL-MANN's totalitarian principle in QM

Everything (that is) not forbidden is compulsory

THANK YOU