# Self accelerating coherent states*

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# Point particle dynamics

- Classical: Accelerated particle motion necessarily occurs in presence of a potential gradient

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x}$$

- Same also seems to hold in the quantum case:

$$m\langle\ddot{x}\rangle = -\langle\frac{\partial V(x)}{\partial x}\rangle$$

for some state $|\psi(t)\rangle$, which evolves as per the Schrödinger equation:

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

- Stationary states $|\psi_E(t)\rangle$, for which $|\psi_E(t)\rangle = e^{-\frac{iEt}{\hbar}}|\psi_E(0)\rangle$
- These seems to defy the logic: NO motion at all

$$|\psi_E(t)|^2 = |\psi_E(0)|^2$$

- Q. Is the converse true ?
  Can one have accelerated motion in *absence* of a potential ?
- The answer is emphatic YES !

# Airy wavepackets

- Berry and Balazs (1979) showed that the wavefunction $\psi(x, t)$ of a nonrelativistic free particle system, which evolves as per the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t},$$

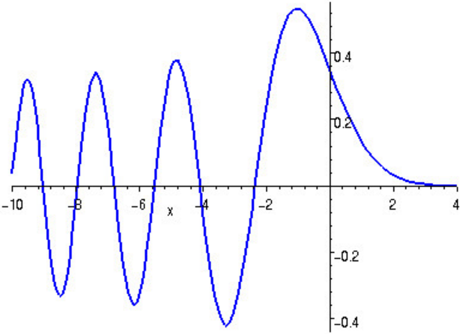
  admits a solution having an Airy function profile:

$$\psi(x, t=0) = \mathrm{Ai}\left(\frac{Bx}{\hbar^{2/3}}\right),$$

- It uniformly accelerates without spreading:

$$|\psi(x,t)|^2 = \mathrm{Ai}^2\left(\frac{B}{\hbar^{2/3}}\left(x - \frac{B^3t^2}{4m^2}\right)\right)$$

# Airy wavepackets




- NOT in conflict with the Ehrenfest theorem, these are like scattering states
- Not square-integrable, giving rise to ill-defined averages $\langle x \rangle$ and $\langle x^2 \rangle$
- $|\psi(x, t)|^2$ does not represent probability density of a particle

# Airy wavepackets

- Assumed to be pathological, mathematical curiosity, and were largely ignored
- Only a couple of studies from different viewpoints
- Greenberger (1980) showed that these are $e^{i(kx-wt)}$ solutions BUT viewed by a uniformly accelerating observer
- Rau/Unnikrishnan (1996) showed that these are unique - no other solution exists having these properties
- Were realised in optical experiments by Siviloglou *et. al.* in 2007
- Since then have been extensively studied in experiments
- Prof. R. P. Singh in PRL-Ahmedabad have done some interesting work with these beams

# Airy wavepackets

- This remind one of SHO *coherent (wavepacket) states* $\phi(x, t)$
- Evolve in time in a manner so as to preserve its shape:

$$|\phi(x,t)|^2 = |\phi(x - f(t), 0)|^2$$

- $f(t)$ solves the classical equation of motion for the harmonic oscillator
- The Airy wavepackets also evolve in time similar to these coherent states, preserving their shape:

$$|\psi(x,t)|^2 = |\psi(x - g(t), 0)|^2$$

  where $g(t) = \frac{B^3 t^2}{4m^2}$
- One naturally wonders: Airy wavepackets have any connection with the harmonic oscillator coherent states ?
- YES, they are actually generalised coherent states

# SHO Coherent states

- Heisenberg-Weyl algebra:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{1}{\sqrt{2\hbar m\omega}}\hat{p},$$
$$[\hat{a}, \hat{a}^{\dagger}] = \hat{I}$$

- Coherent state is obtained from ground state $|0\rangle$:

$$\hat{a}|0\rangle = 0,$$
$$|\alpha\rangle = \exp\left(\alpha\hat{a}^{\dagger} - \frac{|\alpha|^2}{2}\hat{I}\right)|0\rangle$$

- Temporal stability: coherent state is stable under time evolution, which only changes the value of $\alpha$:

$$e^{-i\frac{\hat{H}t}{\hbar}}|\alpha\rangle = |\alpha e^{-i\omega t}\rangle$$

# Generalised coherent states

- Many generalisations exist for these coherent states in different quantum systems
- We are interested in Perelomov construction
- If operators $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ form a closed (Lie) algebra, then the Perelomov coherent state $|a, b\rangle$:

$$\hat{X}|x\rangle = x|x\rangle \quad \text{(fiducial state)}$$

$$|a, b\rangle = \exp\left(a\hat{Y} + b\hat{Z}\right)|x\rangle$$

# Perelomov coherent state

- We shall work with free particle Schrödinger equation $\hat{H} = \frac{\hat{p}^2}{2m}$
- It has Galilean invariance: two inertial observers observe same physics
- Transformation from one frame to the other is via a unitary transformation:

$$\hat{U}(t) = e^{i\frac{v}{\hbar}\hat{K}(t)}$$
$$\hat{K}(t) = t\hat{p} - m\hat{x} \quad \text{(Galilean boost generator)}$$

- The set of operators $\{\ \hat{I},\ \hat{x},\ \hat{p},\ \frac{\hat{p}^2}{2}\ \}$ with the commutation relations:

$$[\hat{x}, \hat{p}] = i\hbar\hat{I}, \quad [\hat{x}, \frac{\hat{p}^2}{2}] = i\hbar\hat{p}, \quad [\hat{p}, \frac{\hat{p}^2}{2}] = 0,$$

  constitute a closed algebra
- This algebra is a generalisation of harmonic oscillator algebra of $\{\hat{a}, \hat{a}^\dagger, \hat{I}\ \}$, which is expressed using the operators $\{\hat{I},\ \hat{x},\ \hat{p}\}$

# Perelomov coherent state

- Consider $|\Phi\rangle = |x = 0\rangle$ as the fiducial state, and construct coherent state:

$$|\xi; t\rangle = \exp\left(-\frac{it}{2\hbar m}\hat{p}^2 + \frac{i}{\hbar}\frac{\xi}{m}\hat{p}\right)|\Phi\rangle$$

- Interestingly one finds that these states are eigenstates of $\hat{K}(t)$:

$$\hat{K}(t)|\xi; t\rangle = \xi|\xi; t\rangle$$

- In $x$-representation:

$$\psi_\xi(x, t) = \langle x|\xi; t\rangle = \exp\left(\frac{i}{\hbar}\left(\frac{mx^2}{2t} - \frac{\xi x}{t}\right)\right)$$

# Perelomov coherent state

- The set of operators $\{\hat{I},\ \hat{x},\ \hat{p},\ \frac{\hat{p}^2}{2},\ \frac{\hat{p}^3}{6}\}$, with non-trivial commutation relations:

$$[\hat{x}, \hat{p}] = i\hbar\hat{I}, \quad [\hat{x}, \frac{\hat{p}^2}{2}] = i\hbar\hat{p}, \quad [\hat{x}, \frac{\hat{p}^3}{6}] = i\hbar\frac{\hat{p}^2}{2},$$

  form a closed algebra

- A coherent state $|\varepsilon, \xi; t\rangle$, from the fiducial state $|\Phi\rangle = |x = 0\rangle$, using the operators $\{\hat{p},\ \frac{\hat{p}^2}{2},\ \frac{\hat{p}^3}{6}\}$:

$$|\varepsilon, \xi; t\rangle = \exp\left(-\frac{i\varepsilon}{6\hbar m^2}\hat{p}^3 - \frac{it}{2\hbar m}\hat{p}^2 + \frac{i\xi}{\hbar m}\hat{p}\right)|\Phi\rangle,$$

  for any real $\varepsilon$ and $\xi$

# Perelomov coherent state

- Interestingly it turns out that this state is an eigenstate of $\hat{K}(t) + \varepsilon \hat{H}$:

$$\left(\hat{K}(t) + \varepsilon \hat{H}\right) |\varepsilon, \xi; t\rangle = \xi |\varepsilon, \xi; t\rangle$$

- This property gives them a unique behaviour
- At $t = 0$ the action of time evolution operator gives:

$$e^{-\frac{i\tau}{\hbar}\hat{H}} |\varepsilon, \xi; 0\rangle = \left(e^{-\frac{i\tau\xi}{\hbar\varepsilon}} e^{-\frac{im\tau^3}{3\hbar\varepsilon^3}}\right) e^{\frac{i\tau}{\hbar\varepsilon}\hat{K}(0)} e^{\frac{i\tau^2}{2\hbar\varepsilon}\hat{p}} |\varepsilon, \xi; 0\rangle$$

# Airy wavepackets

- The effect of time evolution operator gives rise to a constant acceleration $\frac{1}{\varepsilon}$
- In x-representation this is apparent:

$$\psi_{\varepsilon,\xi}(x,\tau) = e^{-\frac{i\tau\xi}{\hbar\varepsilon}} e^{-\frac{i}{\hbar}(\frac{m\tau^3}{3\varepsilon^2}+\frac{mx\tau}{\varepsilon})}\psi_{\varepsilon,\xi}(x+\frac{\tau^2}{2\varepsilon},0)$$

- The time evolution is shape preserving: $|\psi_{\varepsilon,\xi}(x,\tau)|^2 = |\psi_{\varepsilon,\xi}(x+\frac{\tau^2}{2\varepsilon},0)|^2$

# Airy wavepackets

- The explicit form of $\psi_{\varepsilon,\xi}(x,t)$:

$$\psi_{\varepsilon,\xi}(x,t) = \frac{1}{\sqrt{\hbar m}}\left(\frac{2\hbar m^2}{\varepsilon}\right)^{\frac{1}{3}} \exp\left(\frac{i}{\hbar}\frac{(\xi - mx)t}{\varepsilon} - \frac{i}{\hbar}\frac{mt^3}{3\varepsilon^2}\right) \times \mathrm{Ai}\left(-\frac{1}{\hbar}\left(\frac{2\hbar m^2}{\varepsilon}\right)^{\frac{1}{3}}\left(x - \frac{\xi}{m} + \frac{t^2}{2\varepsilon}\right)\right)$$

- BUT this is exactly the Airy wavepacket state discovered by Berry/Balazs !
- Airy wavepackets are Perelomov coherent states
- Clarifies the origin of 'acceleration' (without force) and shape-preservation
- The peak of the probability density of these coherent states does not traverse a classical trajectory
- BUT rather traverses along a caustic of classical trajectories (Berry/Balazs)

# Airy wavepackets

- The state of zero acceleration can be obtained by taking the limit $\varepsilon \to \infty$ with a fixed finite value of $\xi$
- This limit yields:

$$e^{-\frac{i\tau}{\hbar}\hat{H}}|\varepsilon \to \infty, \xi; 0\rangle = |\varepsilon \to \infty, \xi; 0\rangle$$

- Shows that the state $|\varepsilon \to \infty, \xi; \tau\rangle$ is zero energy state $|p = 0\rangle$
- The state $|\varepsilon \to 0, \xi; 0\rangle$ goes over to become the eigenstate of $\hat{K}$
- The accelerating coherent states $|\varepsilon, \xi; \tau\rangle$ interpolate smoothly and continuously from the ground state $|p = 0\rangle$ to $\hat{K}$ eigenstate $|\varepsilon = 0, \xi; \tau\rangle$

# Summary

- The accelerating Airy solutions of free Schrödinger equation are Perelomov coherent states
- Origin of acceleration and shape preservation is an outcome of their definition
- Unique to free Schrödinger equation owing to Galilean invariance