Fermionic vacua and entanglement in hyperbolic de Sitter spacetime

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Introduction & overview

- de Sitter spacetime : Soln. of the Einstein eq. with a positive cosmological constant $\Lambda \rightarrow G_{ab} + \Lambda g_{ab} = 0$. Inclusion of a mass with charge and or angular momentum $\rightarrow$ various star and black solutions in the de Sitter universe.

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- Why de Sitter? **a)** high degree of homogeneity+ isotropy of the large scale universe $\to$ early universe underwent a very rapid phase of accelerated expansion $\to$ the inflation. For slow role $\to$ de Sitter is a very well motivated and successful model to describe the inflationary scenario and various primordial perturbations built over it **b)** recent phase of accelerated expansion of the universe $\to$ a small but a positive Λ. ΛCDM $\to$ the simplest and phenomenologically most successful description of the modern cosmology (e.g., S. Weinberg, (2007)). Various alternative gravity models (e.g., C. Skordis *et al*, Phys. Rep., (2012)), but no phenomenological/conceptual superiority over ΛCDM so far.

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- dS spacetime : A maximally symmetric spacetime with a constant positive curvature, $R = 2n\Lambda/(n-2)$ in n-spacetime dim. Could be constructed via compactifying an $(n+1)$-d Minkowski spacetime, $-T^2 + X^2 + Y^2 + Z^2 + W^2 + \cdots = 3/\Lambda \rightarrow$ an S^n if we continue T to imaginary values $\Rightarrow$ isometry group is $SO(n,1)$. Will take $n = 4$.

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- Most popular global covering of dS $\rightarrow$ $R^1 \times S^3$ $\rightarrow$ contracts from $\tau \rightarrow -\infty$ to $\tau = 0$ and then expands (e.g. J. B. Griffiths and J. Podolsky, 2009)

The geometry of the de Sitter universe

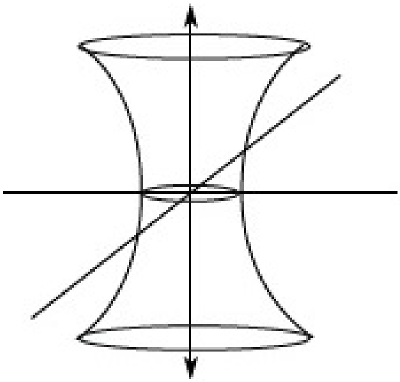

Figure: Schematic global dS. Vertical direction is 'time'. The spatial section is S^3.

The geometry of the de Sitter universe

- Only expanding part interesting for cosmology $\Rightarrow$ half of S^3, the cosmological dS :
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- One of the symmetries $\rightarrow \tau \rightarrow \tau + \tau_0$ *and* $R \rightarrow e^{-H\tau_0} R$. $\Rightarrow$ generator : $\partial_t = \partial_\tau \pm HR\,\partial_R$ Killing vector, timelike for $R\,e^{H\tau} \leq H^{-1}$. Defining also $r = e^{H\tau} R$,
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- Coordinates not well defined for $r \geq H^{-1}$. H^{-1} is the cosmological event horizon $\rightarrow$ maximum limit of the *visible* universe, set due to the accelerated expansion. CH has thermal properties like black hole horizon (G. W. Gibbons, S. W. Hawking, PRD, 1976; A. Higuchi, K. Yamamoto,1808.02147). There are stationary BH soln.s within CH $\rightarrow$ non-trivial vacuum structure due to existence of two temperatures (SB, 1810.13260 and Refs. therein).

The hyperbolic de Sitter spacetime

- Global, cosmological and static coordinates $\rightarrow$ most popular coordinate coverings of dS. Static patch $\rightarrow$ near horizon geometry is Rindler $\rightarrow$ Hilbert space inside and outside the horizon $\rightarrow$ entanglement entropy for a quantum field using Replica trick by tracing dof beyond CH $\rightarrow$ Area/4 (S. N. Solodukhin, Liv. Rev. Rel., 2011).

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- Question : how to discuss the long range quantum correlations in dS, for observers located outside each other's CH? $\rightarrow$ hyperbolic dS (M. Sasaki et al, PRD, 1995).

The Penrose diagramme of the hyperbolic de Sitter

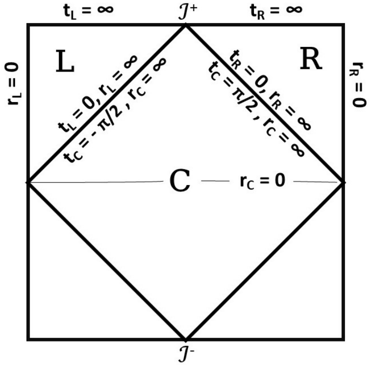

Figure: The hyperbolic de Sitter spacetime

The metric

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- Analytically continue to real coordinate values to cover the whole dS :

$$t_R = i\left(\tau - \tfrac{\pi}{2}\right), \qquad r_R = i\rho, \qquad (t_R \geq 0,\ \ r_R \geq 0)$$
$$t_C = \tau, \qquad r_C = i\left(\rho - \tfrac{\pi}{2}\right), \qquad \left(-\tfrac{\pi}{2} \leq t_C \leq \tfrac{\pi}{2},\ \ 0 \leq r_C < \infty\right)$$
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- The metrics :

$$ds^2_{R,L} = H^{-2}\left(-dt^2_{R,L} + \sinh^2 t_{R,L}(dr^2_{R,L} + \sinh^2 r_{R,L} d\Omega^2)\right)$$
$$ds^2_C \ \ = H^{-2}\left(dt^2_C + \cos^2 t_C(-dr^2_C + \cosh^2 r_C d\Omega^2)\right)$$

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- Three causally disconnected regions. R, $L \rightarrow$ not separated by a wall but by the entire region $C \Rightarrow$ natural set up to investigate long range quantum correlation between them.

The entanglement entropy

- Field theoretic derivation of EE for for a massive scalar with non-minimal coupling ζ (J. Maldacena, G. L. Pimentel, JHEP, 2013). Field quantisation (M. Sasaki et al, PRD, 1995) $\rightarrow$ consider modes having supports in R and L regions only (local modes) :

$$\phi(x) = \int dp \sum_{lm}[a^R_{plm}\, u^R_{plm}(R) + a^{R\dagger}_{plm}\, u^{\star R}_{plm}(R) + a^L_{plm}\, u^L_{plm}(L) + a^{L\dagger}_{plm}\, u^{\star L}_{plm}(L)]$$

$$u^{R/L}_{plm}(x) = \frac{H\, P^{ip}_{\nu}(z_{R/L})}{\sqrt{z^2_{R/L}-1}}\, Y_{plm}(r,\theta,\phi)$$

$$z_{R/L} = \cosh t_{R/L},\ \nu^2 = 9/4 - (m^2 + 12\zeta H^2)/H^2, \quad p \geq 0$$

$a^{R/L}|R/L\rangle = 0$ (local vacua; u positive freq. in asymp. past.).

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- Also global modes $\rightarrow$ found via analytically continuing R modes to L and vice versa. Gives a measure of the quantum correlation. $z_R = 1$ is continued through C to $z_L = 1$. Spatial part remains intact $\Rightarrow$ take P^{ip}_{ν} from rhs of its branch point $z = +1$ to lhs of another branch point $z = -1$ along $Im(z) > 0 \Rightarrow P^{ip}_{\nu}(z_R) \rightarrow P^{ip}_{\nu}(-z_L)$ for $R \rightarrow L$. Likewise for $L \rightarrow R$.

The entanglement entropy

- $P_\nu^{ip}(-z_L) = e^{-\pi p}[(e^{i\pi\nu} + \frac{i\sin\pi(\nu+ip)}{\sinh\pi p})P_\nu^{ip}(z_L) - \frac{i\sin\pi(\nu+ip)\,\Gamma(\nu+1+ip)}{\sinh\pi p\,\Gamma(\nu+1-ip)}P_\nu^{-ip}(z_L)]$
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- Orthogonality of the mode functions : KG inner product
  $(u_1, u_2) = i\int d\Sigma^a(u_2^\star\partial_a u_1 - u_1\partial_a u_2^\star)$ For local modes $\rightarrow$ $t = \mathrm{const.}$ hypersurafce in respective regions. Global modes $\rightarrow$ supports in both regions $\rightarrow$ two hypersurfaces in R and L connected through C. Latter could be deformed such that it has vanishing contribution. Turns out that global modes obtained via analytic continuation of local modes are orthogonal $\rightarrow$ normalise using KG inner product and the Wronskian $\Rightarrow$ field expansion in terms of the orthonormal global modes. Mixing of positive and negative frequencies in global modes $\Rightarrow$ Bogoliubov relations $\Rightarrow$ global and local vacua have to be different.

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- The Bogoliubov coeffs. $\Rightarrow |0\rangle_G = N\exp(\sum_{i,j=R,L} m_{ij}a_i^\dagger a_j^\dagger)|R\rangle\otimes|L\rangle$
  $m_{ij} \rightarrow$ by annihilating $|0\rangle_G$ by global annihilation operator and using the Bogoliubov relations.

The entanglement entropy

- Matrix representation of $\rho_{\text{global}} = |0\rangle_G\langle 0|_G \rightarrow$ trace over R or L to obtain the reduced density matrix $\rightarrow$ EE $= \mathrm{Tr}(\rho_R \ln \rho_R)$ $\rightarrow$ long range correlation, not area, final result obtained by numerically integrating over momentum eigenvalues (J. Maldacena, G. L. Pimentel, JHEP, 2013).

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- Other measures of long range correlations such as : α-vacua, Bell's inequality violation, decoherence, negativity and discord investigated in this framework (S. Kanno, J. Soda PRD, 2017; A. Albrecht et al, 1802.08794; S. Choudhury et al, 1809.02732 and Refs. therein); quantum complexity (A. Reynolds, S. F. Ross, 1706.03788).

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- Possible observational effects : imprints of quantum correlation in the primordial gravitational perturbations (S. Kanno, J. Soda, 1810.07604 and Refs. therein); effective action and its consequences on the power spectrum by summing over fermion interaction with inflaton (D. Boyanovsky 1804.07967 and Refs. therein)

The case of fermions

- Fermions can play important role during inflation, chiefly via Yukawa coupling to the inflaton. Wish to compute EE for Dirac fermions in this framework. Dirac eq. in curved spacetime : $i\gamma^a\nabla_a\Psi - m\Psi = 0$. $\nabla_a \equiv \partial_a - i\omega_{a\mu\nu}[\gamma^\mu, \gamma^\nu]/8$, $\omega_{a\mu\nu} := e^b_\mu \partial_a e_{b\nu} - \Gamma^b_{ac} e_{b\mu}\, e^c_\nu$. Local R, L modes (no normalisable modes in C) (R. Camporesi, A. Higuchi, gr-qc/9505009; S. Kanno et al, JHEP, 2017) :

$$\Psi^{+R}_{(-)} = \begin{pmatrix} u_p(z) \\ v_p(z) \end{pmatrix} \chi^{(-)}_{pjm} \qquad \Psi^{-R}_{(-)} = \begin{pmatrix} -v^\star_p(z) \\ u^\star_p(z) \end{pmatrix} \chi^{(-)}_{pjm}.$$

$$\Psi^{-R}_{(+)} = \begin{pmatrix} u^\star_p(z) \\ -v^\star_p(z) \end{pmatrix} \chi^{(+)}_{pjm} \qquad \Psi^{+R}_{(+)} = \begin{pmatrix} v_p(z) \\ u_p(z) \end{pmatrix} \chi^{(+)}_{pjm}.$$

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$$u_p(z) = \left(z^2 - 1\right)^{-\frac{3}{4}} \left(\frac{z+1}{z-1}\right)^{\frac{ip}{2}} F\left(-\frac{im}{H}, \frac{im}{H}, \frac{1}{2} - ip, \frac{1-z}{2}\right)$$

$$v_p(z) = \left(z^2 - 1\right)^{-\frac{1}{4}} \left(\frac{z+1}{z-1}\right)^{\frac{ip}{2}} F\left(1-\frac{im}{H}, 1+\frac{im}{H}, \frac{3}{2} - ip, \frac{1-z}{2}\right).$$

spatial part, $\chi^{\pm} \rightarrow$ orthonormal, spin-1/2 weighted 2-component harmonics over H^3. $p \geq 0$ and continuous. u, v positive freq. in asymp. past.

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- Dirac inner product (Local normalisation integral) :

$$\left(\Psi_{(a)}, \Psi_{(b)}\right) := \int (z^2-1)^{3/2} \sqrt{h}\, dr d\theta d\phi\, \Psi^{\dagger}_{(a)} \Psi_{(b)}$$

$\rightarrow$ all local modes are orthonormal + asymp. positive/negative frequency behaviour $\Rightarrow$ make the field expansion with creation/annihilation op.s with suitable anti-commutation reln. imposed.

The global fermionic modes

- Analytic continuation of the local modes from $R \to L$ and vice versa. u_p, v_p have branch points at $z = \pm 1, \infty$. Choose the cut properly + analytic properties of the hypergeometric fn. (suppressing $\chi^{\pm}$) $R \to L$ (S. Kanno et al, JHEP, 2017) :

$$\Psi^{+R}_{(-)}(z_R) = \begin{pmatrix} u_p(z_R) \\ v_p(z_R) \end{pmatrix}, \quad \Psi^{+R}_{(-)}(z_L) = \begin{pmatrix} \lambda_1 \, v_p(z_L) + \lambda_2 \, u_p^\star(z_L) \\ -\lambda_1 \, u_p(z_L) + \lambda_2 \, v_p^\star(z_L) \end{pmatrix}$$

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- Likewise for $L \to R$ (suppressing $\chi^{\pm}$) :

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$$\lambda_1 = \frac{\sinh \frac{m\pi}{H}}{\cosh \pi p}, \qquad \lambda_2 = \frac{e^{-\pi p} \left(\Gamma\left(\frac{1}{2} - ip\right)\right)^2}{\Gamma\left(\frac{1}{2} - ip - \frac{im}{H}\right) \Gamma\left(\frac{1}{2} - ip + \frac{im}{H}\right)}.$$

Constructing global orthonormal modes

- Unlike the scalar field, the modes thus found do not form orthogonal set, e.g. $\left(\Psi^{+L}_{(-)}, \Psi^{-R}_{(-)}\right)_{\mathrm{G}} = \left(\Psi^{+L}_{(-)}, \Psi^{-R}_{(-)}\right)_{z=z_L} + \left(\Psi^{+L}_{(-)}, \Psi^{-R}_{(-)}\right)_{z=z_R} = -2\lambda_2^\star$

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- Orthogonalise first by forming intra-pair linear combinations. For example, $\left\{\widetilde{\Psi}^{+R}_{(-)}, \Psi^{-L}_{(-)}\right\}$, where $\widetilde{\Psi}^{+R}_{(-)} = \Psi^{+R}_{(-)} - \frac{(\Psi^{+R}_{(-)}, \Psi^{-L}_{(-)})_{\rm G}}{(\Psi^{-L}_{(-)}, \Psi^{-L}_{(-)})_{\rm G}}\, \Psi^{-L}_{(-)}$

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- Recall global modes mix +ve and -ve frequencies $\rightarrow$ the above do *not* keep an equal footing between them $\Rightarrow$ introduce a spacetime indep. parameter $0 \leq \theta_{\rm RL} \leq \pi/2$, $\rightarrow$ 1-parameter family of orthogonal fn.,

$$\Psi' = \Psi^{+R}_{(-)} - \frac{2\lambda_2 \Delta\theta_1}{N_b^2}\Psi^{-L}_{(-)} \qquad \Psi'' = \Psi^{-L}_{(-)} - \frac{2\lambda_2^\star \Delta\theta_2}{N_b^2}\Psi^{+R}_{(-)}, \quad \text{where}$$

$$\Delta\theta_1 = \frac{\cos^2\theta_{\rm RL}}{1 - \frac{2|\lambda_2|}{N_b^2}\sin^2\theta_{\rm RL}} \quad \Delta\theta_2 = \frac{\sin^2\theta_{\rm RL}}{1 + \frac{2|\lambda_2|}{N_b^2}\cos^2\theta_{\rm RL}} \qquad \left(0 \leq \theta_{\rm RL} \leq \frac{\pi}{2}\right)$$

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- All other pairs can be orthogonalised in the same spirit. Could have introduced four different parametrisation for four different pairs $\rightarrow$ such an abundance of parameter would weaken the theory's predictability. Now normalise and make the field expansions with new creation/annihilation op.s $\rightarrow$ find out the Bogoliubov coefficients $\rightarrow$

The Bogoliubov coefficients

$$
\begin{aligned}
a_1 &= \frac{1}{N_1 N_b}\left[(1-\alpha^\star\lambda_2)\, c^R_{(-)} + \lambda_1 c^L_{(-)} + \alpha^\star\lambda_1 d^{R\dagger}_{(-)} + (\lambda_2^\star - \alpha^\star)\, d^{L\dagger}_{(-)}\right] \\
a_2 &= \frac{1}{N_2 N_b}\left[(1-\beta^\star\lambda_2)\, c^R_{(+)} + \lambda_1 c^L_{(+)} + \beta^\star\lambda_1 d^{R\dagger}_{(+)} + (\lambda_2^\star - \beta^\star)\, d^{L\dagger}_{(+)}\right] \\
a_3 &= \frac{1}{N_1 N_b}\left[-\lambda_1 c^R_{(-)} + (1-\alpha^\star\lambda_2)\, c^L_{(-)} - (\lambda_2^\star - \alpha^\star)\, d^{R\dagger}_{(-)} + \alpha^\star\lambda_1 d^{L\dagger}_{(-)}\right] \\
a_4 &= \frac{1}{N_2 N_b}\left[-\lambda_1 c^R_{(+)} + (1-\beta^\star\lambda_2)\, c^L_{(+)} - (\lambda_2^\star - \beta^\star)\, d^{R\dagger}_{(+)} + \beta^\star\lambda_1 d^{L\dagger}_{(+)}\right] \\
b_1^\dagger &= \frac{1}{N_1 N_b}\left[-\alpha\lambda_1 c^R_{(+)} - (\lambda_2 - \alpha)\, c^L_{(+)} + (1-\alpha\lambda_2^\star)\, d^{R\dagger}_{(+)} + \lambda_1 d^{L\dagger}_{(+)}\right] \\
b_2^\dagger &= \frac{1}{N_2 N_b}\left[-\beta\lambda_1 c^R_{(-)} - (\lambda_2 - \beta)\, c^L_{(-)} + (1-\beta\lambda_2^\star)\, d^{R\dagger}_{(-)} + \lambda_1 d^{L\dagger}_{(-)}\right] \\
b_3^\dagger &= \frac{1}{N_1 N_b}\left[(\lambda_2 - \alpha)\, c^R_{(+)} - \alpha\lambda_1 c^L_{(+)} - \lambda_1 d^{R\dagger}_{(+)} + (1-\alpha\lambda_2^\star)\, d^{L\dagger}_{(+)}\right] \\
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\end{aligned}
$$

where $\alpha = 2\lambda_2\Delta\theta_1/N_b^2, \beta = 2\lambda_2\Delta\theta_2/N_b^2$, N_1, N_2, N_b are normalisations.

The θ_{RL} global vacua

- The global vacuum :

$$|0\rangle_{\text{global}} = N[|0_c\rangle_R \otimes |0_d\rangle_R \otimes |0_c\rangle_L \otimes |0_d\rangle_L$$
$$+\xi_1 \{|1_c\rangle_R \otimes |1_d\rangle_R \otimes |0_c\rangle_L \otimes |0_d\rangle_L + |0_c\rangle_R \otimes |0_d\rangle_R \otimes |1_c\rangle_L \otimes |1_d\rangle_L\}$$
$$+\xi_2 \{|1_c\rangle_R \otimes |0_d\rangle_R \otimes |0_c\rangle_L \otimes |1_d\rangle_L + |0_c\rangle_R \otimes |1_d\rangle_R \otimes |1_c\rangle_L \otimes |0_d\rangle_L\}$$
$$+(\xi_1^2 + \xi_2^2)|1_c\rangle_R \otimes |1_d\rangle_R \otimes |1_c\rangle_L \otimes |1_d\rangle_L]$$

$$\xi_1 = -\frac{2\lambda_1\lambda_2^\star\left(\lambda_1^2 + 2|\lambda_2|\cos 2\theta_{\text{RL}} + |\lambda_2|^2 + 1\right)}{4|\lambda_2|\left(\lambda_1^2+1\right)\cos 2\theta_{\text{RL}} + |\lambda_2|^2\left(2\lambda_1^2 + \cos 4\theta_{\text{RL}} + 1\right) + 2\left(\lambda_1^2+1\right)^2}$$
$$\xi_2 = -\frac{\lambda_2^\star\left(2\left(\lambda_1^2+1\right)\cos 2\theta_{\text{RL}} + 2|\lambda_2|^2\cos 2\theta_{\text{RL}} + |\lambda_2|\left(\cos 4\theta_{\text{RL}} + 3\right)\right)}{4|\lambda_2|\left(\lambda_1^2+1\right)\cos 2\theta_{\text{RL}} + |\lambda_2|^2\left(2\lambda_1^2 + \cos 4\theta_{\text{RL}} + 1\right) + 2\left(\lambda_1^2+1\right)^2}$$

Any observable A, $\langle A\rangle = \text{Tr}(\rho_{\text{global}}A)$ would depend upon θ_{RL}. However since this dS construction is more apt to describe two faraway causally disconnected regions, will look for EE only.

Matrix representation of the reduced density operator

- $\rho_{\text{global}} \rightarrow$ partial trace over either R or L $\rightarrow$ for the 4×4 matrix representation,

$$\rho_R \equiv |N|^2 \begin{pmatrix} 1 + |\xi_1|^2 & 0 & 0 & \xi_1^\star + \xi_1 \left(\xi_1^{\star 2} + \xi_2^{\star 2}\right) \\ 0 & |\xi_2|^2 & 0 & 0 \\ 0 & 0 & |\xi_2|^2 & 0 \\ \xi_1 + \xi_1^\star \left(\xi_1^2 + \xi_2^2\right) & 0 & 0 & |\xi_1|^2 + \left|\xi_1^2 + \xi_2^2\right|^2 \end{pmatrix}$$

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- EE per mode per unit vol : $S_p = \sum_{i=1}^{4} \lambda_i \ln \lambda_i$. Total EE : integrate over p and volume. Vol. integration needs regularisation. Also computed the Rényi entropy, a 1-parameter generalisation of EE :
$S_q = \dfrac{1}{1-q} \ln {\rm Tr}\rho^q, \qquad q > 0.$

The entanglement entropy

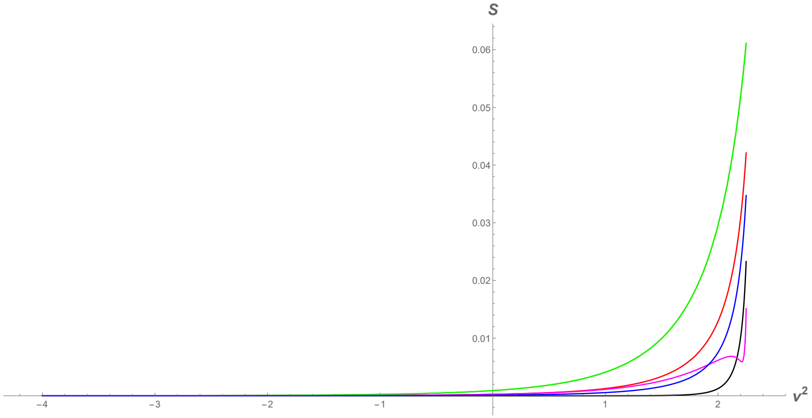

Figure: EE Vs. $\nu^2 = 9/4 - m^2/H^2$ plots for various $\theta_{\rm RL}$. Green curve corresponds to $\theta_{\rm RL} = 0, \pi/2$, red $\theta_{\rm RL} = \pi/6$, blue $\theta_{\rm RL} = \pi/5$, black $\theta_{\rm RL} = \pi/4$, pink $\theta_{\rm RL} = \pi/3$. The result is symmetric under the interchange of R and L (but no $\theta_{\rm RL} \to \pi/2 - \theta_{\rm RL}$ symmetry)

Discussions and outlook

- Discussed a 1-parameter freedom of the fermionic field theory in the hyperbolic dS. Arises merely due to the necessity of orthonormalising the global mode functions. Discussed EE. Such one-parameter family in dS is analogous to the so called α-vacua, constructed as a one parameter family of vacua over the Bunch-Davies vacuum (e.g. H. Collins, PRD, 2005). The qualitative difference of this from our case that the $\theta_{\rm RL}$'s are *fundamental necessity* of the theory, for the sake of preserving the canonical structure. Also it does not seem to be present for other coordinatisation of dS nor for real or complex scalar, massive or massless vectors in any interesting coordinatisation of dS $\Rightarrow$ a very unique feature of fermions in hyperbolic dS (SB, S. Chakrabortty, S. Goyal, to be submitted, 2018).

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- Need to explore further to understand the full implication of the parametrisation. Other measures of quantum correlations. Also global excited states.