Footprint of spatial noncommutativity in resonant detectors of gravitational wave

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Introduction

- Gravitational waves are 'ripples' in the fabric of space-time.
- GW detectors primarily consists of ground-based (LIGO, VIRGO, GEO,TAMA etc.) and space-based (LISA) interferometers.
- The study of resonant-bar detectors is fundamental because it focuses on how GW interacts with elastic matter.
- The present day gravitational wave (GW) detectors strive to detect the length variation $\mathscr{O} \sim 10^{-18} - 10^{-21}$ meter.

- The uncertainty between spatial coordinates can be realized by imposing the Noncommutativity between the spatial coordinates.

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} = i\theta\varepsilon_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0; \quad i, j = 1, 2$$

  θ is the NC parameter and ε_{ij} is an antisymmetric tensor.

- It has been estimated from the error bars in the measurement of various physical quantities that the above spatial noncommutative parameter can have an upperbound $\mathscr{O} \sim 10^{-18} - 10^{-20}$ meter.
- A good possibility of detecting the NC structure of space would be in the GW detection experiments as it may as well pick up the NC signature.

- Interaction of GW with elastic matter causes vibrations.
- The vibrations are nothing but quantum mechanical forced harmonic oscillators (HO).
- Thus the response of a resonant detector to GW can be quantum mechanically described as GW-HO interaction.
- **With this motivation, we have studied the interaction of GW(s) with simple matter systems in a NCQM framework.**

Brief Methodology

- The Lagrangian of the system

$$\mathscr{L} = \frac{1}{2}m(\dot{x}_j)^2 - m\Gamma^j{}_{0k}\dot{x}_j x^k - \frac{1}{2}m\varpi^2(x_j)^2$$

- The Hamiltonian therefore reads

$$H = \frac{1}{2m}\left(p_j + m\Gamma^j_{0k}x^k\right)^2 + \frac{1}{2}m\varpi^2 x_j^2 \ .$$

- The gravitational wave is taken care of by $\Gamma^j_{0k} = \frac{\dot{h}_{jk}}{2}$
- The linearly polarized GW can be expressed as

$$h_{jk}(t) = 2f\left(\varepsilon_\times \sigma^1_{jk} + \varepsilon_+ \sigma^3_{jk}\right)$$

- where $2f$ is the amplitude of the GW and $(\varepsilon_\times, \varepsilon_+)$ are the two possible polarization states of the GW satisfying the condition $\varepsilon_\times^2 + \varepsilon_+^2 = 1$ for all t.

- NCQM: Classical Hamiltonian $\rightarrow$ canonical variables change to NC Heisenberg operators.
- This NC space algebra $\rightarrow$ Standard Heisenberg algebra spanned by the operators $\hat{X}_i$ of the ordinary QM through the transformation equations $\hat{x}_i = \hat{X}_i - \frac{1}{2\hbar}\theta\varepsilon_{ij}\hat{P}_j$.
- With this map, the Hamiltonian in terms of the commutative variables is

$$\hat{H} = \left(\frac{P_j^2}{2m} + \frac{1}{2}m\varpi^2 X_j^2\right) + \Gamma^j_{0k}X_jP_k - \frac{m\varpi^2}{2\hbar}\theta\varepsilon_{jm}X^jP_m - \frac{\theta}{2\hbar}\varepsilon_{jm}P_mP_k\Gamma^j_{0k} + \mathscr{O}(\Gamma^2) .$$

- We now define raising and lowering operators in terms of the oscillator frequency ϖ

$$X_j = \sqrt{\frac{\hbar}{2m\varpi}}\left(a_j + a_j^\dagger\right)\,; \quad P_j = \sqrt{\frac{\hbar m\varpi}{2i}}\left(a_j - a_j^\dagger\right)\,.$$

- The Hamiltonian in terms of raising and lowering operator can be recast as

$$\begin{aligned}\hat{H} &= \hbar\varpi(a_j^\dagger a_j + 1) - \frac{i\hbar}{4}\dot{h}_{jk}(t)\left(a_j a_k - a_j^\dagger a_k^\dagger\right) \\ &\quad + \frac{m\varpi\theta}{8}\varepsilon_{jm}\dot{h}_{jk}(t)\left(a_m a_k - a_m a_k^\dagger + C.C.\right) - \frac{i}{2}m\varpi^2\theta\varepsilon_{jk}a_j^\dagger a_k \\ &\equiv \hat{H}_0 + \hat{H}_{\text{int}}(t)\end{aligned}$$

- Our aim in this section is to calculate the perturbed states, their corresponding energy levels and transition probabilities among them. For this we first break the interaction Hamiltonian $\hat{H}_{\rm int}(t)$ as follows

$$\hat{H}_{\rm int}(t) = \hat{H}_1 + \hat{H}_2(t) \ .$$

- The time independent perturbation is

$$\hat{H}_1 = -i\Lambda_\theta \hbar \varepsilon_{jk} a_j^\dagger a_k \ .$$

- Notice that this is a purely NC term with no effect of GW in it and it also introduces a characteristic frequency of noncommutativity

$$\Lambda_\theta = \frac{m\varpi^2\theta}{2\hbar}$$

- The spatial NC structure removes the degeneracy of the second excited state of the 2-dimensional HO.
- Perturbed eigenstates are

$$
\begin{aligned}
\psi_2^{(0)} &= (|2,0\rangle + |0,2\rangle) \\
\psi_2^{(1)} &= (|2,0\rangle - |0,2\rangle + i\sqrt{2}|1,1\rangle) \\
\psi_2^{(2)} &= (|2,0\rangle - |0,2\rangle - i\sqrt{2}|1,1\rangle)
\end{aligned}
$$

with the corresponding energy eigenvalues

$$
E_2^{(0)} = 3\hbar\varpi \quad , \quad E_2^{(1)} = 3\hbar\varpi(1 + \frac{2}{3}\Lambda)
$$

$$
E_2^{(2)} = 3\hbar\varpi(1 - \frac{2}{3}\Lambda) \ .
$$

where

$$
\Lambda = \frac{m\varpi\theta}{2\hbar}
$$

- The time dependent perturbations are given by

$$\hat{H}_2(t) = -\frac{i\hbar}{4}\dot{h}_{jk}(t)\left(a_j a_k - a_j^\dagger a_k^\dagger\right) + \frac{\Lambda}{4}\hbar\varepsilon_{jm}\dot{h}_{jk}(t)\left(a_m a_k - a_m a_k^\dagger + C.C.\right)$$

- Therefore the first possible transition survives between the ground state $|0,0\rangle$ and the perturbed second excited states, given by

$$C_{0\to 2^{(0)}} = 0$$

$$C_{0\to 2^{(1)}} = -\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, e^{2i\varpi(1+\Lambda)t}\hbar\left[iA(\Lambda)\dot{h}_{11}(t) - B(\Lambda)\dot{h}_{12}(t)\right].$$

$$C_{0\to 2^{(2)}} = -\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, e^{2i\varpi(1-\Lambda)t}\hbar\left[iC(\Lambda)\dot{h}_{11}(t) - D(\Lambda)\dot{h}_{12}(t)\right].$$

- where

$$A(\Lambda) = \frac{1}{\sqrt{2}}(1+\Lambda), \quad B(\Lambda) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{3}{2}}\Lambda + 1\right),$$
$$C(\Lambda) = \frac{1}{\sqrt{2}}(1-\Lambda), \quad D(\Lambda) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{3}{2}}\Lambda - 1\right).$$

- Transition probabilities

$$P_{0\to 2} = \left|C_{0\to 2}\right|^2 .$$

Transition probabilities for different types of GW

- First we consider the simple scenario of periodic GW with linear polarization. This can be written as

$$h_{jk}(t) = 2f_0 \cos \Omega t \left(\varepsilon_\times \sigma^1_{jk} + \varepsilon_+ \sigma^3_{jk} \right).$$

The amplitude varies sinusoidally with a single frequency Ω.

- The transition rates then take the form

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} P_{0\to 2^{(1)}} &= (\pi f_0 \Omega)^2 \left[A(\Lambda)^2 \varepsilon_+{}^2 + B(\Lambda)^2 \varepsilon_\times{}^2 \right] \times \delta\left(2\varpi_+ - \Omega\right) \\ \lim_{T\to\infty} \frac{1}{T} P_{0\to 2^{(2)}} &= (\pi f_0 \Omega)^2 \left[C(\Lambda)^2 \varepsilon_+{}^2 + D(\Lambda)^2 \varepsilon_\times{}^2 \right] \times \delta\left(2\varpi_- - \Omega\right). \end{aligned}$$

$$\varpi_+ = \varpi + \Lambda_\theta = \varpi(1+\Lambda), \quad \varpi_- = \varpi - \Lambda_\theta = \varpi(1-\Lambda).$$

- From the above results we can make following observations:

1. The transition rates will be peaked around the frequencies $\Omega = 2\varpi_{+}$ and $\Omega = 2\varpi_{-}$. Thus we should get two resonant points if the space has an underlying NC geometry.
2. The transition probability $P_{0\rightarrow 2^{(1)}}$ is larger than $P_{0\rightarrow 2^{(2)}}$.
3. Both linear and quadratic terms in the dimensionless NC parameter Λ will appear in the transition probabilities.
4. Both the $+$ and the $\times$ polarizations includes the effects of the NC structure of space.

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