Hidden Symmetries:

Gauge and Integrable Theories on Loop Spaces

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Current Developments in Quantum Field Theory and Gravity

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Main Topics

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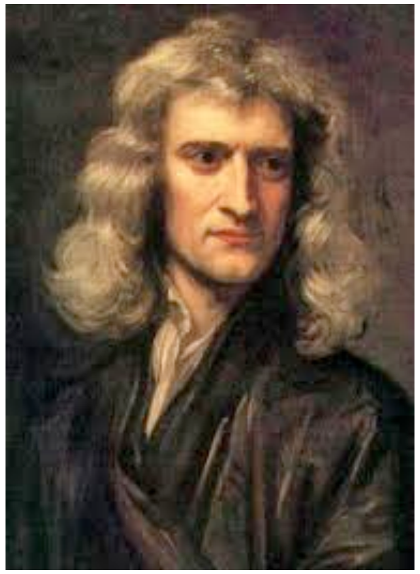
- Truly gauge invariant conserved charges

Hidden Symmetries in a nutshell: Conservative Forces



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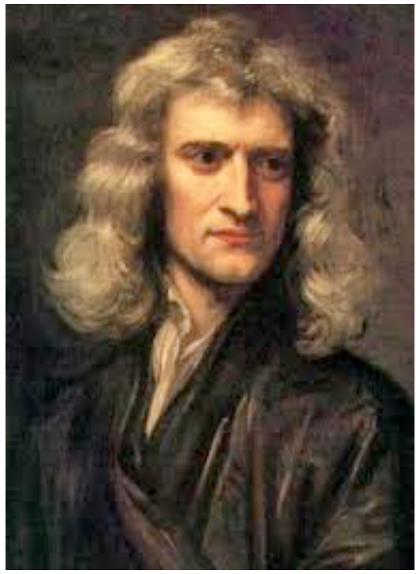

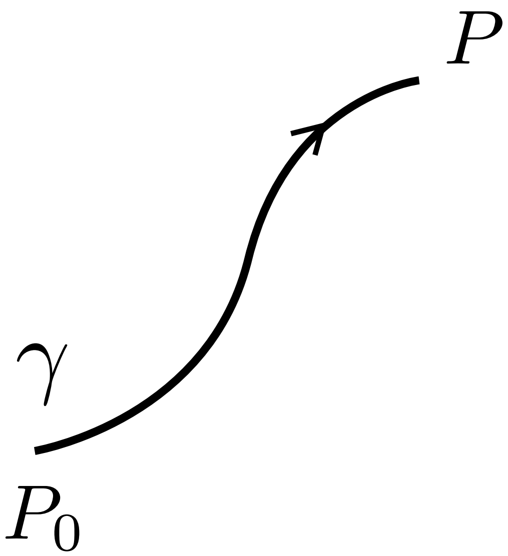

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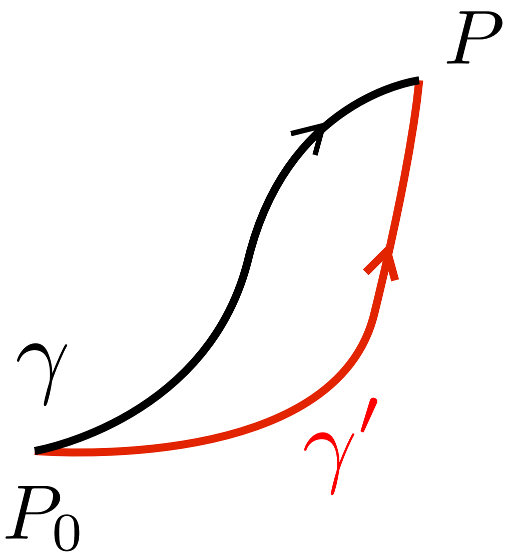

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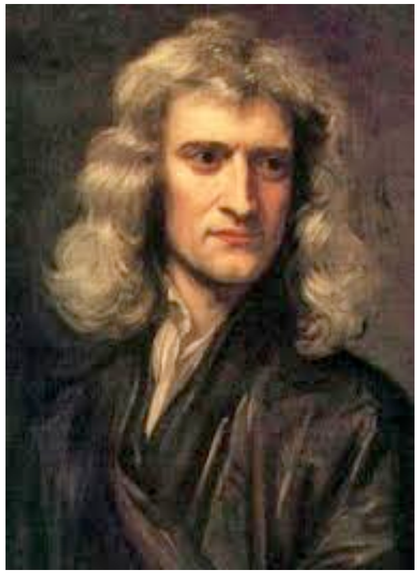

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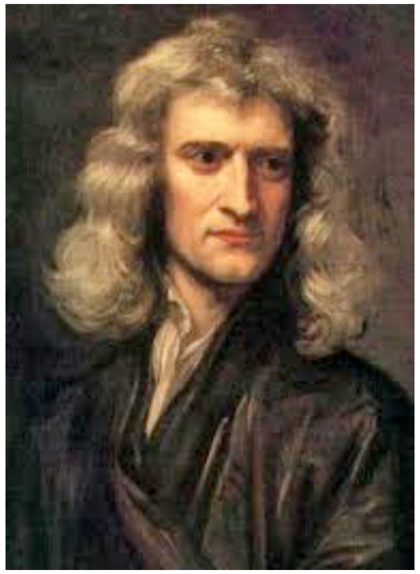

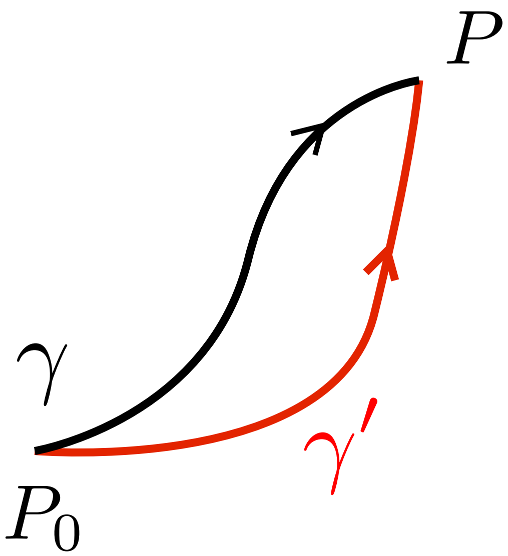

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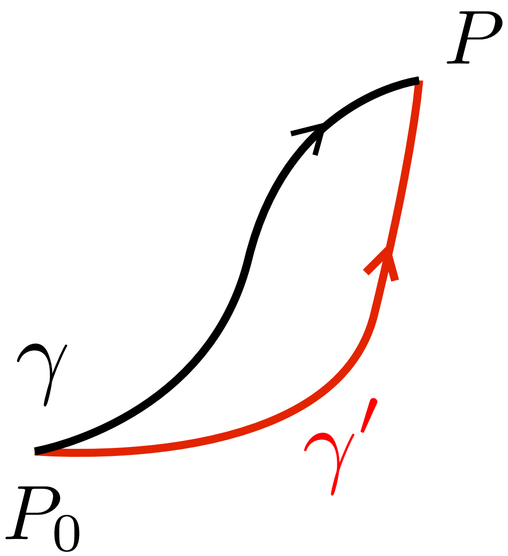

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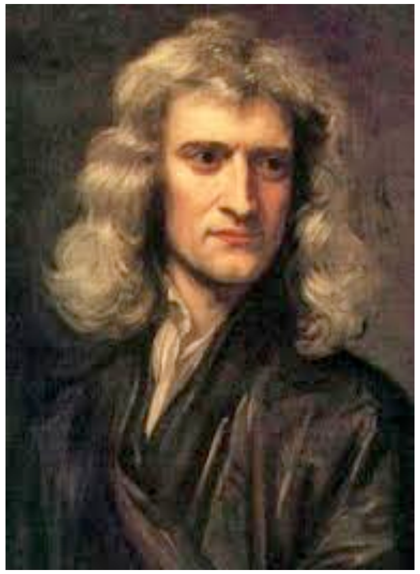

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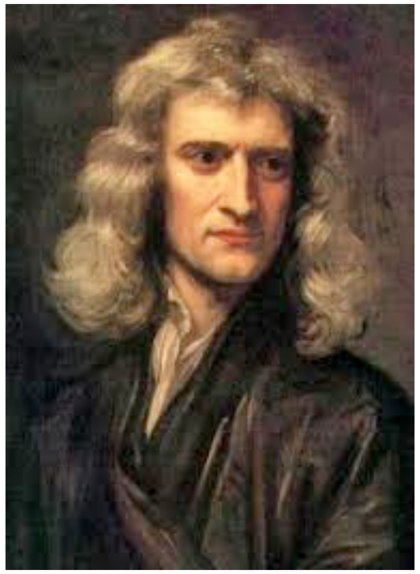

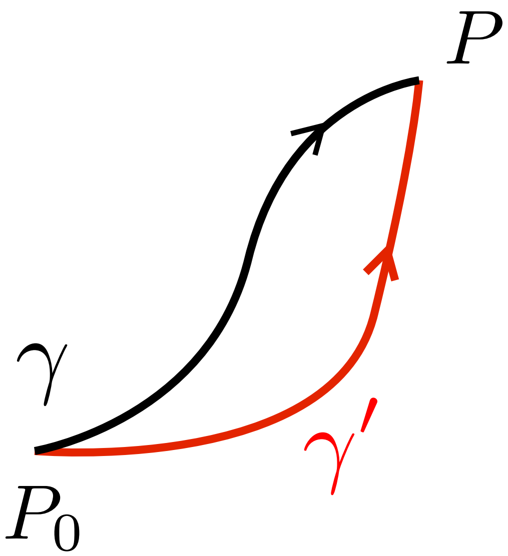

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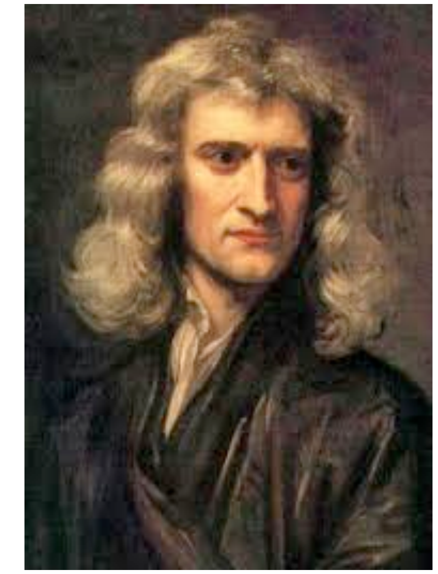

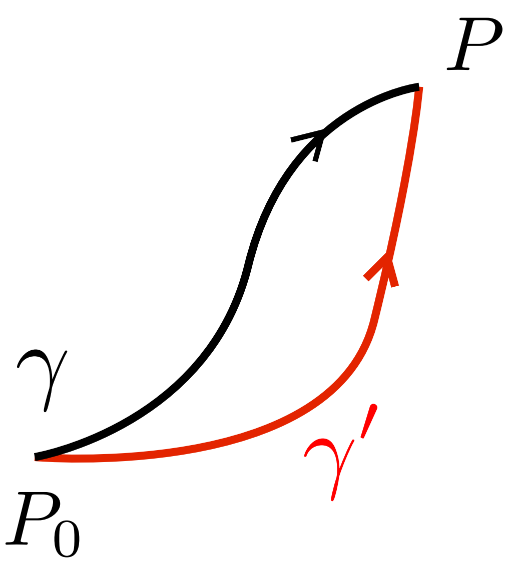

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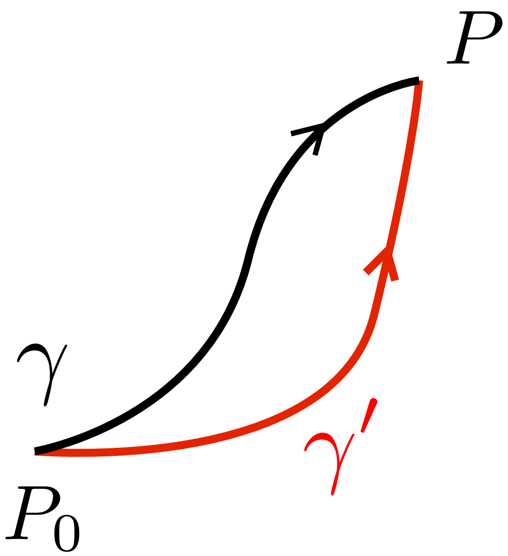

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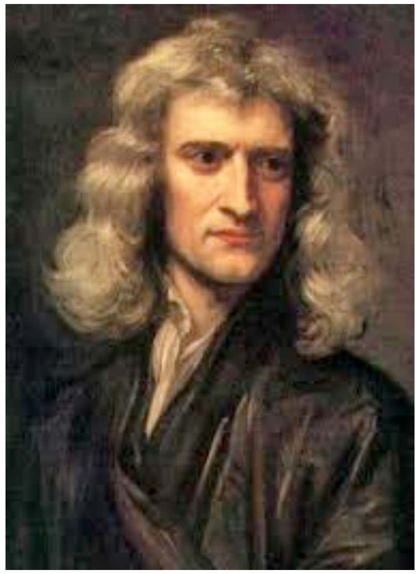

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Conservation of mechanical energy $\quad E = \dfrac{p^2}{2\,m} + U$

Energy as an eigenvalue

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Take $$B = -\frac{d\,U}{d\,t}U^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \frac{d\,K}{d\,q} \\ -\frac{d\,K}{d\,q} & 0 \end{pmatrix}$$

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$$\frac{dA}{dt} - [A, B] = \frac{1}{2}\left[\frac{dp}{dt} + \frac{dU}{dq}\right]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

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Hidden symmetry

$$A \to g\,A\,g^{-1}$$
$$B \to g\,B\,g^{-1} - \frac{dg}{dt}g^{-1}$$

$$g \in SU(2)$$

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Introduce "fake" variable x and denote $A_x \equiv A$ $A_t \equiv B$

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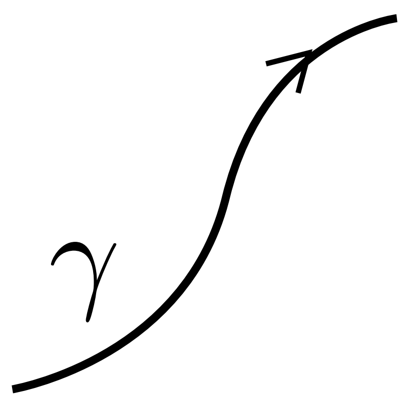
Wilson line $\quad \frac{d\,W}{d\,t} + A_\mu \frac{d\,x^\mu}{d\,\sigma} W = 0 \longrightarrow W = P\,e^{-\int_\gamma d\sigma\,A_\mu \frac{d\,x^\mu}{d\,\sigma}}$

$$W = 1 - \int_{\sigma_0}^{\sigma} d\sigma'\,A_\mu \frac{dx^\mu}{d\sigma'} + \int_{\sigma_0}^{\sigma} d\sigma'\,A_\mu(\sigma')\frac{dx^\mu}{d\sigma'}\int_{\sigma_0}^{\sigma'} d\sigma''\,A_\nu(\sigma'')\frac{dx^\nu}{d\sigma''} - \ldots$$



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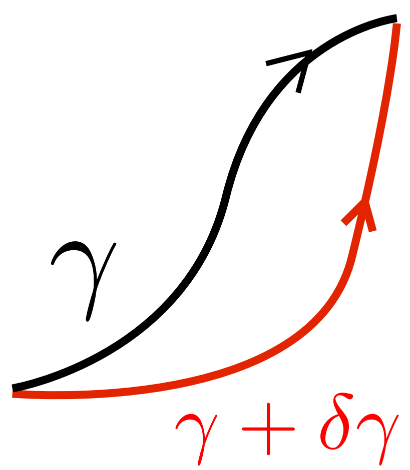

Newton's equation

$$W^{-1}\,\delta W = \int_\gamma d\sigma\, W^{-1} F_{\mu\nu}\, W \frac{dx^\mu}{d\sigma}\, \delta x^\nu \;=\; 0$$

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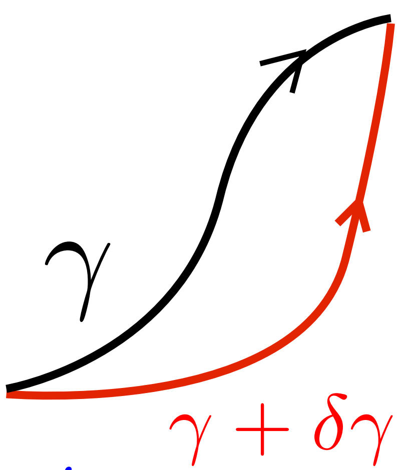

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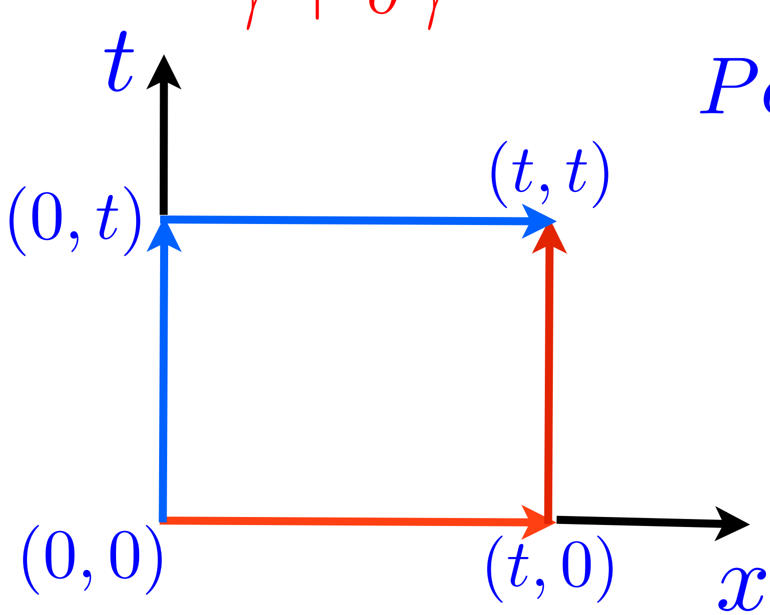

$$Pe^{-\int_0^t dx\, A(t)}\, Pe^{-\int_0^t dt'\, B(t')} = Pe^{-\int_0^t dt'\, B(t')}\, Pe^{-\int_0^t dx\, A(0)}$$

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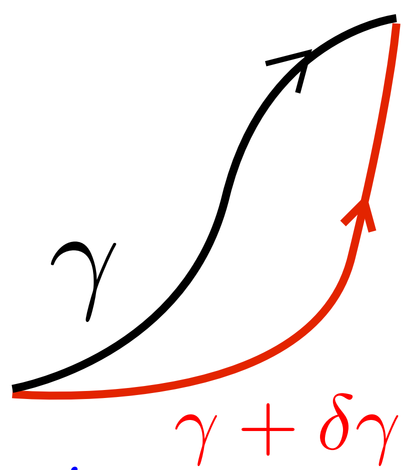

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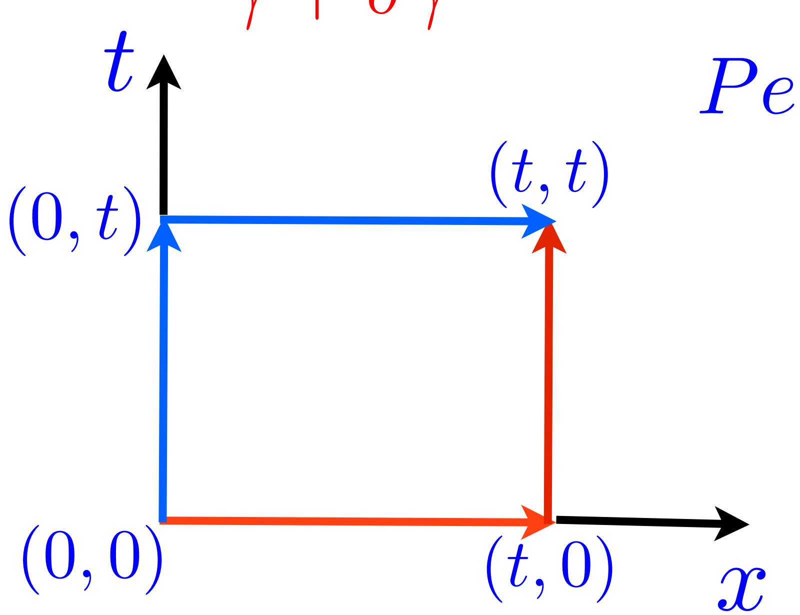

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$$U(t) = Pe^{-\int_0^t dt'\, B(t')}$$

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The general solution

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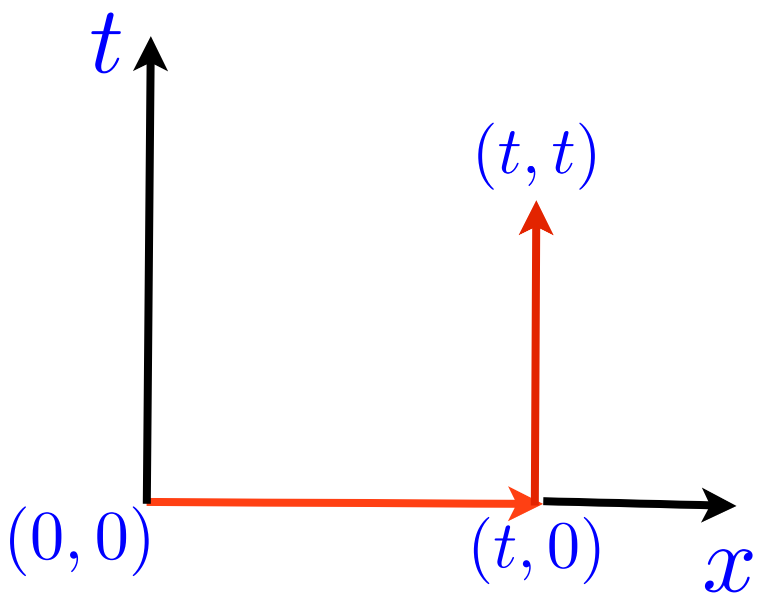

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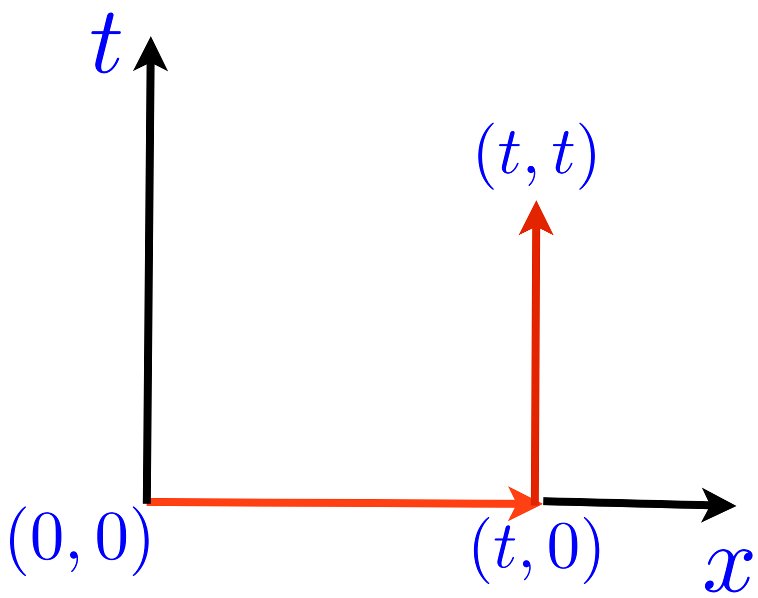

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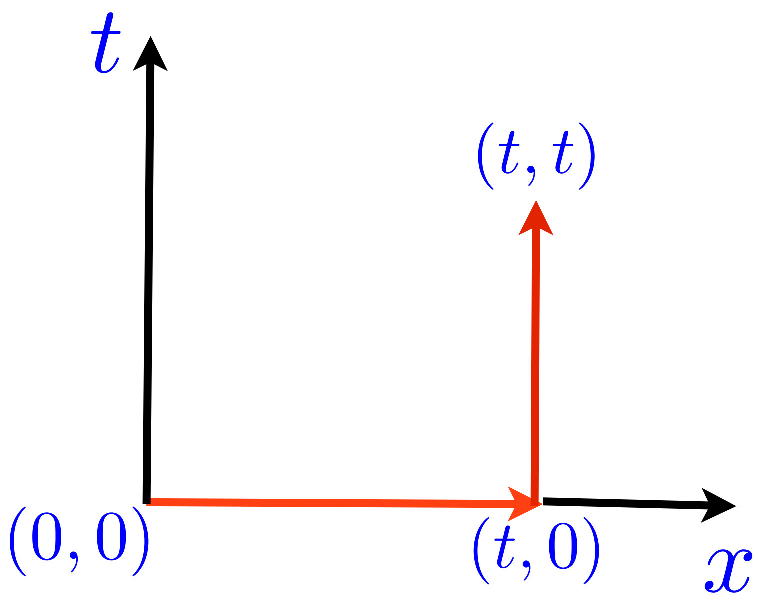

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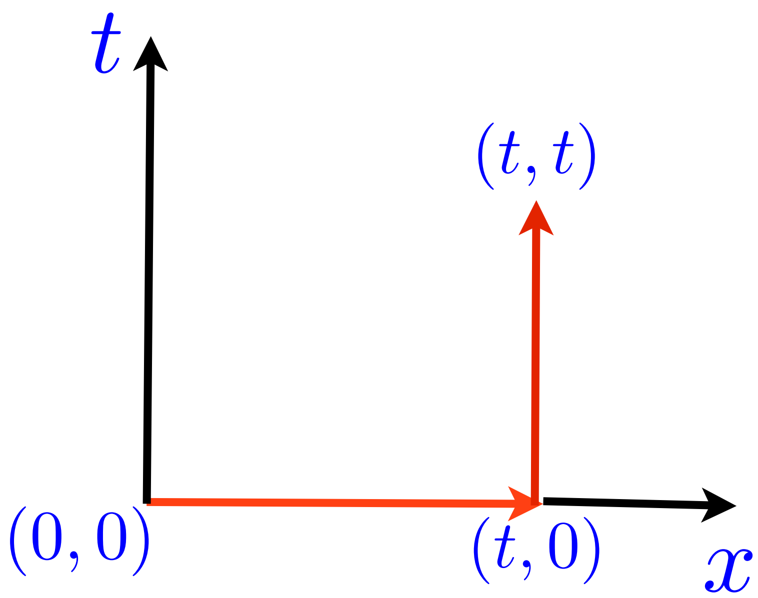

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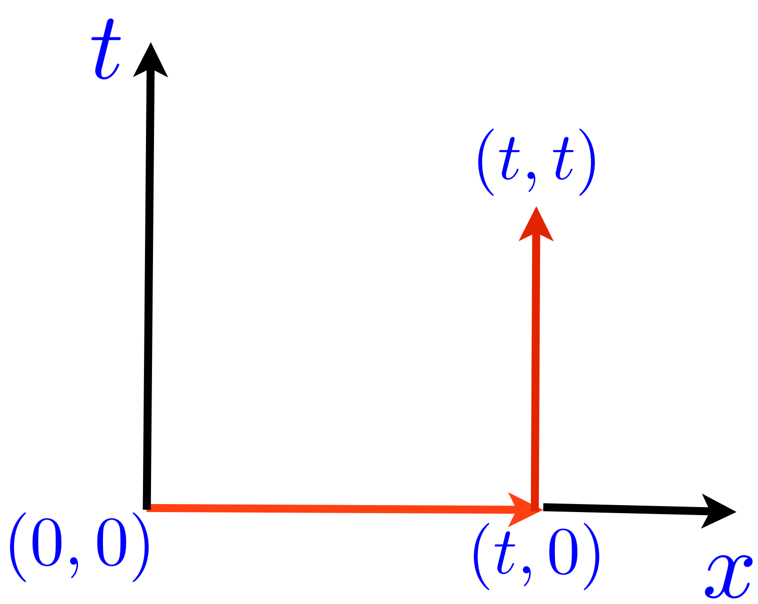

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Parametrizes the symmetric space $SU(2)/U(1)$

Phase space of a particle in 1-d

Hidden Symmetry in a coconut shell: Pendula on a clothes line

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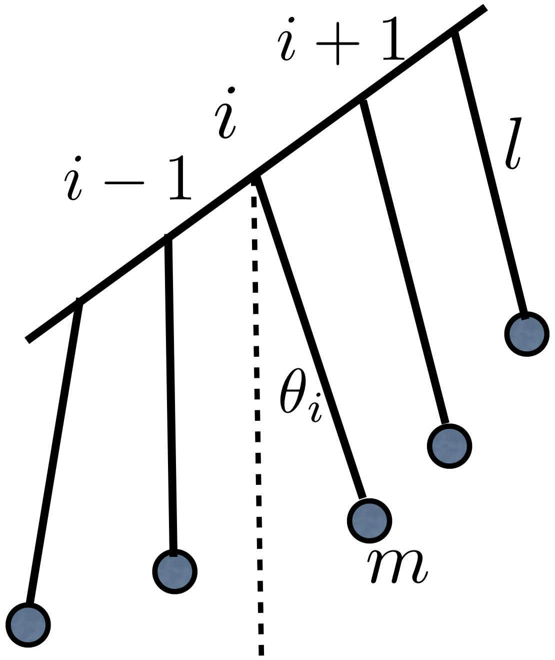

Newton's equation ($s_i \equiv l\,\theta_i$)

$$m\frac{d^2 s_i}{dt^2} = F_i^g + F_i^\tau$$

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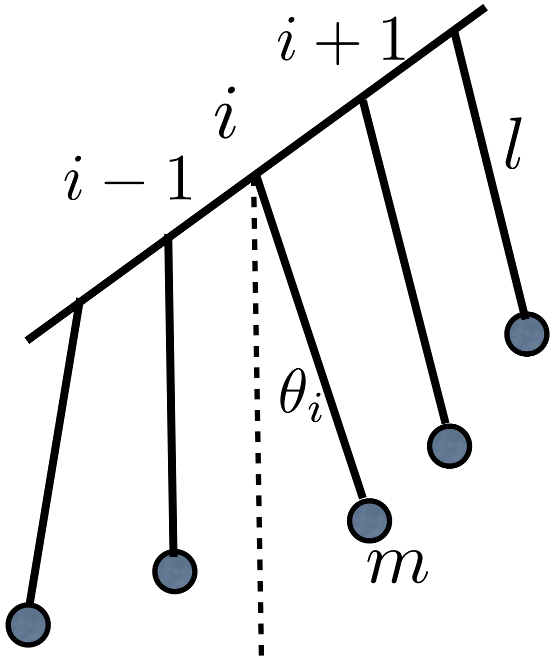

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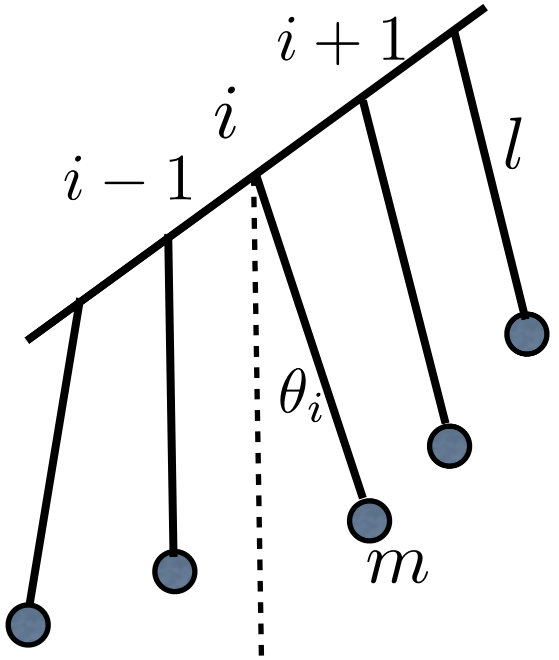

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$$-\frac{\alpha}{l}\,(\theta_i - \theta_{i-1}) + \frac{\alpha}{l}\,(\theta_{i+1} - \theta_i)$$

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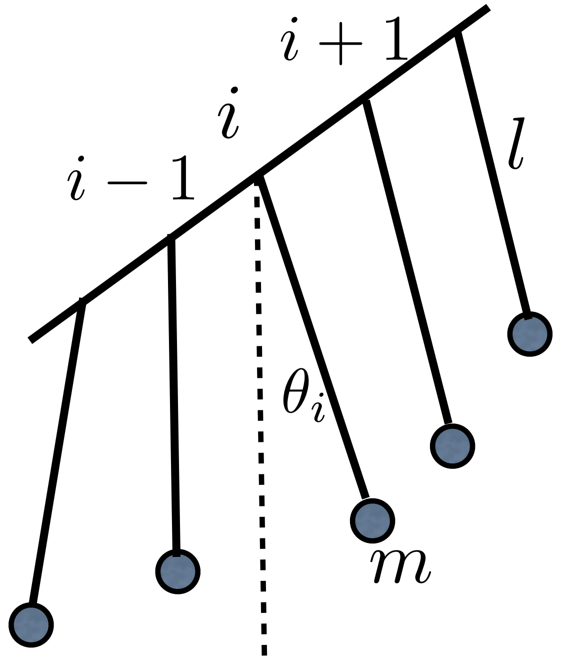

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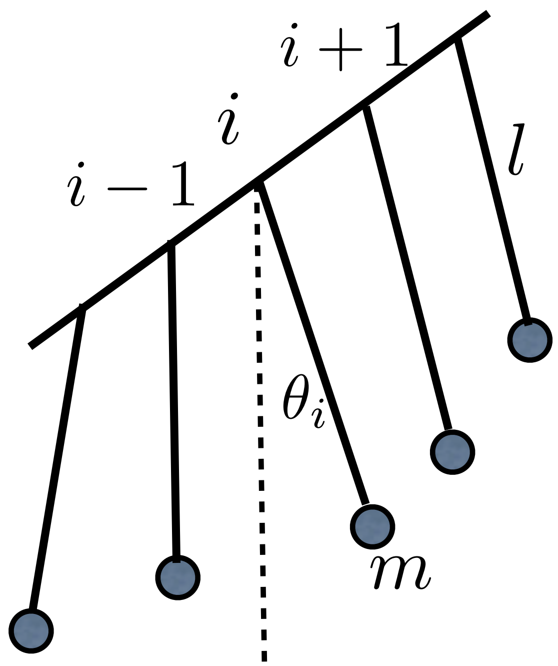
$$F_i^g \to -m\,g\,\sin\theta_i$$

$$F_i^\tau \to -\frac{\alpha}{l}\,(\theta_i - \theta_{i-1}) + \frac{\alpha}{l}\,(\theta_{i+1} - \theta_i)$$

Define $\quad \theta_i \equiv \theta(x_i) \qquad x_i = i\,\Delta \qquad \Delta \equiv \text{spacing between pendula}$

$$\frac{\theta_{i+1} - 2\,\theta_i + \theta_{i-1}}{\Delta^2} \to \frac{\partial^2\theta}{\partial x^2} \qquad \Delta \to 0$$

Hidden Symmetry in a coconut shell: Pendula on a clothes line



Newton's equation $(s_i \equiv l\,\theta_i)$

$$m\frac{d^2 s_i}{dt^2} = F_i^g + F_i^\tau$$

$$F_i^g \to -m\,g\,\sin\theta_i \qquad F_i^\tau \to -\frac{\alpha}{l}\,(\theta_i - \theta_{i-1}) + \frac{\alpha}{l}\,(\theta_{i+1} - \theta_i)$$

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In the limit $\Delta \to 0$, and $\alpha \to \infty$, with $\Delta^2\,\alpha \equiv$ finite, we get

sine-Gordon eq. $$\frac{1}{c^2}\,\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} = -\frac{\omega^2}{c^2}\,\sin\theta \qquad \left(c^2 = \frac{\alpha\,\Delta^2}{m\,l^2}\;;\;\omega^2 = \frac{g}{l}\right)$$

The magic of it ...

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$$A_0 = \frac{i}{4} \begin{pmatrix} \frac{\partial \theta}{\partial x^1} & \frac{\omega}{c}\left(e^{i\,\theta/2} + \frac{1}{\lambda}\, e^{-i\,\theta/2}\right) \\ \frac{\omega}{c}\left(e^{i\,\theta/2} + \lambda\, e^{-i\,\theta/2}\right) & -\frac{\partial \theta}{\partial x^1} \end{pmatrix}$$

$$A_1 = \frac{i}{4} \begin{pmatrix} \frac{\partial \theta}{\partial x^0} & \frac{\omega}{c}\left(e^{i\,\theta/2} - \frac{1}{\lambda}\, e^{-i\,\theta/2}\right) \\ -\frac{\omega}{c}\left(e^{i\,\theta/2} - \lambda\, e^{-i\,\theta/2}\right) & -\frac{\partial \theta}{\partial x^0} \end{pmatrix}$$

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Note that λ is arbitrary

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Loop Algebra $\quad \left[T_i^m\,,\,T_j^n\right] = i\,\varepsilon_{ijk}\,T_k^{m+n} \qquad (T_i^n = \lambda^n\,T_i)$

hidden symmetries

Kac-Moody Alg. $\left[T_i^m\,,\,T_j^n\right] = i\,\varepsilon_{ijk}\,T_k^{m+n} + C\,m\,\delta_{m+n,0}\,\delta_{i,j}$

Path independency

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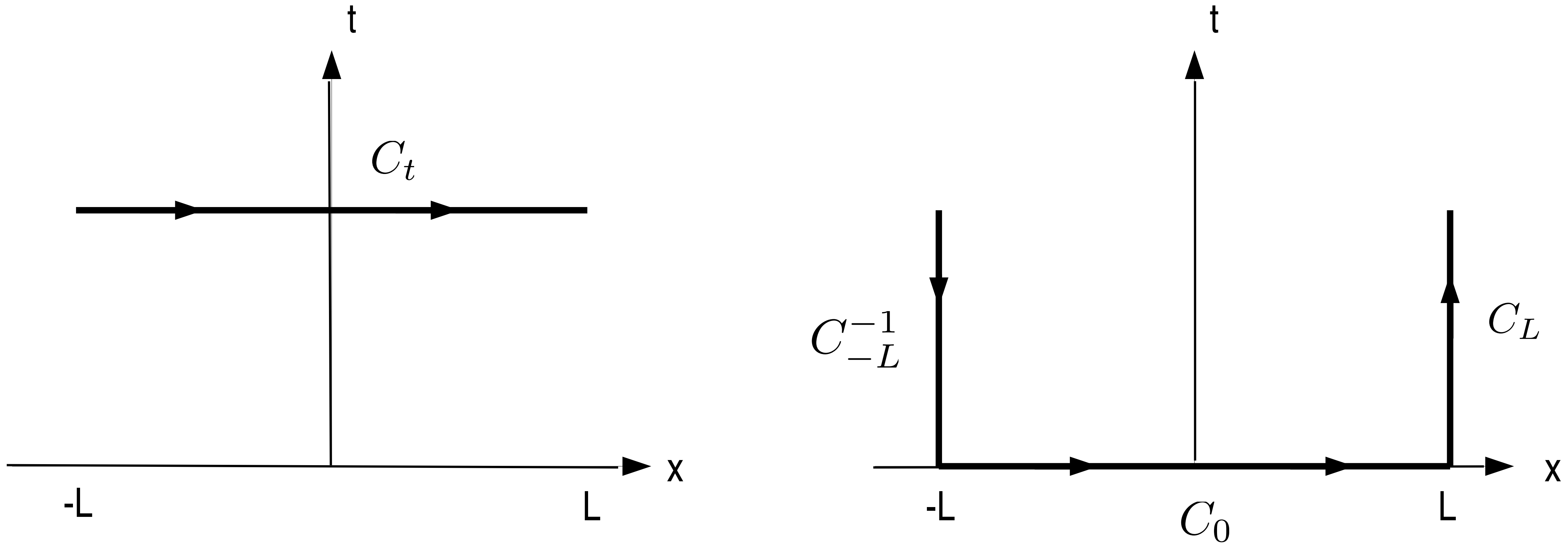

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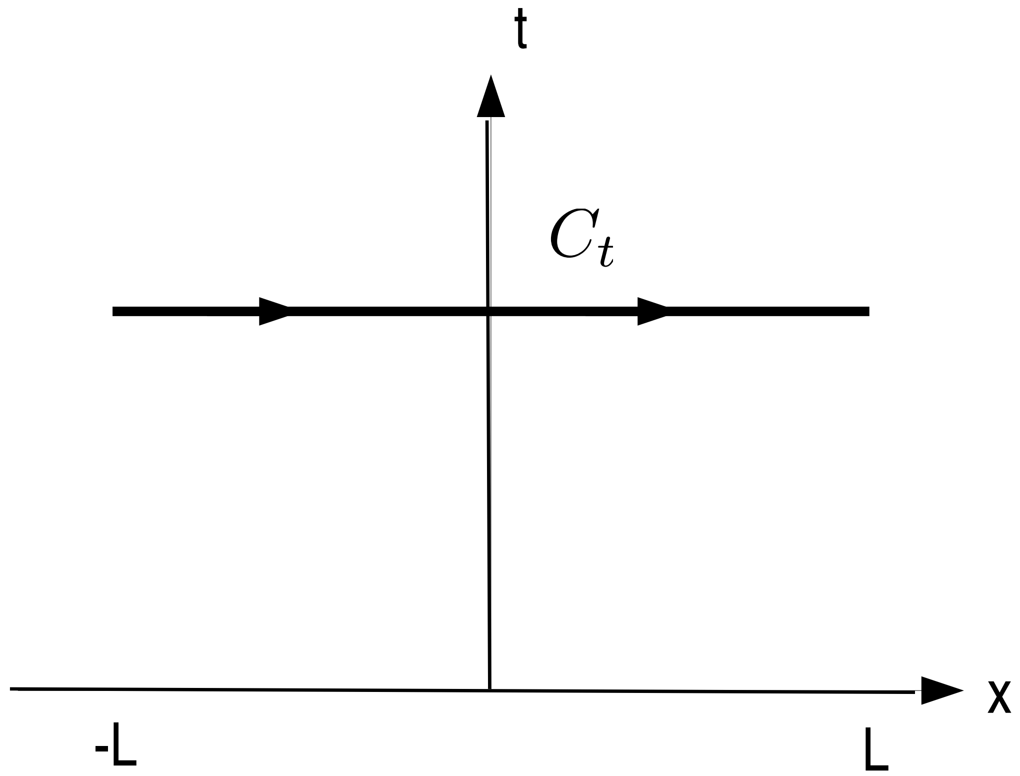

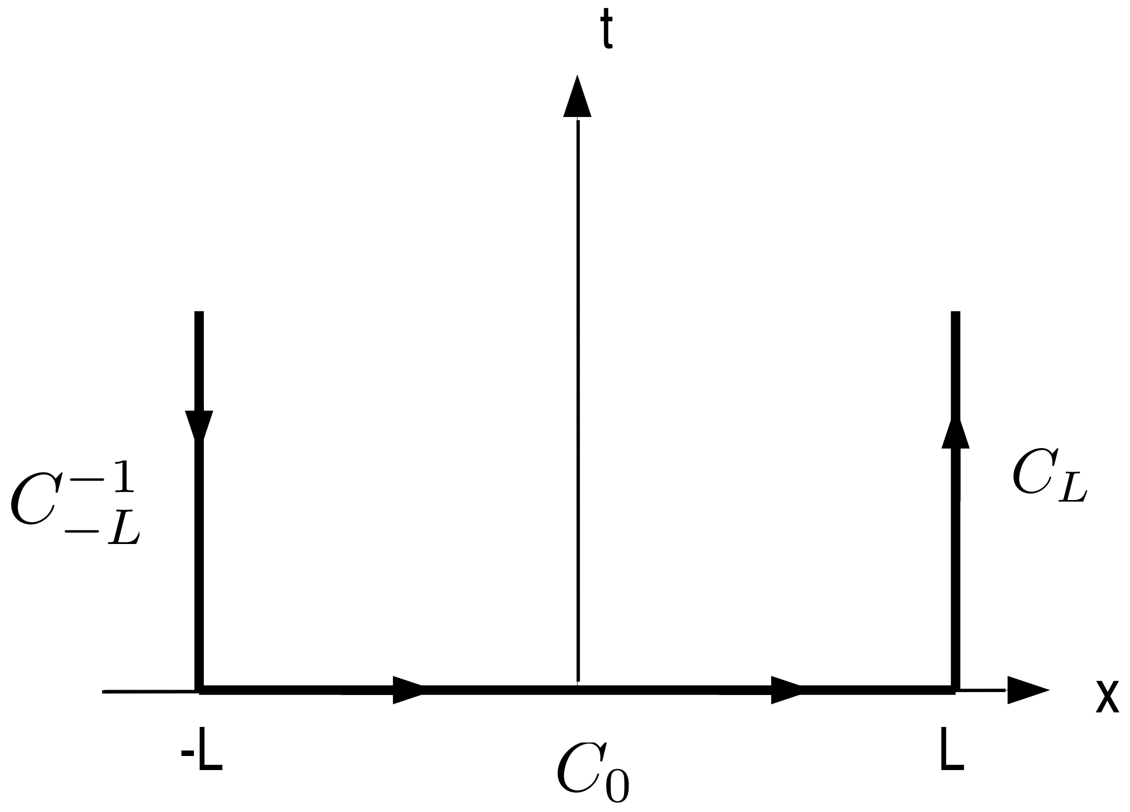

Boundary Condition $\quad A_t(-L,t) = A_t(L,t)$

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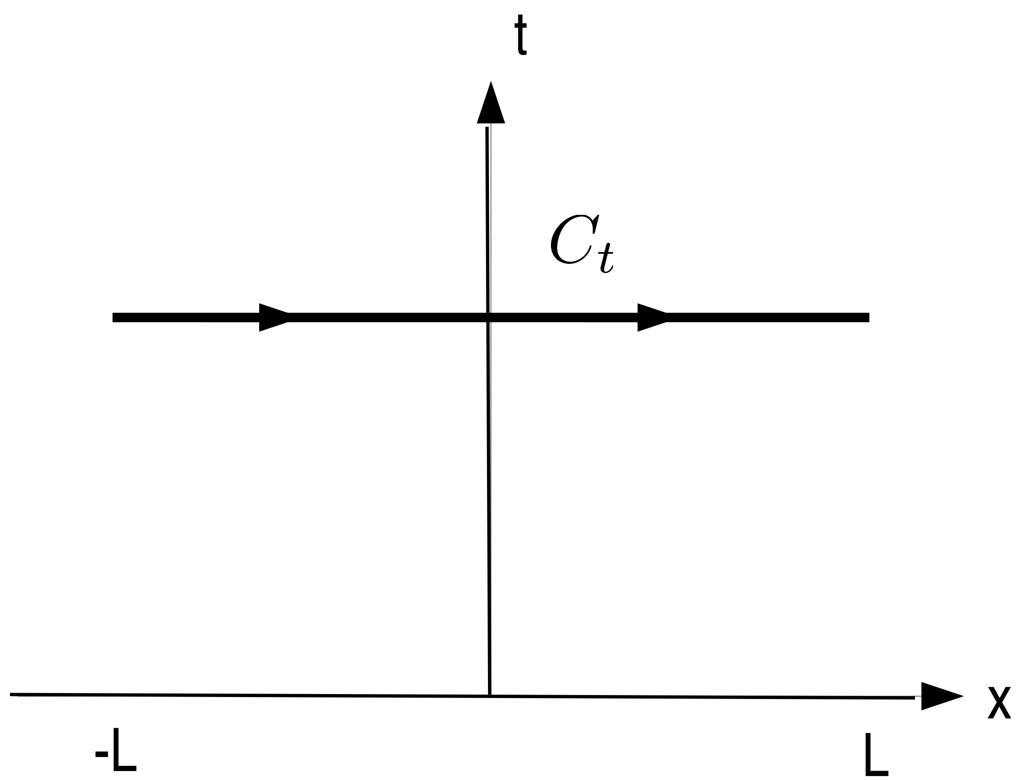

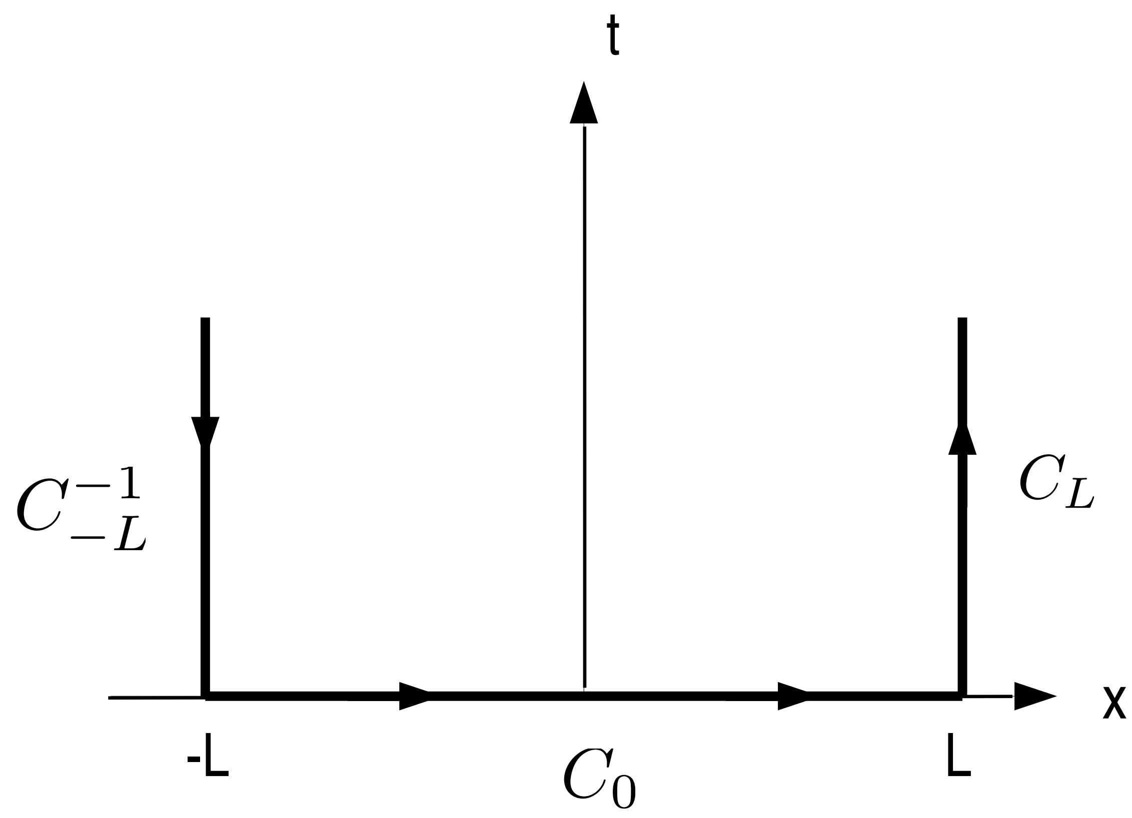

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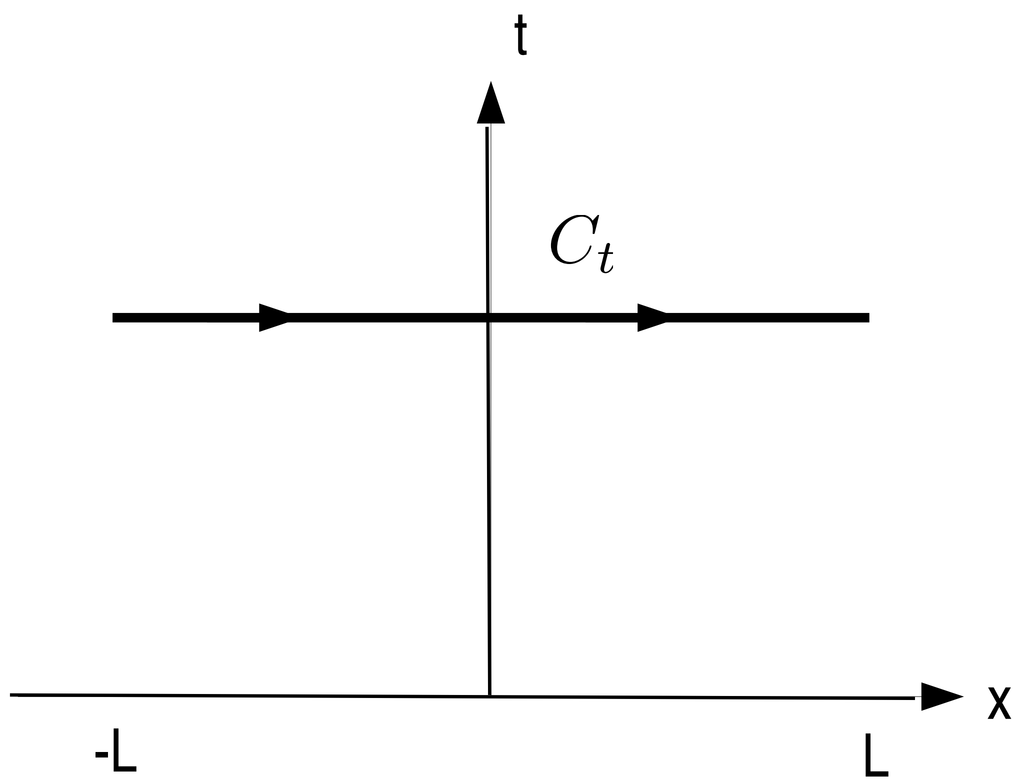

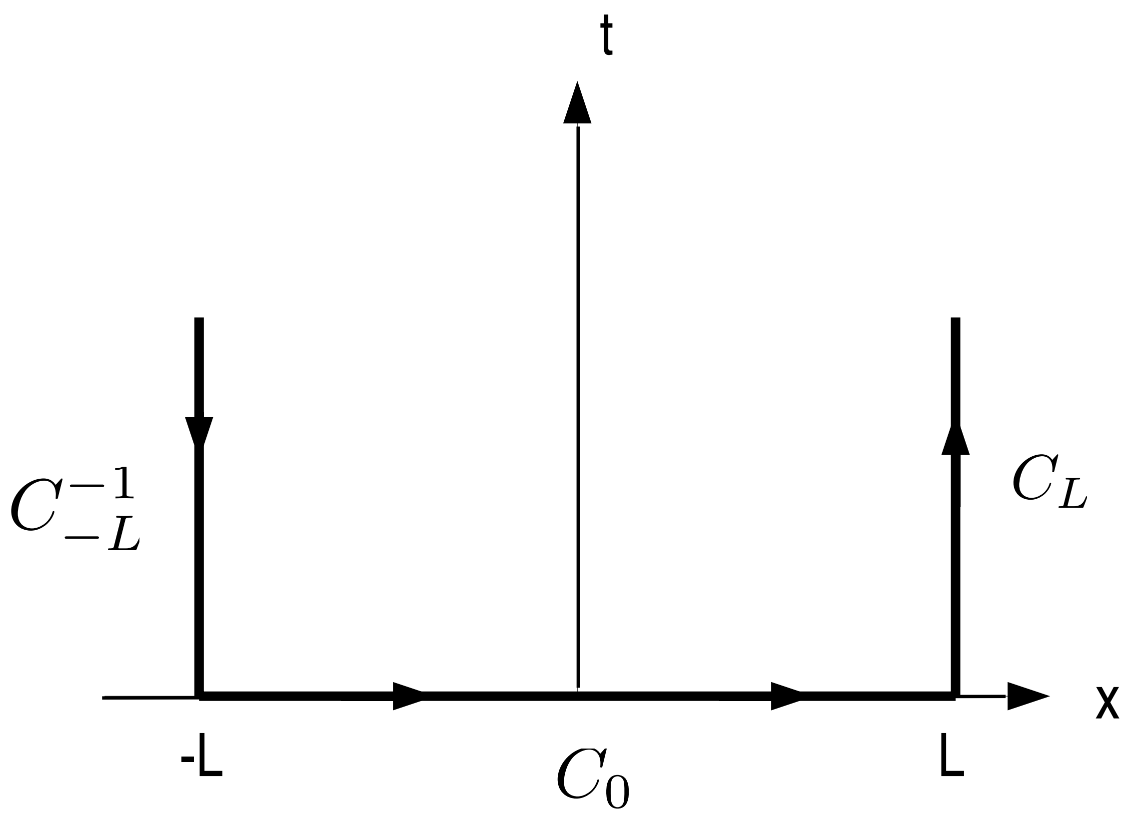

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power series in λ: infinite number of conserved quantities

$2 + 1$ dimensions

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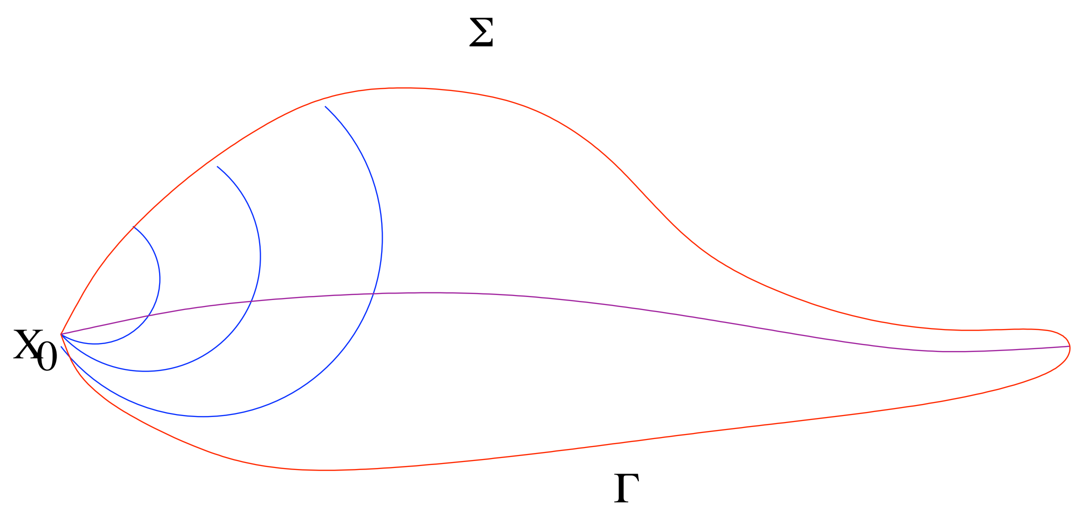
Charges should be integrals over 2d space



space-time surface

$2+1$ dimensions

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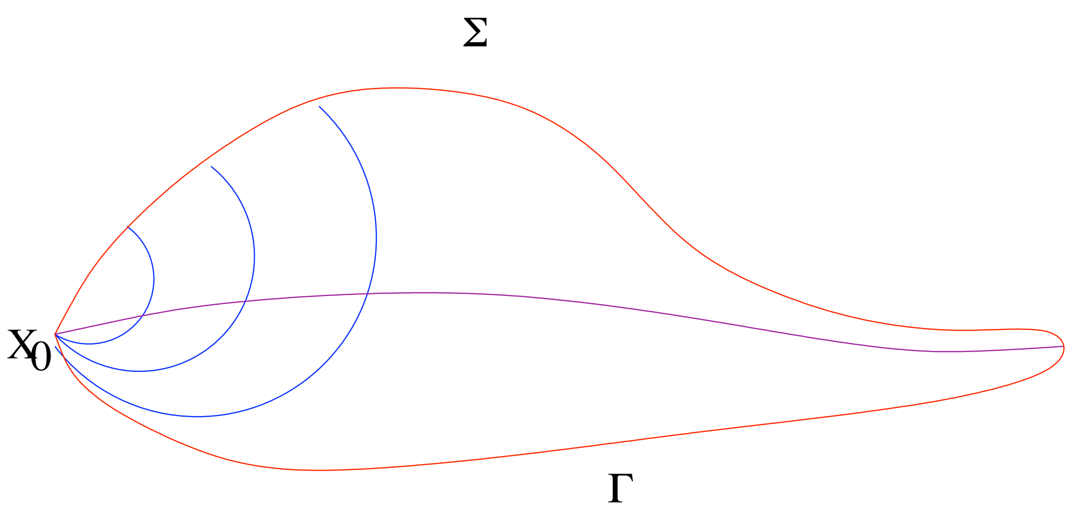

space-time surface

Loop Space: $\Omega^{(1)} = \{f : S^1 \to M \mid \text{north pole } \to x_0\}$

$2+1$ dimensions

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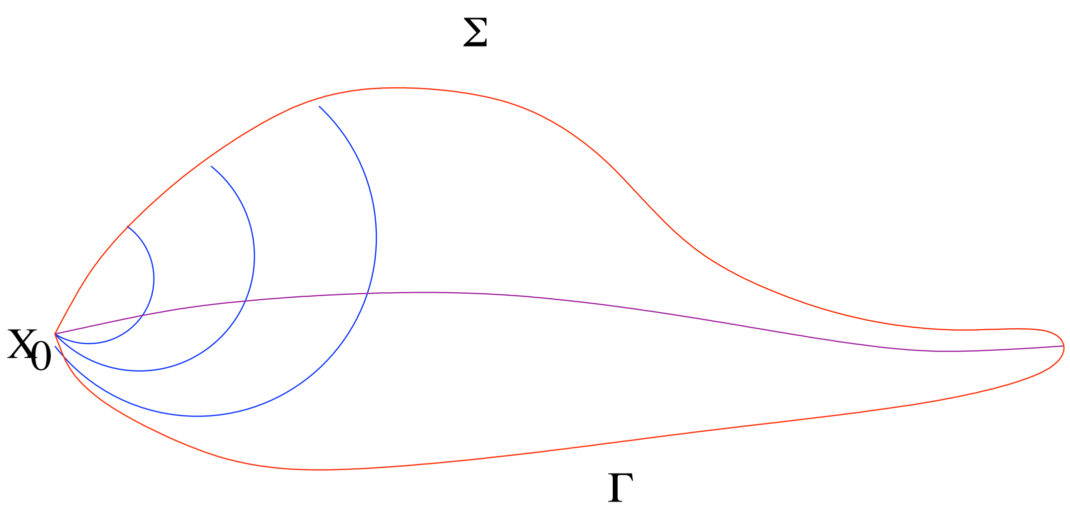

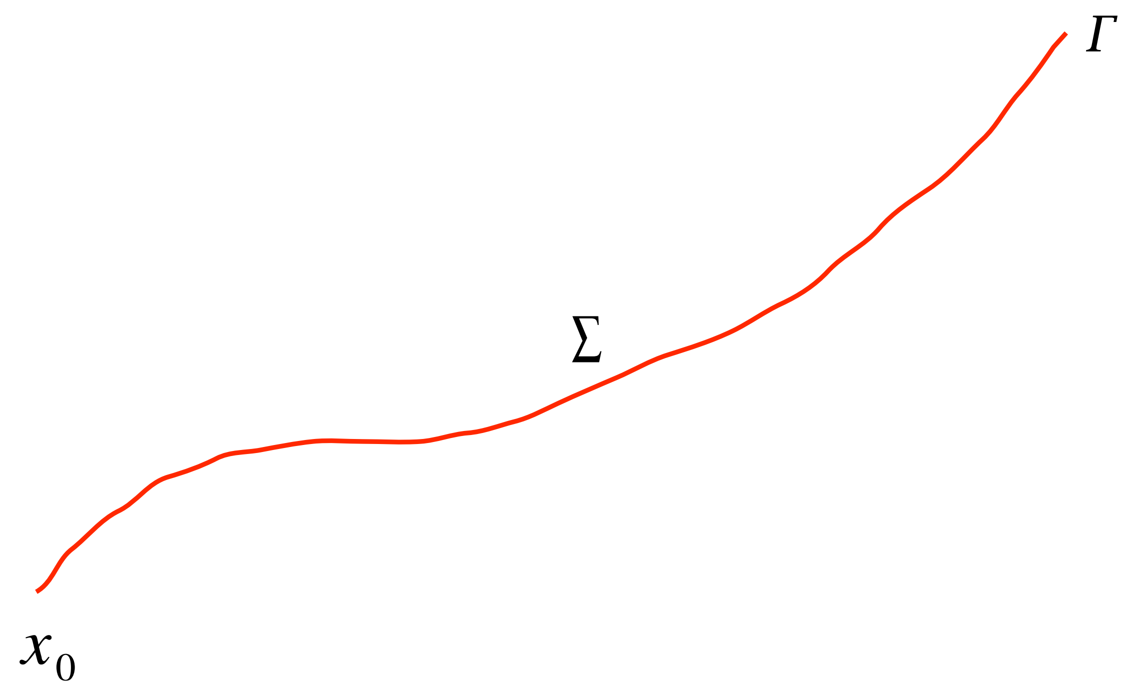

space-time surface $\longleftrightarrow$ path in loop space

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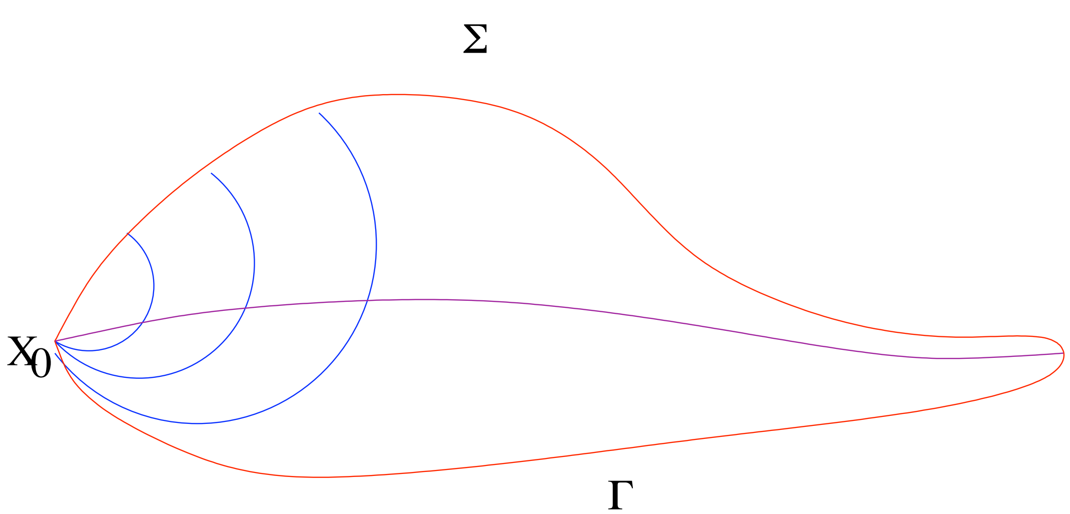

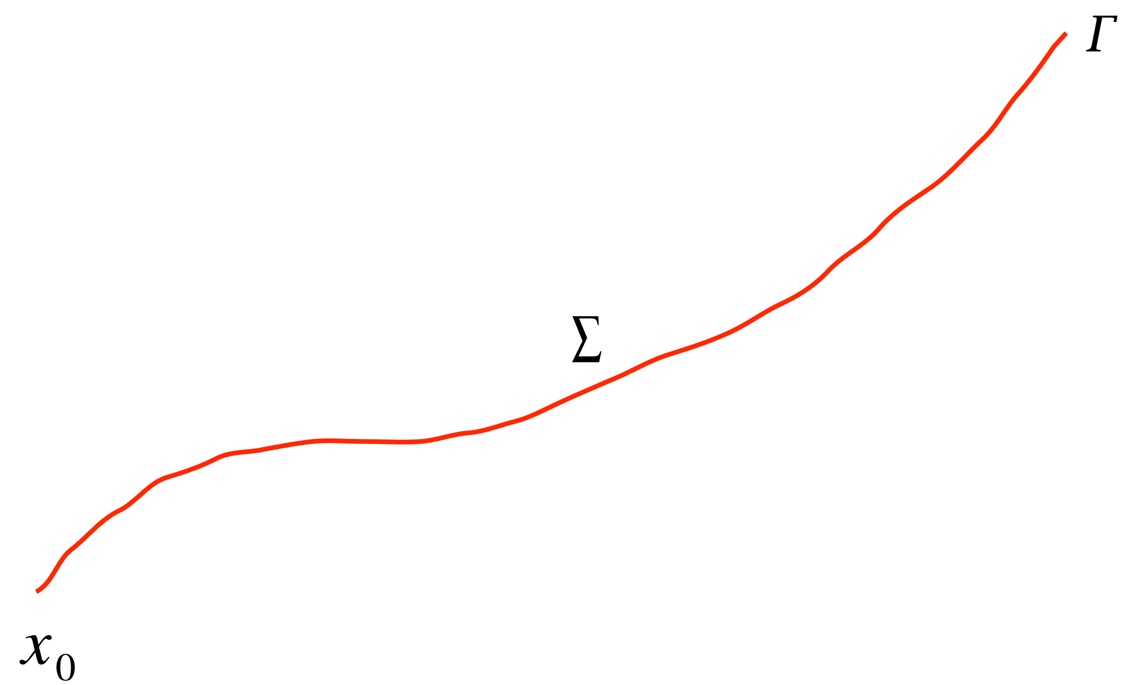

space-time surface $\longleftrightarrow$ path in loop space

Loop Space: $\Omega^{(1)} = \{ f : S^1 \to M \mid \text{north pole } \to x_0 \}$

Introduce a flat connection $\mathcal{A}$ in loop space

$$\mathcal{F} = \delta \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

Construct the charges using path independency!

The one-form connection on loop space

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$$\mathcal{A}[x(\sigma)] = \int_0^{2\pi} d\sigma \; W(\sigma)^{-1} B_{\mu\nu}(x(\sigma)) W(\sigma) \frac{dx^\mu}{d\sigma} \delta x^\nu(\sigma)$$

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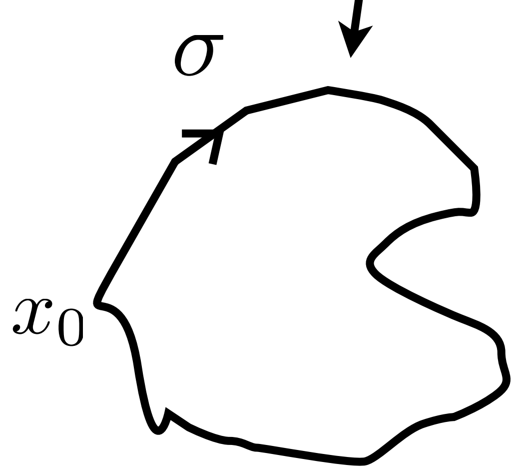
integrated over a loop



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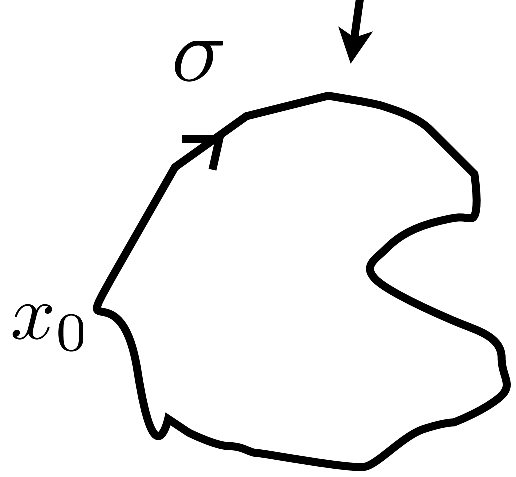
antisymmetric tensor

integrated over a loop



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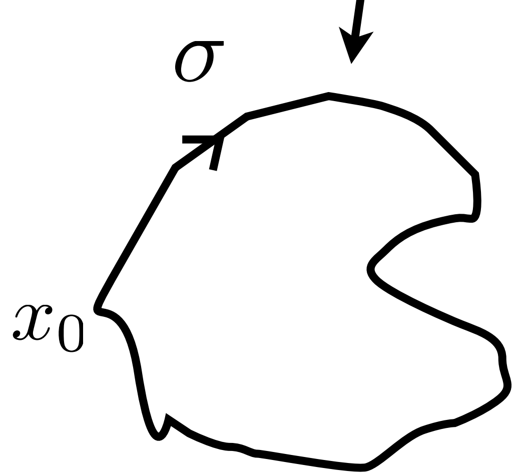

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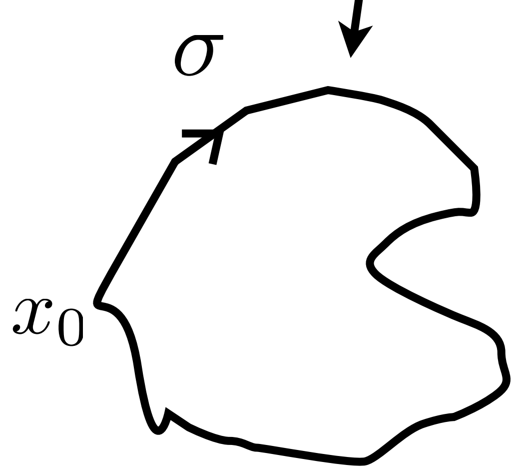
variations perpendicular to the loop



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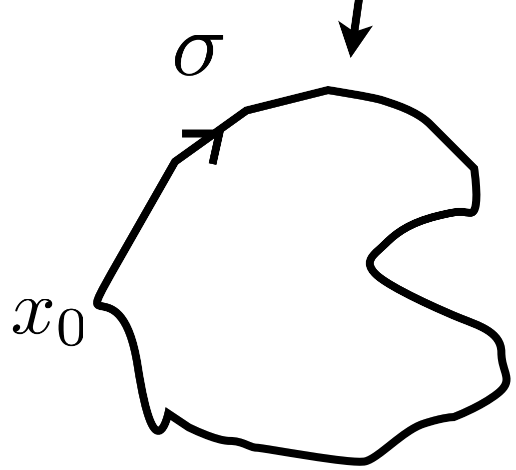

variations perpendicular to the loop

$$A_\mu \to g\,A_\mu\,g^{-1} + \frac{i}{e}\,\partial_\mu g\,g^{-1} \qquad \text{then} \qquad W \to g(x)\,W\,g^{-1}(x_0)$$

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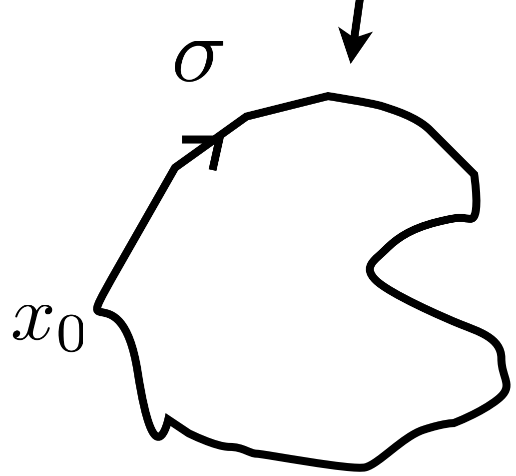

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The curvature on loop space

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$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

$$\begin{aligned}
\mathcal{F} = \quad & -\frac{1}{2}\int_0^{2\pi} d\sigma\, W^{-1}(\sigma)\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right](x(\sigma))W(\sigma)\frac{dx^\lambda}{d\sigma}\;\delta x^\mu(\sigma) \wedge \delta x^\nu(\sigma) \\
+ \quad & \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' \left[B^W_{\kappa\mu}(x(\sigma')) - F^W_{\kappa\mu}(x(\sigma')), B^W_{\lambda\nu}(x(\sigma))\right] \frac{dx^\kappa}{d\sigma'}\frac{dx^\lambda}{d\sigma}\;\delta x^\mu(\sigma') \wedge \delta x^\nu(\sigma)
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$$D_\mu * \equiv \partial_\mu * + i\, e\, [A_\mu \,,\, *]$$

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Problems to have $\mathcal{F} = 0$:

- Non-locality
- Dependency upon reparameterization
- Hard to reconcile with local field theories

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$$[T_a\, ,\, T_b] = i\, f_{abc}\, T_c$$

$$[T_a\, ,\, P_i] = P_j\, R_{ji}(T_a)$$

$$[P_i\, ,\, P_j] = 0$$

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CP^1-model
Skyrme model
Skyrme-Faddeev model
Self-dual YM, etc

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Connects to:
- Gerbes
- Two-form connections
- Higher spin gauge theories
- etc

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$$+ \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' \left[B^W_{\kappa\mu}(x(\sigma')) - F^W_{\kappa\mu}(x(\sigma')), B^W_{\lambda\nu}(x(\sigma))\right]\frac{dx^\kappa}{d\sigma'}\frac{dx^\lambda}{d\sigma}\,\delta x^\mu(\sigma')\wedge\delta x^\nu(\sigma)$$

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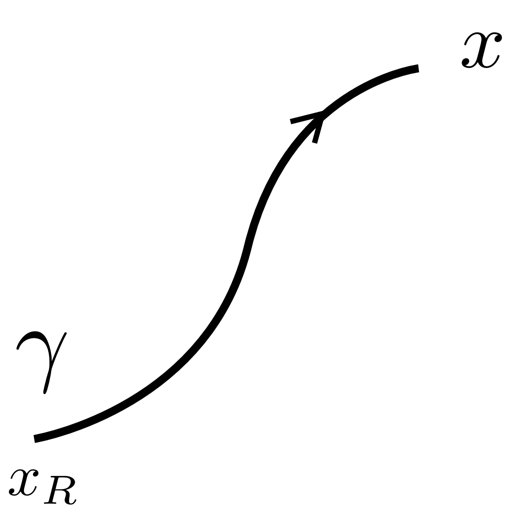
Orlando Alvarez, LAF and J. Sánchez Guillén
hep-th/9710147, *Nucl. Phys.* **B529** (1998) 689-736
IJMPA, **24** (2009) 1825 - 1888; arXiv:0901.1654 [hep-th].

Revisit integrable field theories in $1 + 1$ dimensions

Revisit integrable field theories in $1+1$ dimensions



Let $g(x)$ and A_μ be functionals of the fields of the theory and impose on any curve γ the integral equation

$$g\,(x)\; g^{-1}\,(x_R) = P_1\, e^{-\int_\gamma d\sigma\, A_\mu\, \frac{d\,x^\mu}{d\sigma}}$$

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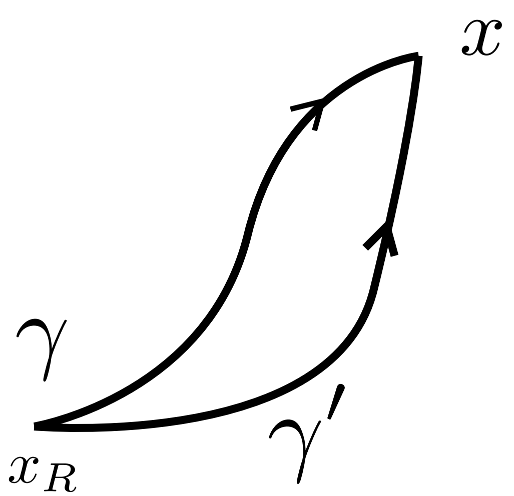

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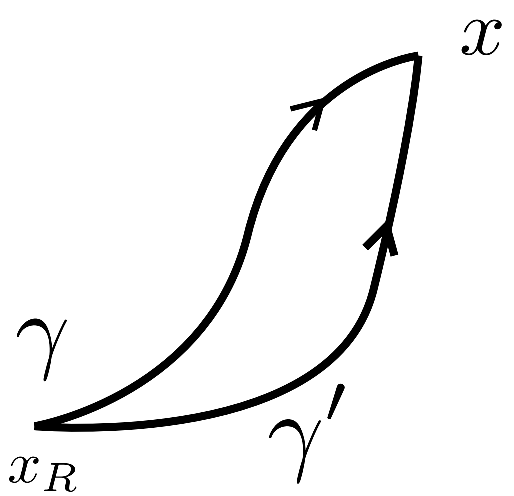

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path independent

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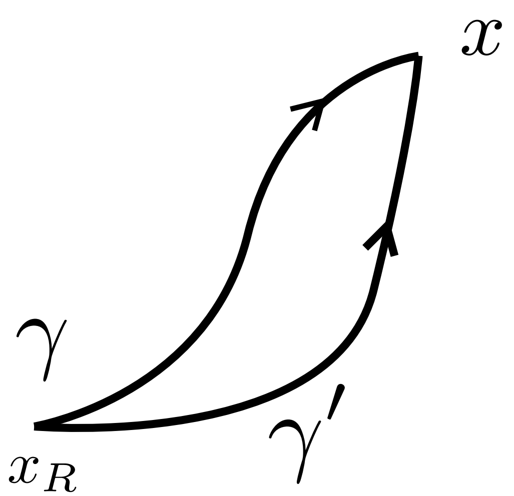

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path independent $\longrightarrow$ conservation laws

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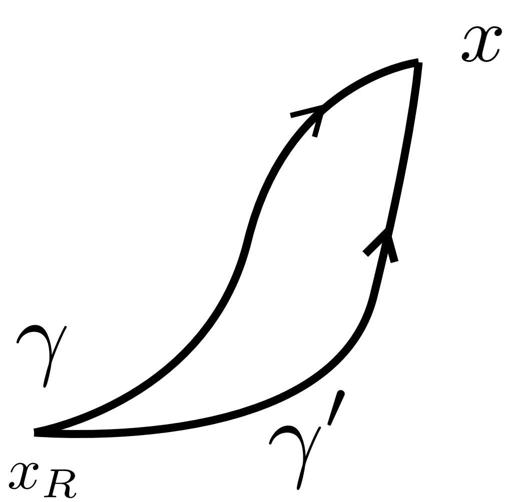

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Take γ infinitesimal

$$A_\mu = -\partial_\mu g\,g^{-1}$$

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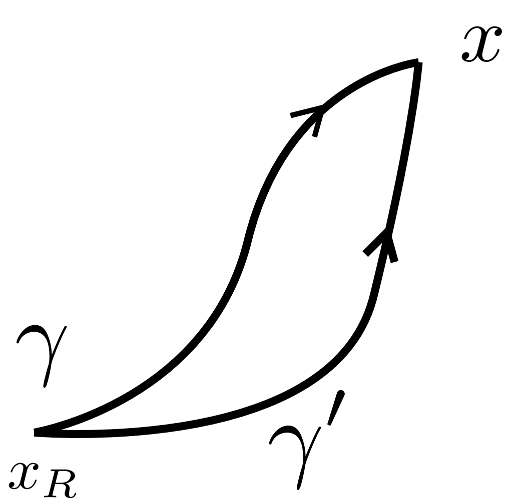

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So, A_μ is flat and we have Lax-Zakharov-Shabat equation

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,,\,A_\nu] = 0$$

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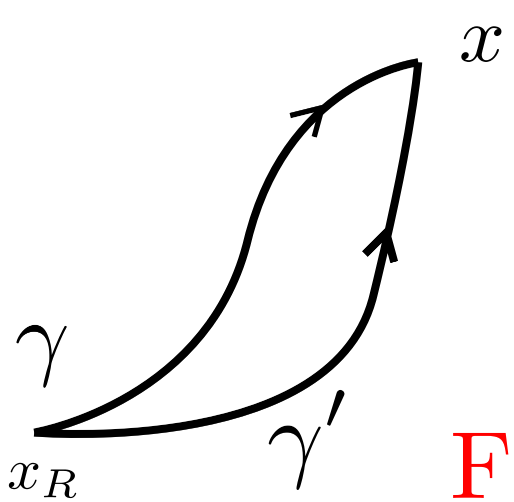

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Flux

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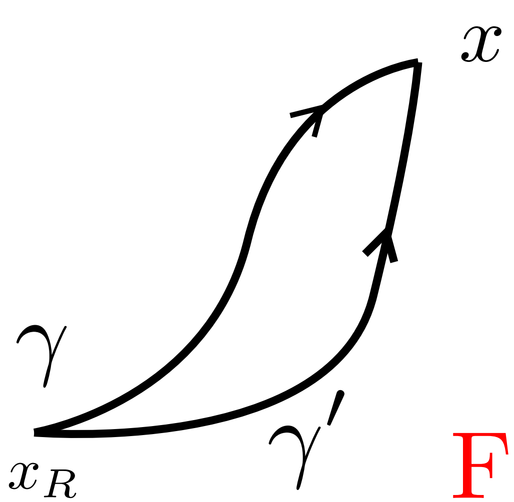

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Look for integral equations!! $P_{d-1}e^{\int_{\partial\Omega}\mathcal{A}} = P_d e^{\int_\Omega \mathcal{F}}$

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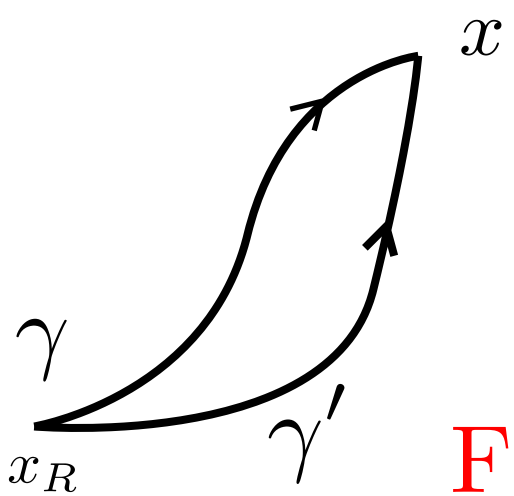

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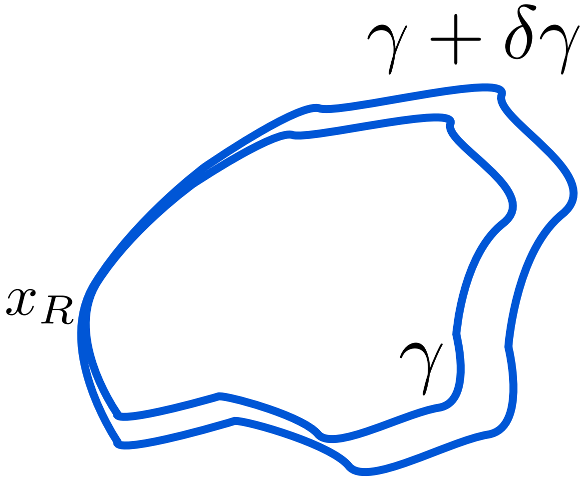
Basic property of gauge theories: Flux=Charge

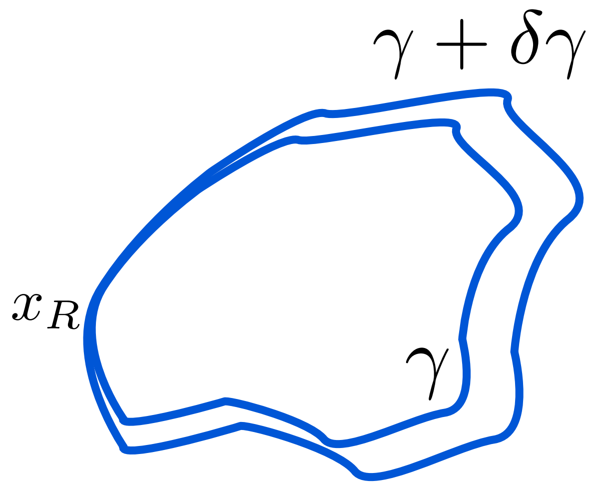
Non-Abelian Stokes Theorem

Non-Abelian Stokes Theorem



$$W^{-1}(\gamma)\delta W(\gamma) = \int_0^{2\pi} d\sigma\, W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma} \delta x^\nu$$

Non-Abelian Stokes Theorem



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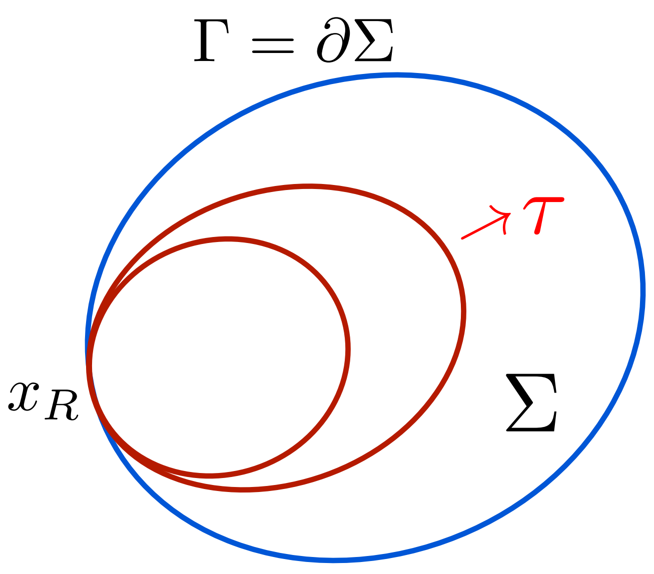

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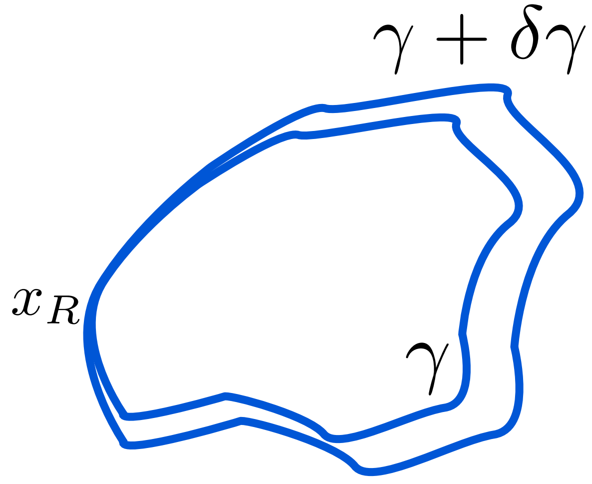

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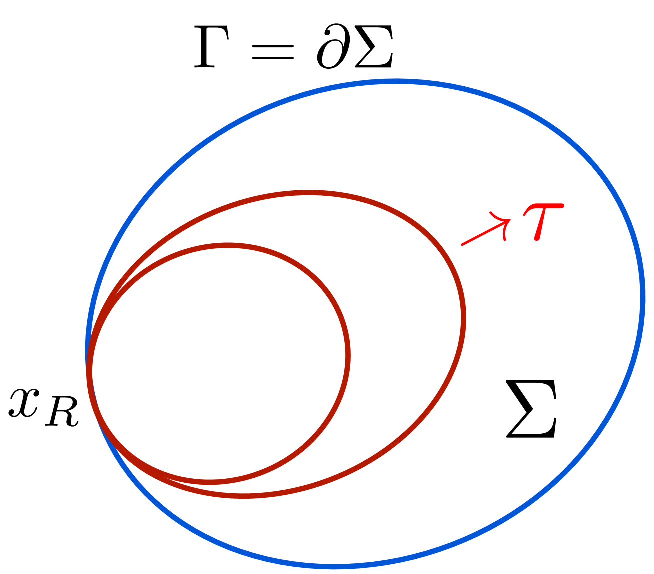
$$\Downarrow$$



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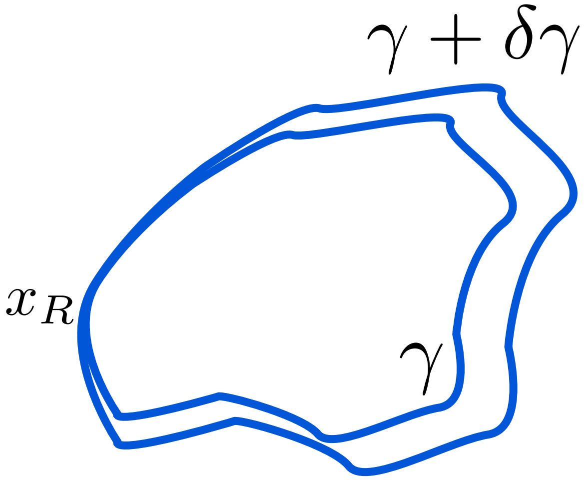

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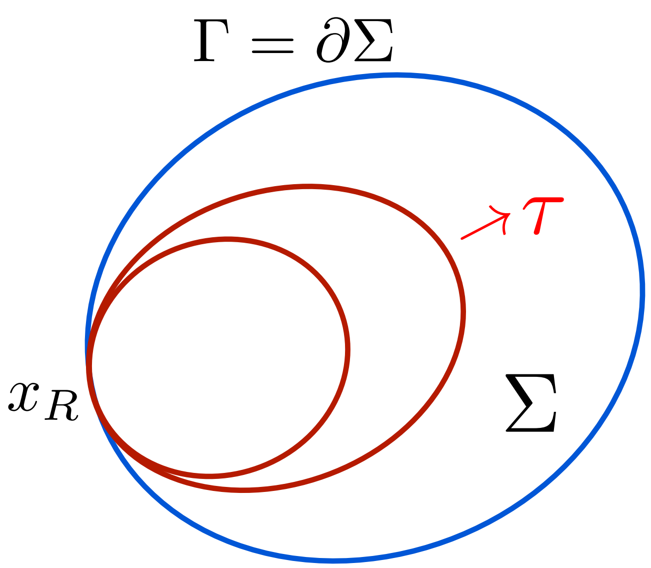

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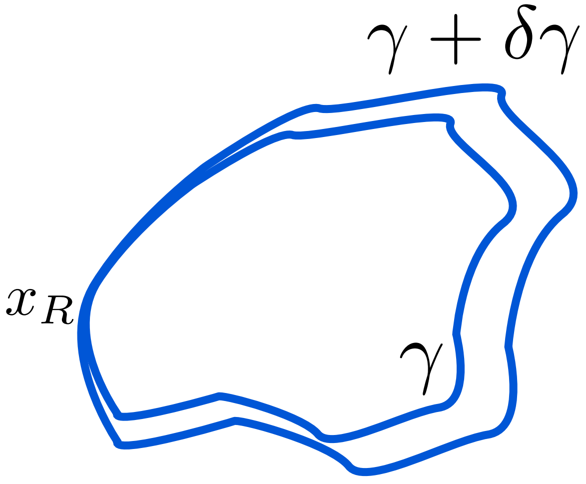

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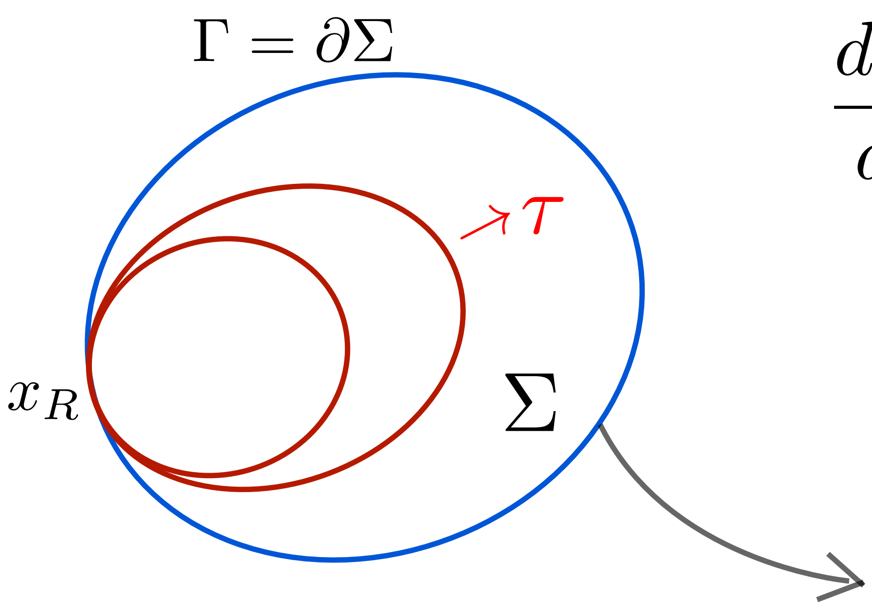

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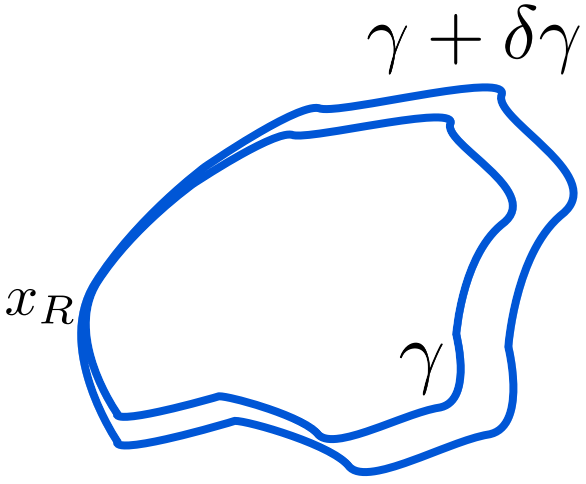

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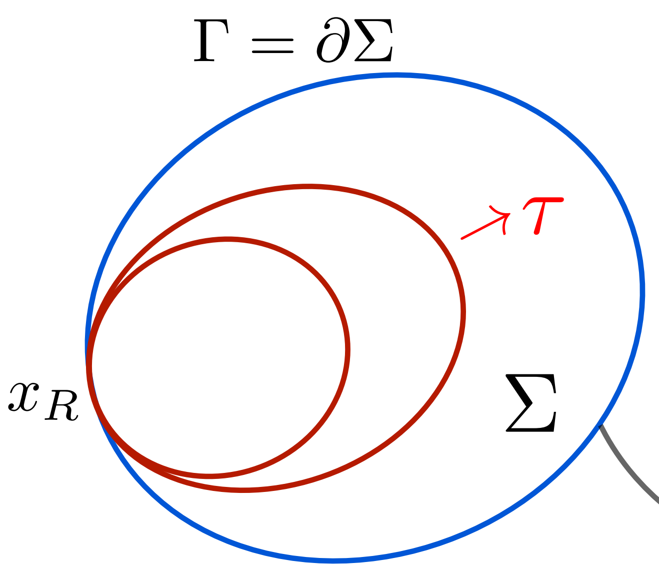

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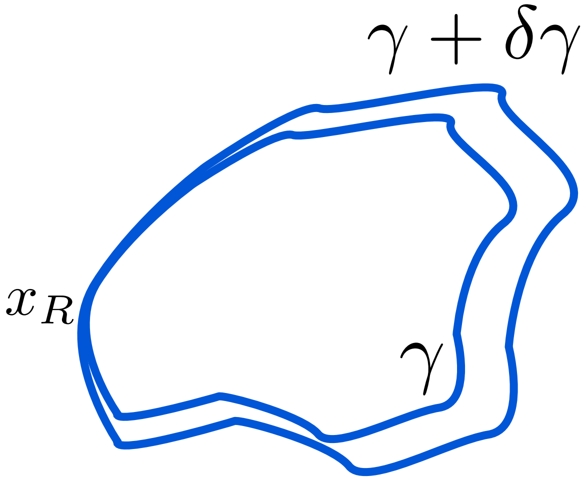

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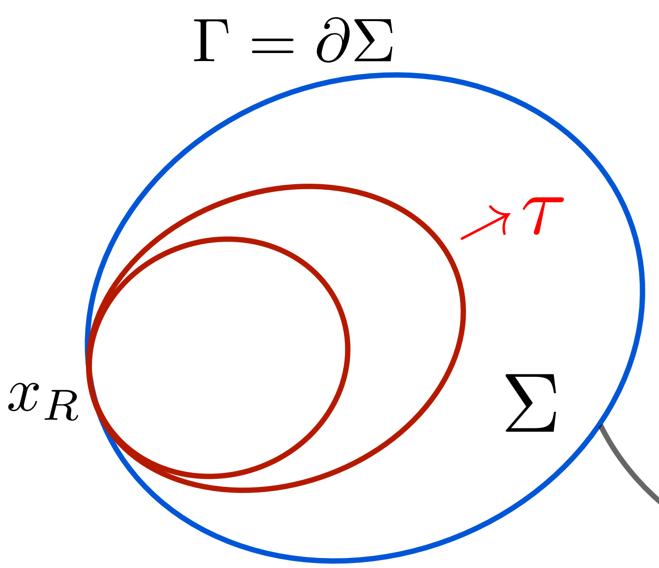

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$$\int_{\partial\Sigma} A = \int_\Sigma d \wedge A$$

The Flatness Condition

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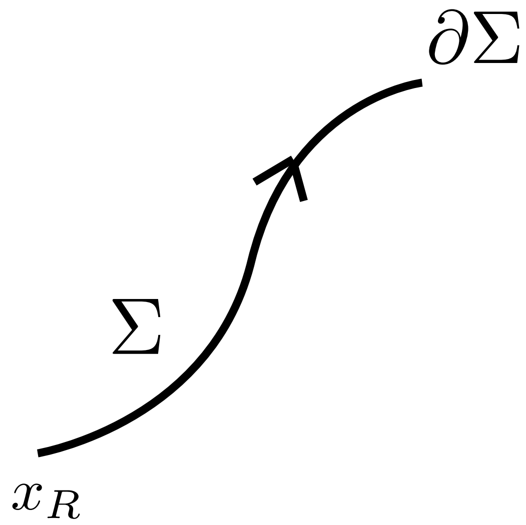

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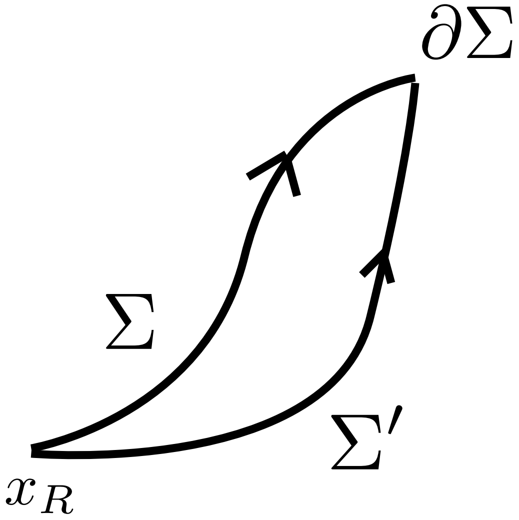

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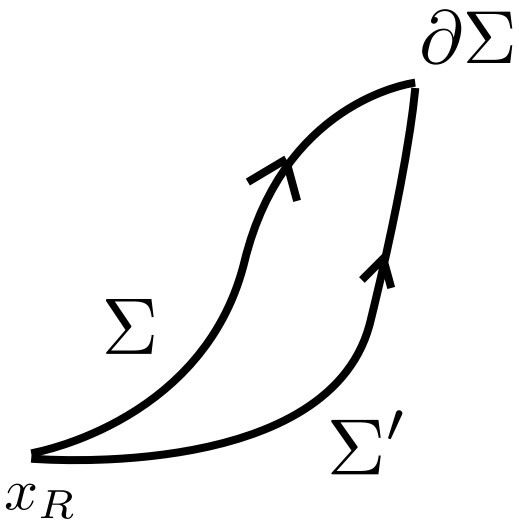

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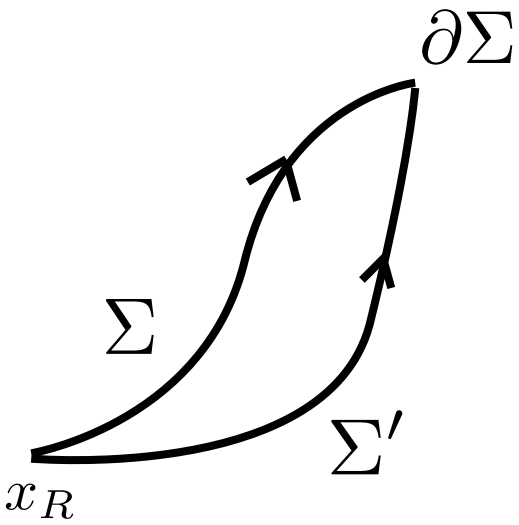

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Flat connection on loop space

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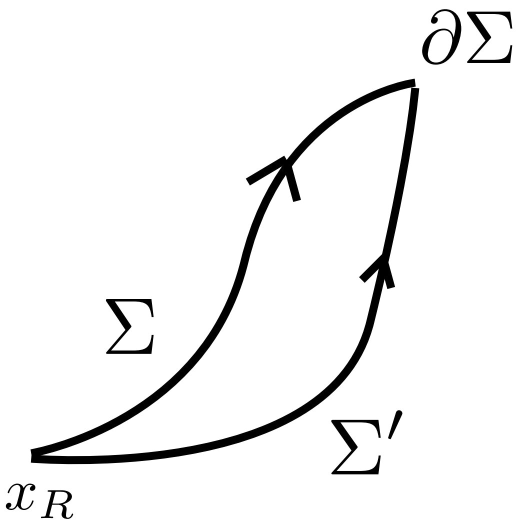

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Flat connection on loop space

Path independency on loop space $\rightarrow$ conservation laws in $2+1$

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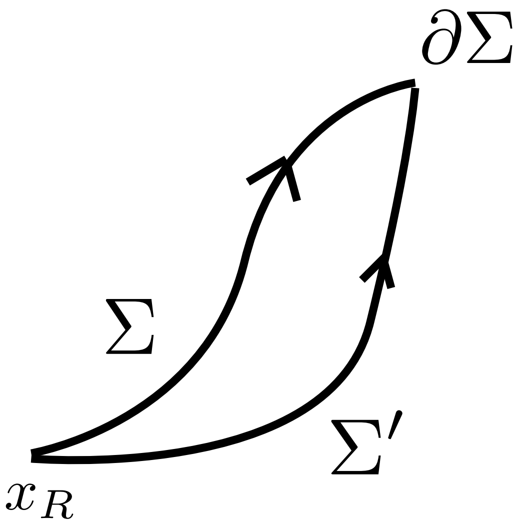

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Examples:

Chern-Simons in $2+1$

Yang-Mills in $2+1$

LAF and G. Luchini, NPB 858 (2012), 336; arXiv 1109.2606

Generalizing Faraday: Non-Abelian integrals

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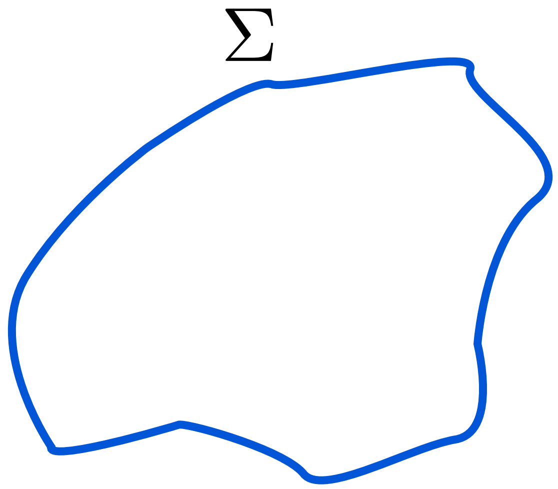

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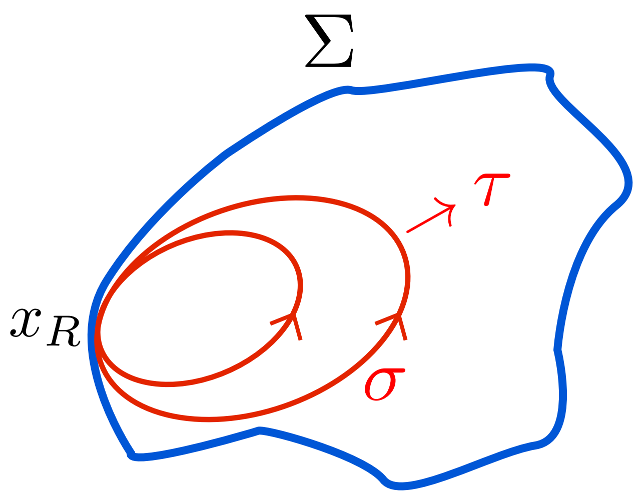

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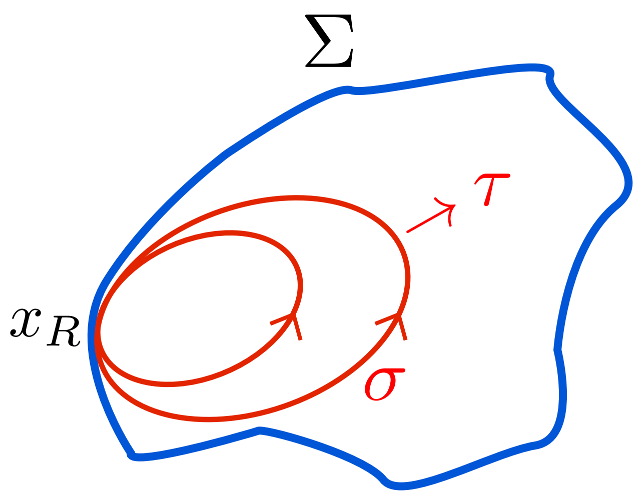
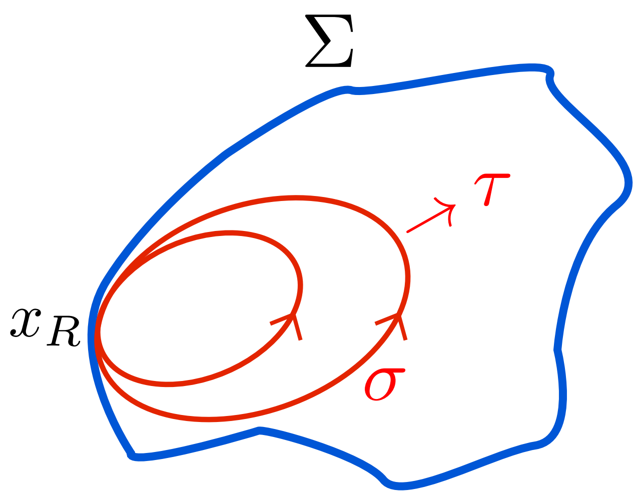


$$\frac{dV}{d\tau} - V\,T\,(A, B, \tau) = 0$$

$$T\,(B, A, \tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}$$

Generalizing Faraday: Non-Abelian integrals



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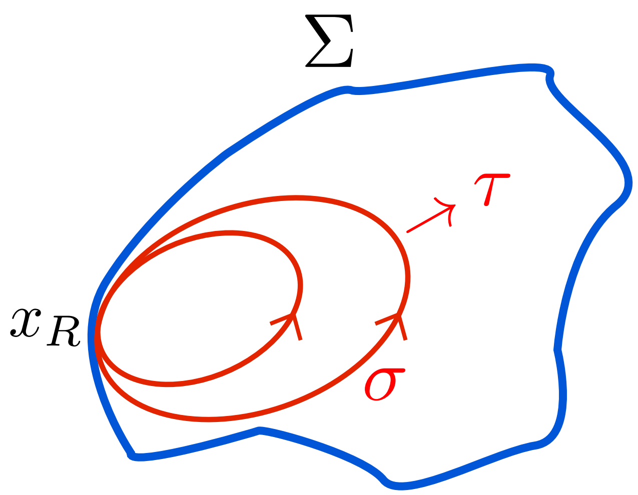
It is a surface ordered integral

$$V(\Sigma) = V_R\, P_2\, e^{\int_\Sigma d\sigma d\tau\, W^{-1}\, B_{\mu\nu}\, W\, \frac{dx^\mu}{d\sigma}\, \frac{dx^\nu}{d\tau}}$$

Generalizing Faraday: Non-Abelian integrals



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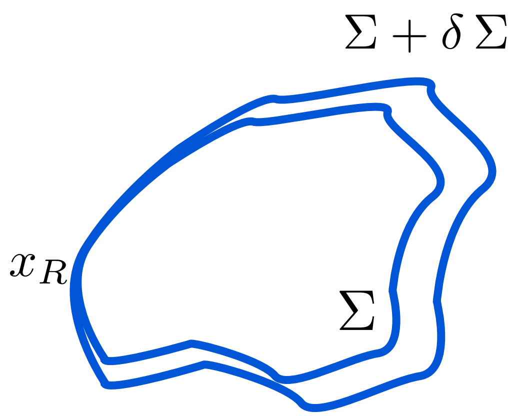
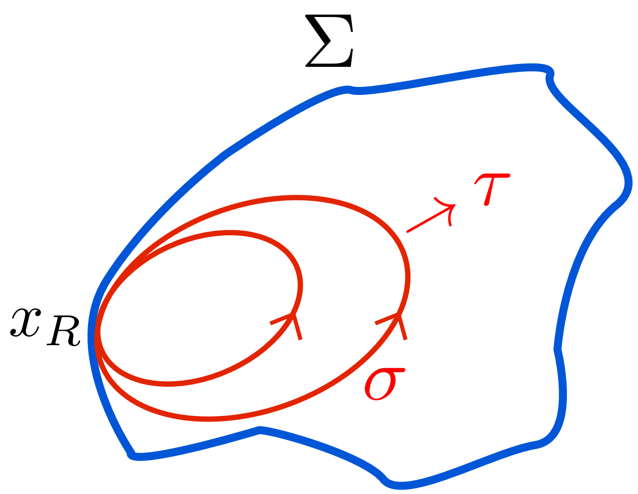
Vary Σ



Generalizing Faraday: Non-Abelian integrals



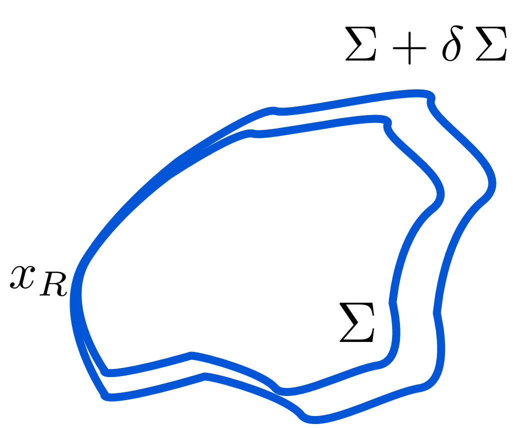
$$\frac{dV}{d\tau} - V\,T\,(A,B,\tau) = 0$$

$$T\,(B,A,\tau) \equiv \int_0^{2\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}$$

It is a surface ordered integral

$$V(\Sigma) = V_R\; P_2\, e^{\int_\Sigma d\sigma d\tau\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}}$$

Vary Σ



$$\delta V\,V^{-1} \equiv \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma\, V\,(\tau)\,\Big\{$$
$$W^{-1}\,\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right] W \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}\,\delta x^\lambda$$
$$-\int_0^\sigma d\sigma'\,\left[\,B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'}\,\frac{dx^\mu}{d\sigma}$$
$$\times\left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\delta x^\nu\,(\sigma) - \delta x^\rho\,(\sigma')\,\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Big\}\,V^{-1}\,(\tau)$$

The generalized non-abelian Stokes Theorem

The generalized non-abelian Stokes Theorem

$$\boxed{V_R \, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta \, \mathcal{K}} V_R}$$

The generalized non-abelian Stokes Theorem

$$\boxed{V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta\, \mathcal{K}}\, V_R}$$

$$\frac{d\,V}{d\,\tau} - V\,T\,(A, B, \tau) = 0$$

$$T\,(B, A, \tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\; \frac{d\,x^\mu}{d\,\sigma}\; \frac{d\,x^\nu}{d\,\tau}$$

The generalized non-abelian Stokes Theorem

$$\boxed{V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta\, \mathcal{K}}\, V_R}$$

$$\frac{d\,V}{d\,\tau} - V\,T\,(A, B, \tau) = 0$$

$$\frac{d\,V}{d\,\zeta} - \mathcal{K}\,V = 0$$

$$T\,(B, A, \tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}$$

$$\begin{aligned}
\mathcal{K} \equiv & \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\; \Big\{ \\
& W^{-1}\, \left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\, W\, \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \frac{d\,x^\lambda}{d\,\zeta} \\
& - \int_0^\sigma d\sigma' \left[\, B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'}\, \frac{dx^\mu}{d\sigma} \\
& \times \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\, \frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\, \frac{d\,x^\nu\,(\sigma)}{d\,\tau} \right) \Big\}\; V^{-1}\,(\tau)
\end{aligned}$$

The generalized non-abelian Stokes Theorem

$$\boxed{V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma}\,\frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta\, \mathcal{K}}\, V_R}$$

$$\frac{d\,V}{d\,\tau} - V\,T\,(A,B,\tau) = 0$$

$$\frac{d\,V}{d\,\zeta} - \mathcal{K}\,V = 0$$

$$T\,(B,A,\tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\,B_{\mu\nu}\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}$$

$$\begin{aligned}\mathcal{K} \equiv \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\,\{ \\ W^{-1}\,\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}\,\frac{d\,x^\lambda}{d\,\zeta} \\ - \int_0^\sigma d\sigma'\,\left[\,B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\,B^W_{\mu\nu}\,(\sigma)\,\right]\,\frac{dx^\kappa}{d\sigma'}\,\frac{dx^\mu}{d\sigma} \\ \times\left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\,\frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\,\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Big\}\,V^{-1}\,(\tau)\end{aligned}$$

Flat connection on the loop space:

$$\Omega^{(2)} = \{f : S^2 \rightarrow M \mid \text{north pole } \rightarrow x_0\}$$

The Integral Equations for Yang-Mills

The Integral Equations for Yang-Mills

$$P_2 e^{ie \int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

The Integral Equations for Yang-Mills

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$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta \widetilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \right.$$
$$\times \left[\left((\alpha - 1) F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right)(\sigma'), \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu} \right)(\sigma) \right]$$
$$\left. \times \frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\frac{d\,x^\nu\,(\sigma)}{d\,\tau} \right) \right\}$$

The Integral Equations for Yang-Mills

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta \widetilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \right.$$
$$\times \left[\left((\alpha - 1) F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right)(\sigma'), \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu} \right)(\sigma) \right]$$
$$\left. \times \frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho(\sigma')}{d\,\tau}\frac{d\,x^\nu(\sigma)}{d\,\zeta} - \frac{d\,x^\rho(\sigma')}{d\,\zeta}\frac{d\,x^\nu(\sigma)}{d\,\tau} \right) \right\}$$

$$B_{\mu\nu} \to \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu}$$

The Integral Equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

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$$B_{\mu\nu} \rightarrow \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu} \qquad D^\mu F_{\mu\nu} = J_\nu \qquad D^\mu \widetilde{F}_{\mu\nu} = 0$$

$$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\,\widetilde{J}_{\nu\rho\lambda}$$

The Integral Equations for Yang-Mills

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta \widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta \widetilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \right.$$
$$\times \left[\left((\alpha - 1) F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right)(\sigma') , \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu} \right)(\sigma) \right]$$
$$\left. \times \frac{d\,x^\kappa}{d\,\sigma'} \frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau} \frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta} \frac{d\,x^\nu\,(\sigma)}{d\,\tau} \right) \right\}$$

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Direct consequence of Stokes theorem and Yang-Mills eqs.

$$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\,\widetilde{J}_{\nu\rho\lambda}$$

The Integral Equations for Yang-Mills

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$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta\widetilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2\int_0^\sigma d\sigma' \right.$$
$$\times\left[\left((\alpha-1)\,F^W_{\kappa\rho}+\beta\widetilde{F}^W_{\kappa\rho}\right)(\sigma')\,,\,\left(\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right)(\sigma)\right]$$
$$\left.\times\frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma}\left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\frac{d\,x^\nu\,(\sigma)}{d\,\zeta}-\frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\right\}$$

$B_{\mu\nu} \rightarrow \alpha F_{\mu\nu} + \beta\widetilde{F}_{\mu\nu}$ $\qquad D^\mu F_{\mu\nu} = J_\nu$ $\qquad D^\mu \widetilde{F}_{\mu\nu} = 0$

Direct consequence of Stokes theorem and Yang-Mills eqs.

Implies Yang-Mills eqs. in the limit $\Omega \rightarrow 0$

$$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\,\widetilde{J}_{\nu\rho\lambda}$$

The Integral Equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V\mathcal{J}V^{-1}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ie\beta\widetilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2\int_0^\sigma d\sigma'\right.$$
$$\times\left[\left((\alpha-1)F^W_{\kappa\rho}+\beta\widetilde{F}^W_{\kappa\rho}\right)(\sigma')\,,\,\left(\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right)(\sigma)\right]$$
$$\left.\times\frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma}\left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\frac{d\,x^\nu\,(\sigma)}{d\,\zeta}-\frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\right\}$$

$$B_{\mu\nu}\rightarrow\alpha F_{\mu\nu}+\beta\widetilde{F}_{\mu\nu} \qquad D^\mu F_{\mu\nu}=J_\nu \qquad D^\mu\widetilde{F}_{\mu\nu}=0$$

Direct consequence of Stokes theorem and Yang-Mills eqs.

Implies Yang-Mills eqs. in the limit $\Omega\rightarrow 0$

L.A. Ferreira and G. Luchini
1) [arXiv:1205.2088 [hep-th]], Phys. Rev. D 86, 085039 (2012)
2) [arXiv:1109.2606 hep-th]], Nuclear Physics B 858PM (2012) 336-365

$$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\,\widetilde{J}_{\nu\rho\lambda}$$

Conserved Charges

Conserved Charges

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

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$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta \widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

If Ω_c is a closed volume $(\partial\Omega_c = 0)$ $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$

Conserved Charges

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

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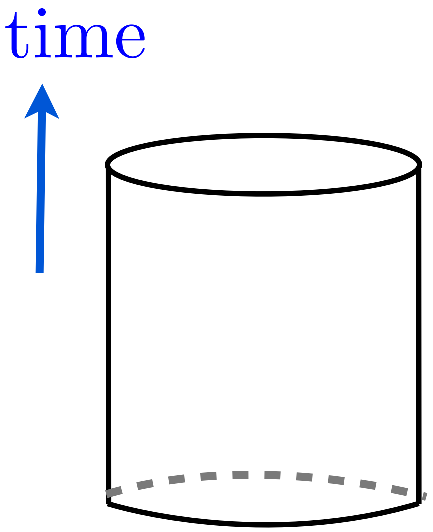

Conserved Charges

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

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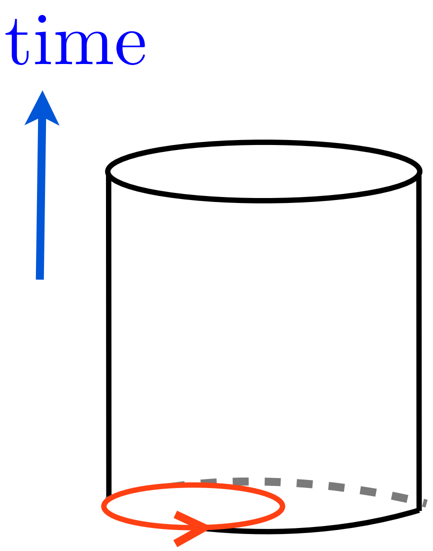
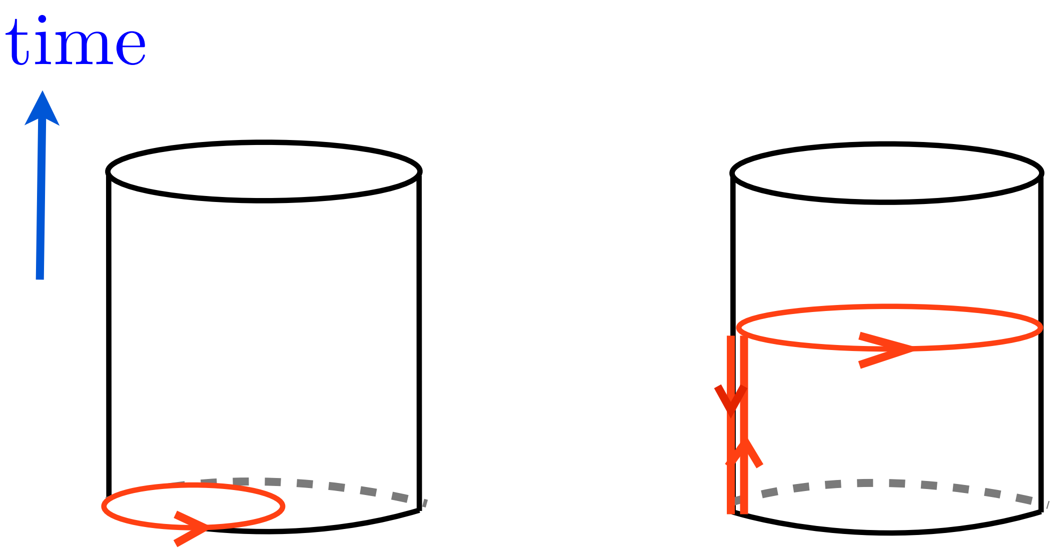


$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$

Conserved Charges

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

If Ω_c is a closed volume $(\partial\Omega_c = 0)$ $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$

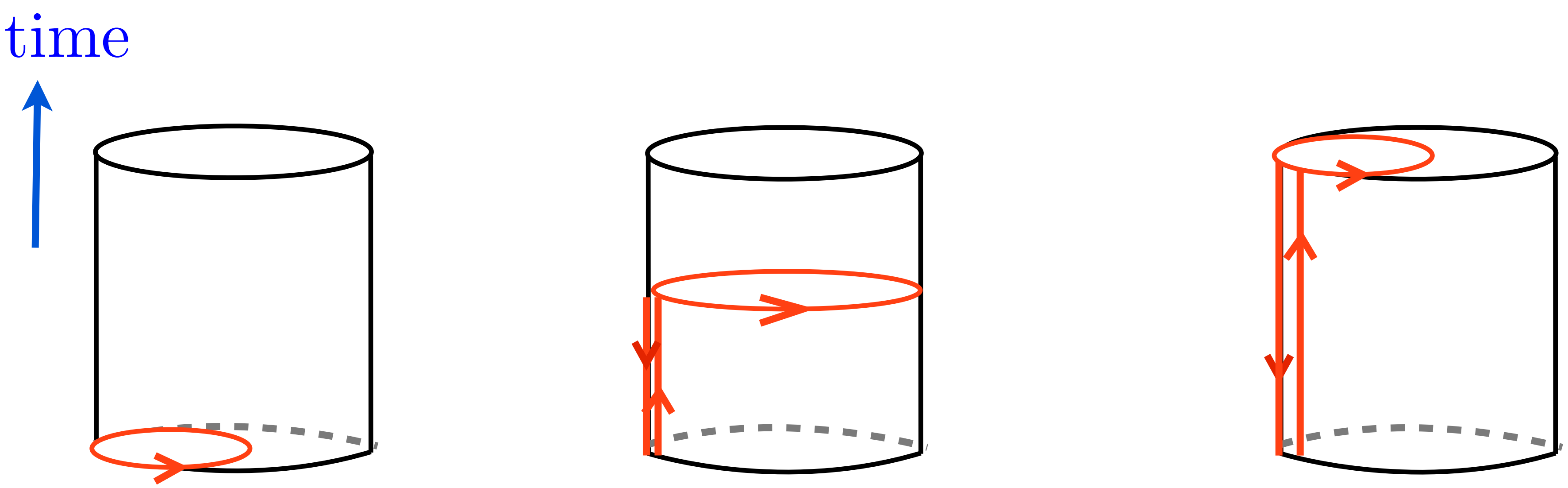


$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$

Conserved Charges

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

If Ω_c is a closed volume $(\partial\Omega_c = 0)$ $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$

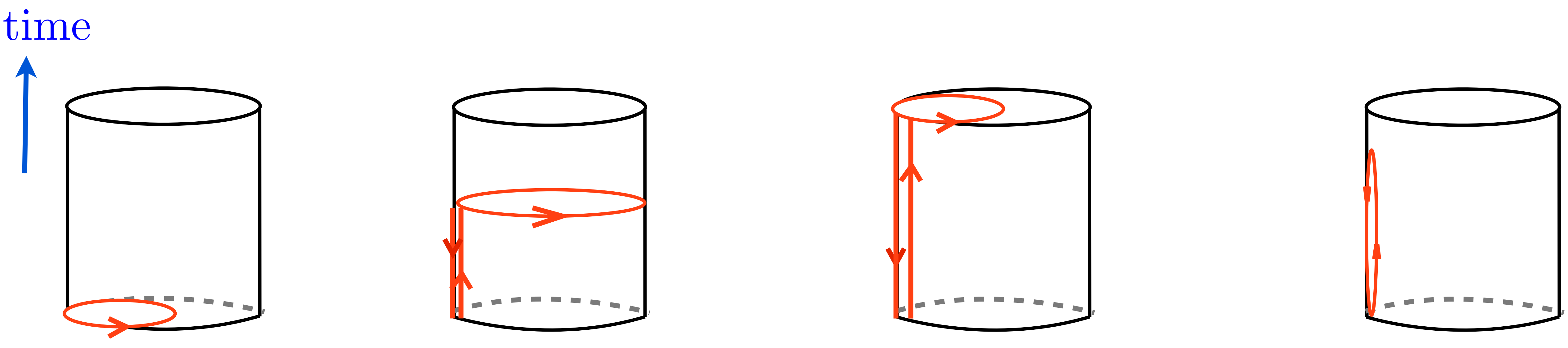


$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V \mathcal{J} V^{-1}}$

Conserved Charges

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V\mathcal{J}V^{-1}}$$

If Ω_c is a closed volume ($\partial\Omega_c = 0$) $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V\mathcal{J}V^{-1}} = \mathbb{1}$



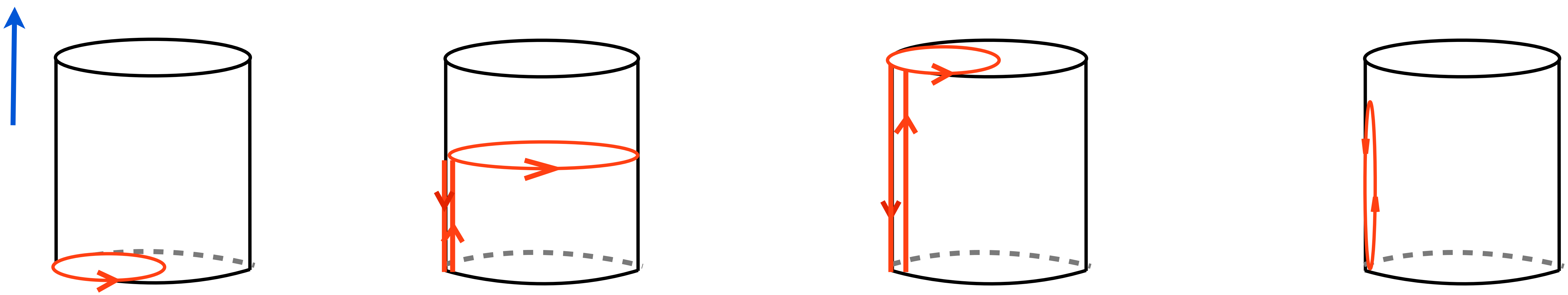
$P_3 e^{\int_{\Omega_0} d\zeta d\tau V\mathcal{J}V^{-1}}$ $\quad P_3 e^{\int_{S^2_\infty\times I} d\zeta d\tau V\mathcal{J}V^{-1}}$ $\quad P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V\mathcal{J}V^{-1}}$ $\quad P_3 e^{\int_{S^2_0\times I} d\zeta d\tau V\mathcal{J}V^{-1}}$

Conserved Charges

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

If Ω_c is a closed volume $(\partial\Omega_c = 0)$ $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$

time



$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_0 \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$

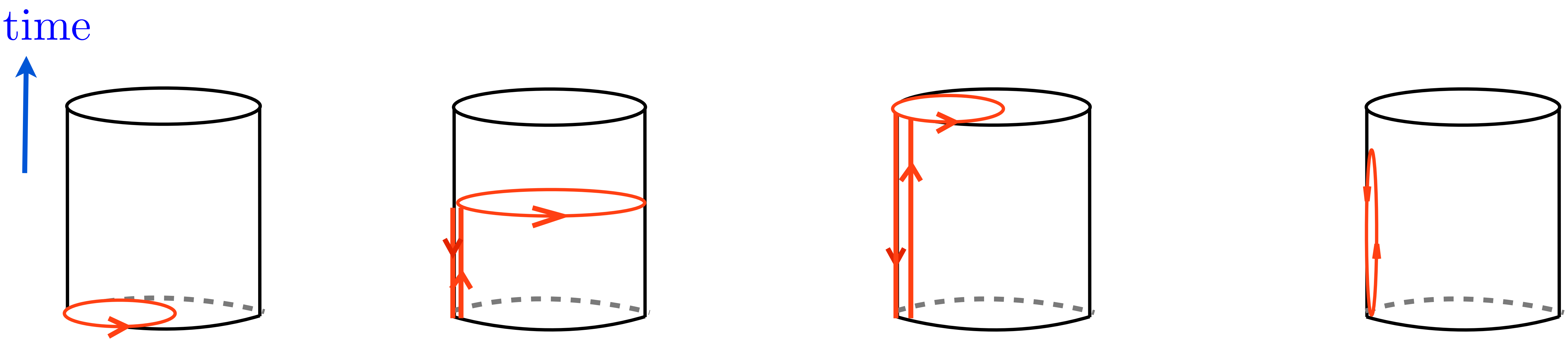
Iso-spectral evolution:

$$\boxed{V(\Omega_t) = U(t)\cdot V(\Omega_0)\cdot U^{-1}(t)}$$

Conserved Charges

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

If Ω_c is a closed volume ($\partial\Omega_c = 0$) $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$



$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_0 \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$

Iso-spectral evolution: $\boxed{V(\Omega_t) = U(t) \cdot V(\Omega_0) \cdot U^{-1}(t)}$

Eigenvalues of $V(\Omega_t)$ are constant in time

Conserved charges are eigenvalues of the operator

$$V(\Omega_t) = P_2 \; e^{ie \int_{\mathcal{S}^{2,(t)}_{\infty}} d\tau \; d\sigma \; \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3 \; e^{\int_{\Omega_t} d\zeta \; d\tau \; V \mathcal{J} V^{-1}}$$

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$$V(\Omega_t) = P_2 \; e^{ie \int_{\mathcal{S}^{2,(t)}_{\infty}} d\tau \; d\sigma \; \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3 \; e^{\int_{\Omega_t} d\zeta \; d\tau \; V \mathcal{J} V^{-1}}$$

Conserved charges are:

Conserved charges are eigenvalues of the operator

$$V(\Omega_t) = P_2 \, e^{ie\int_{\mathcal{S}^{2,(t)}_{\infty}} d\tau \; d\sigma \; \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3 \, e^{\int_{\Omega_t} d\zeta \; d\tau \; V \mathcal{J} V^{-1}}$$

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eigenvalues of Q
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Wu-Yang and 't Hooft-Polyakov monopoles

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At spatial infinity:

$$A_i = -\frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_j}{r}\,T_k \qquad F_{ij} = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,\hat{r}\cdot T \qquad [T_i\,,\,T_j] = i\,\varepsilon_{ijk}\,T_k$$

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- It connects to integrable field theories

Thank You

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1) [hep-th/9710147], NPB 529, 689 (1998)
2) [arXiv:0901.1654 [hep-th]], IJMPA 24, 1825 (2009)
LAF, G. Luchini
3) [arXiv:1205.2088 [hep-th]], PRD 86, 085039 (2012)
4) [arXiv:1109.2606 hep-th]], NPB 858PM (2012) 336-365
C.P. Constantinidis, LAF, G. Luchini
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Integrable Field Theories in $(1+1)$-dimensions

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- Infinite number of conservation laws
- Inverse scattering method
- Dressing method
- Hirota method
- Classical r-matrix
- Quantum R-matrix
- etc

Flatness and path independency

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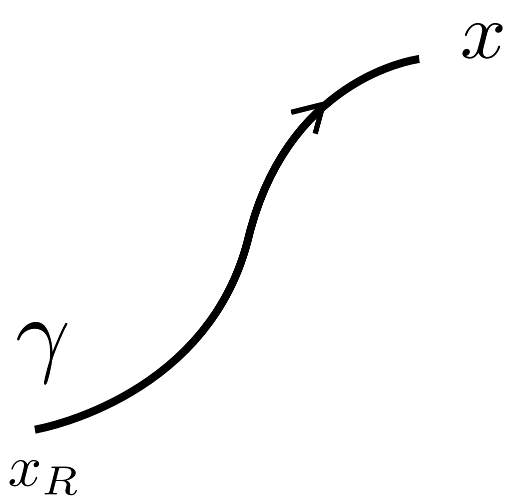

$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0$$

$$W(\gamma) = P_1\, e^{-\int_\gamma A_\mu \frac{dx^\mu}{d\sigma}}\, W_R$$

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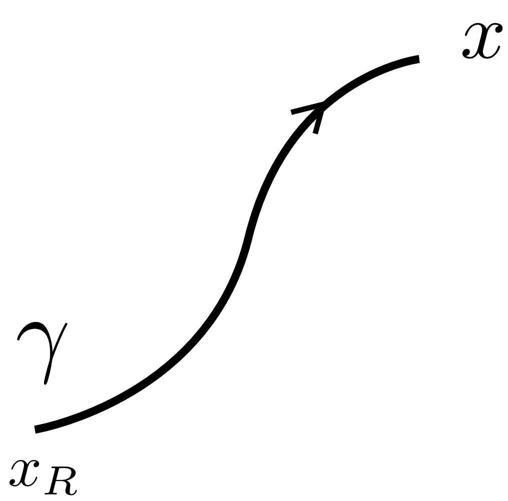

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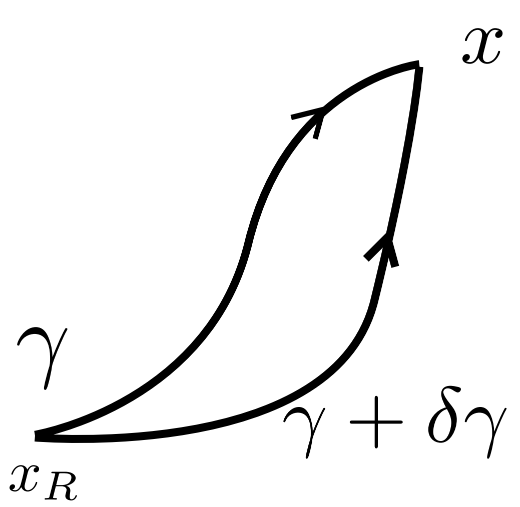

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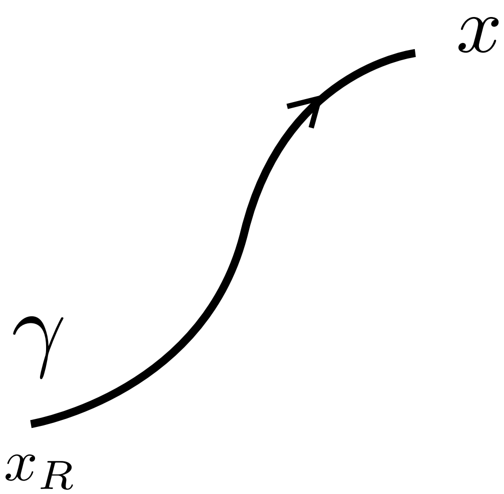

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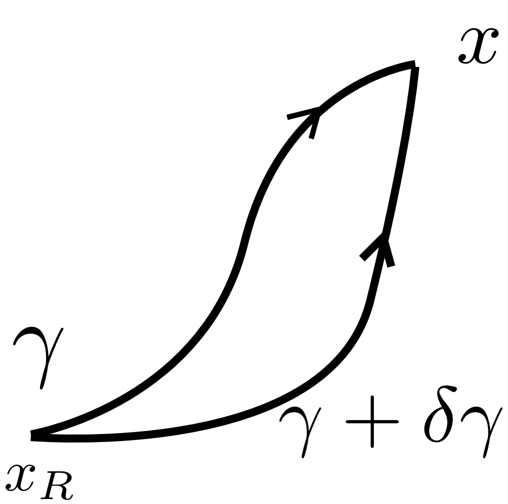

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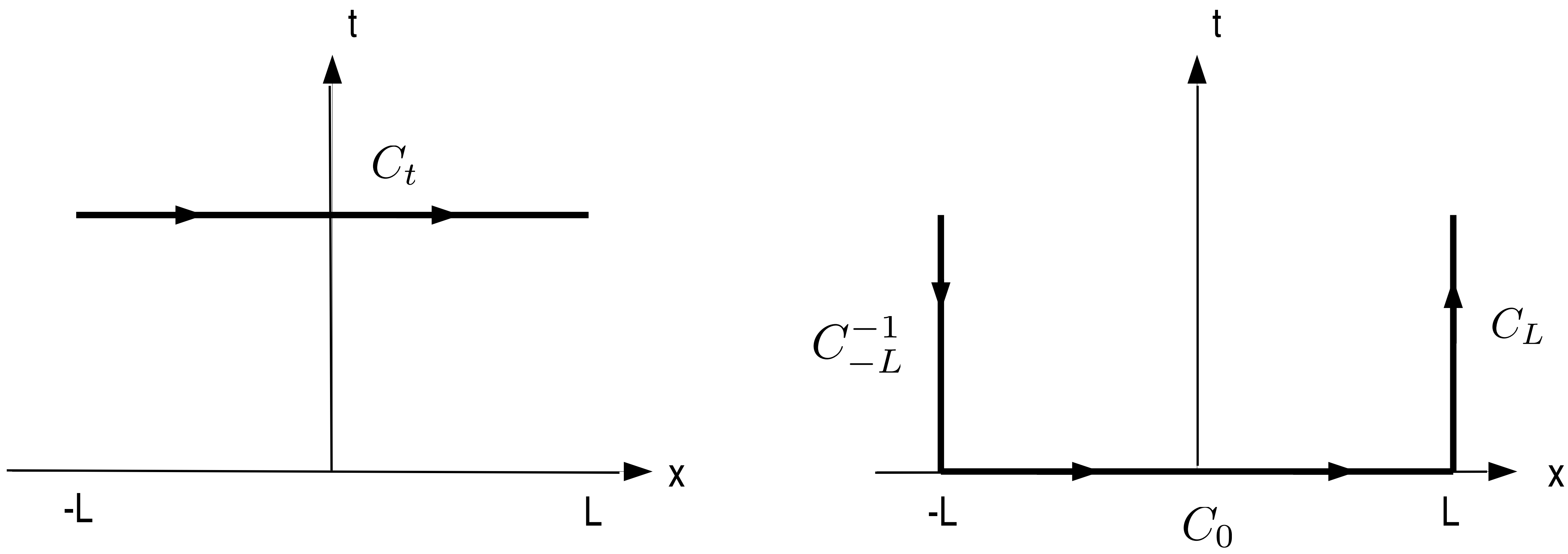

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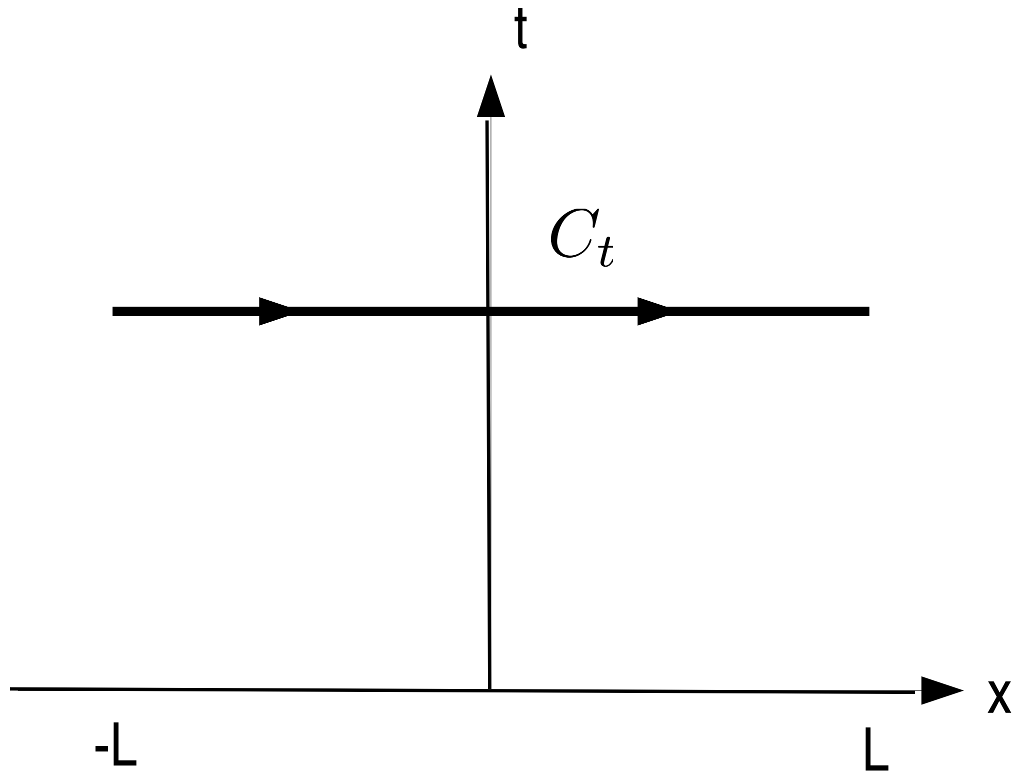

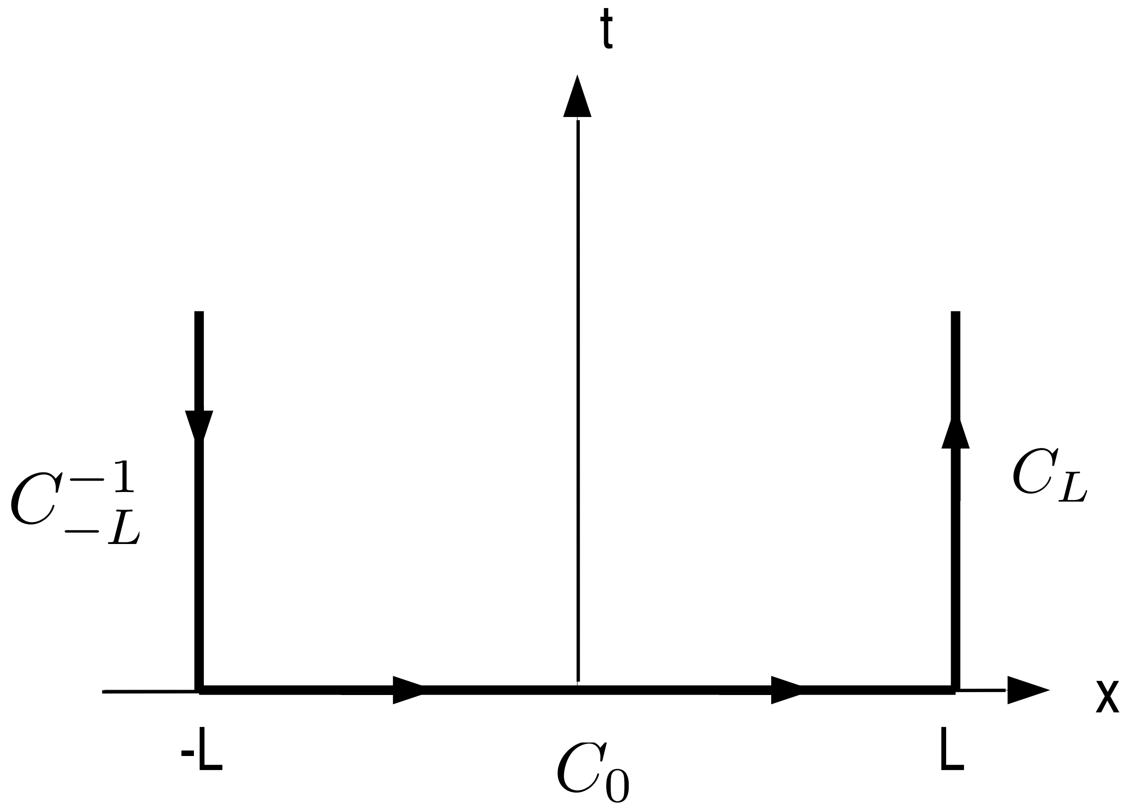

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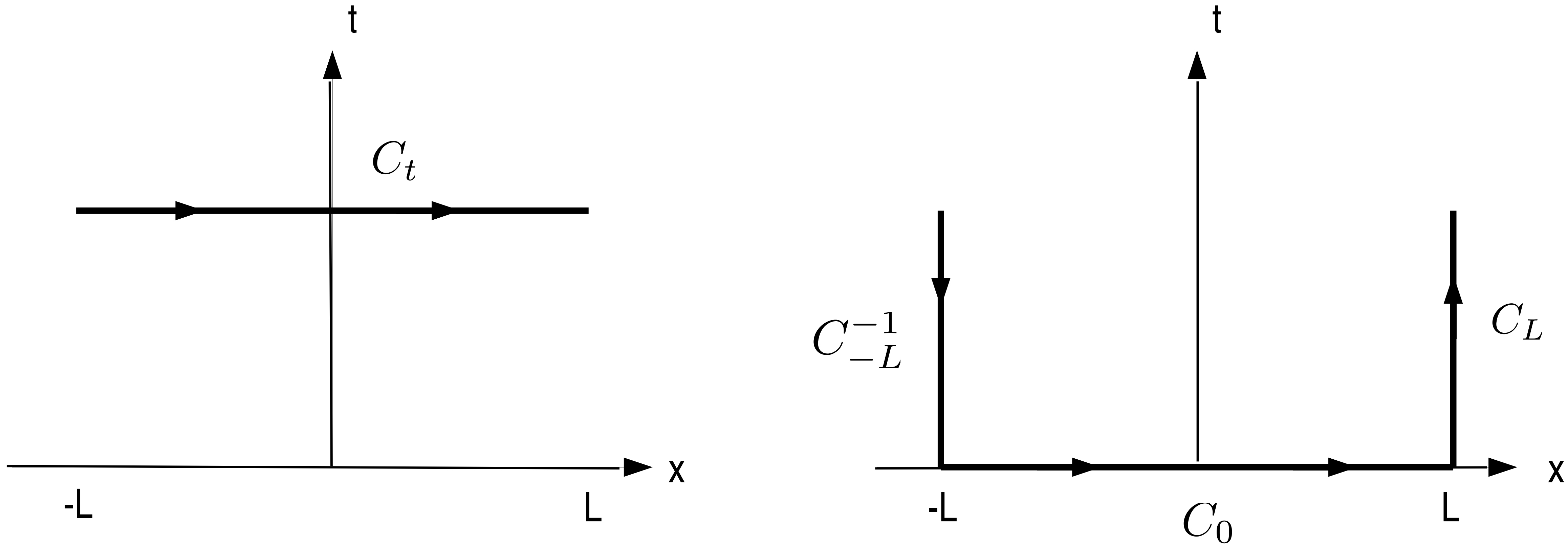

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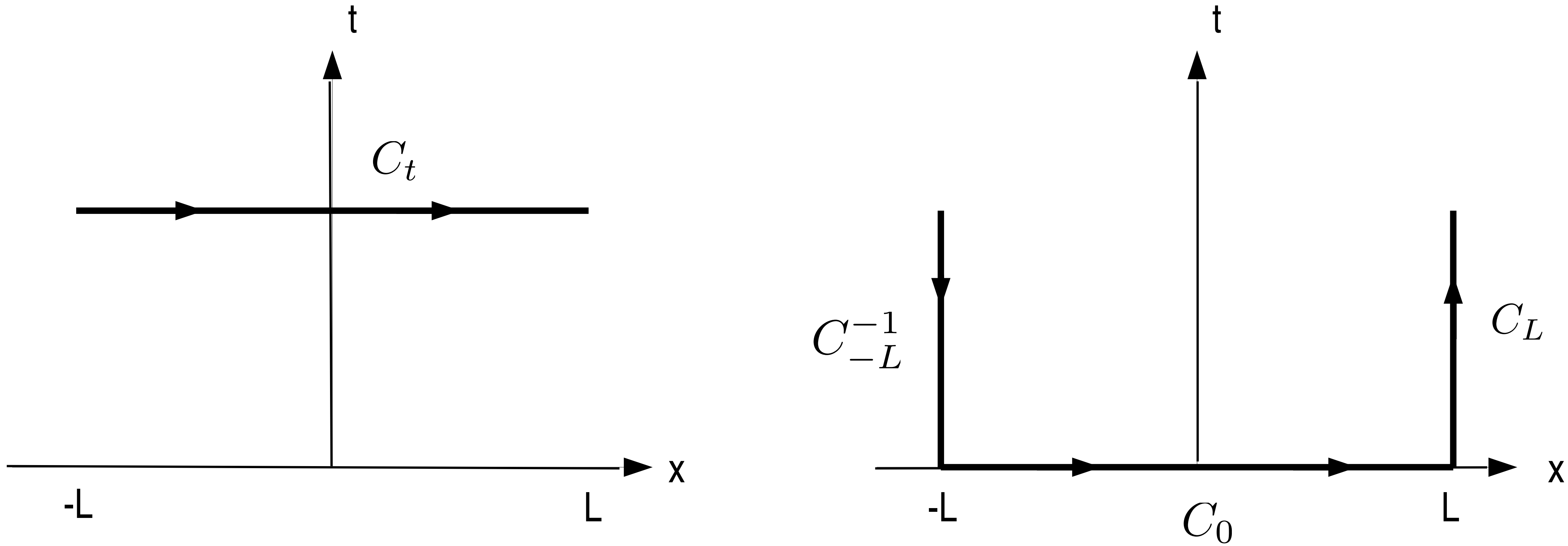

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power series in λ: infinite number of conserved quantities

(no Coleman-Mandula)

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Flux | Charge

For an infinitesimal Σ one gets the differential equation $F_{\mu\nu} = \tilde{J}_{\mu\nu}$

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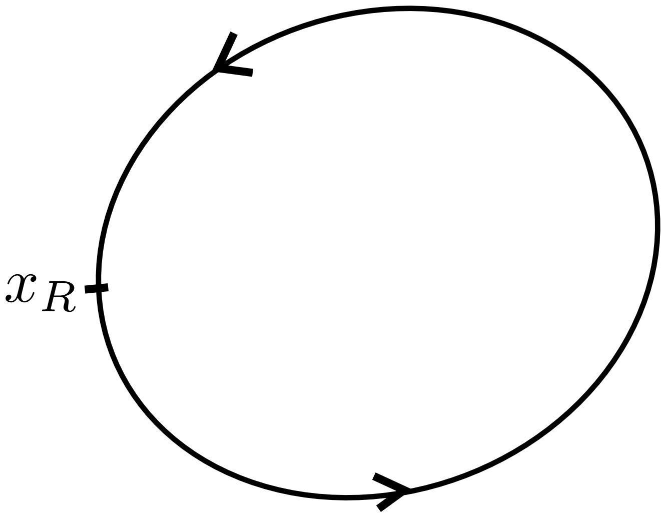

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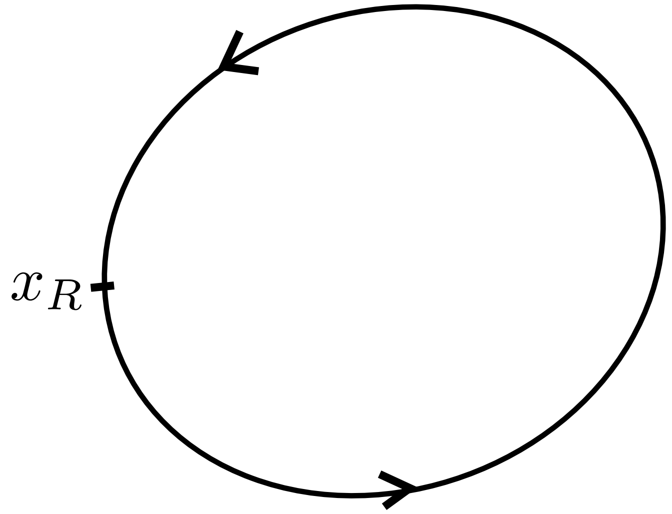

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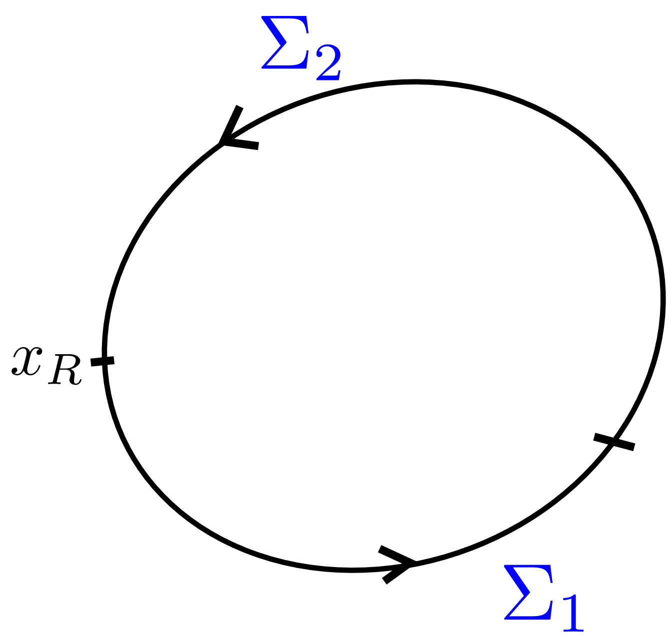

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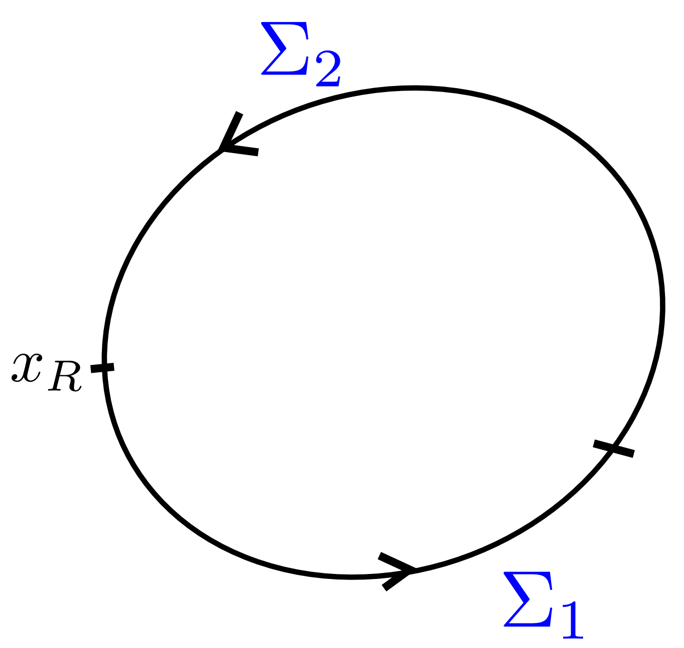

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Path (surface) independency

Construction of conserved charges

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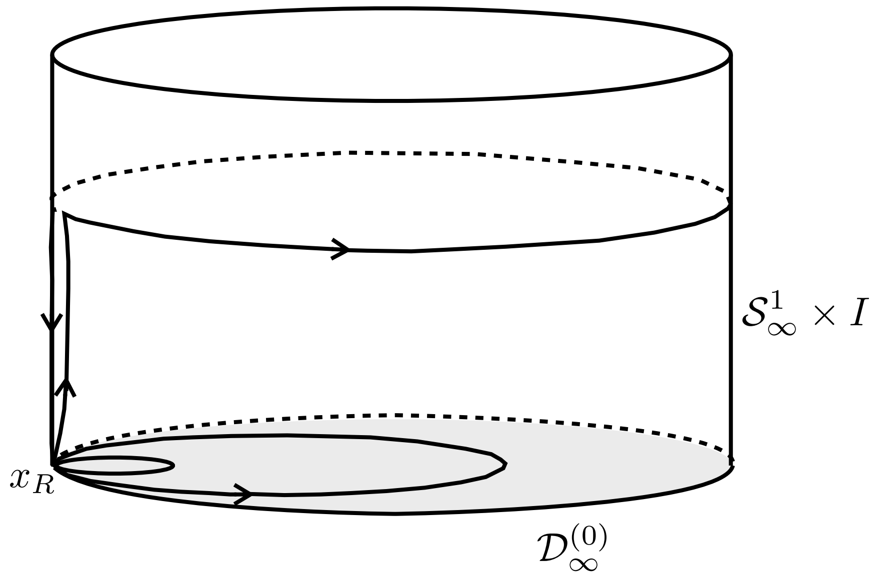
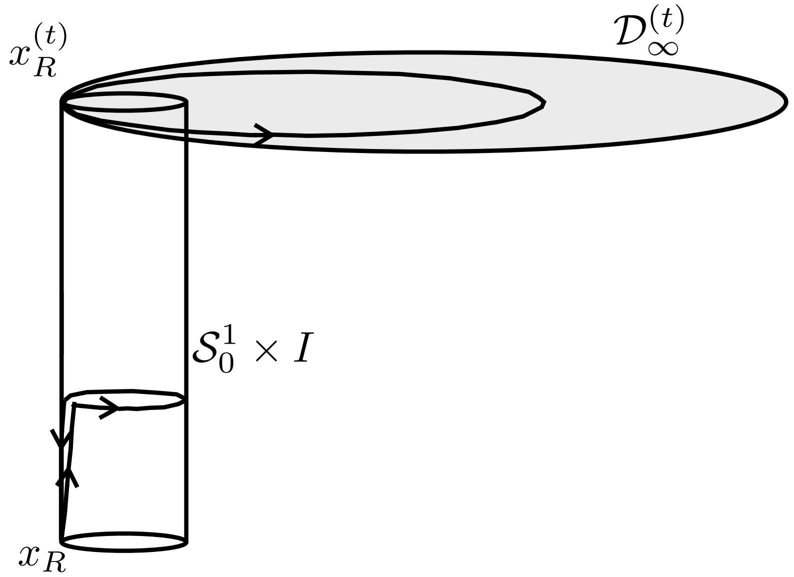


Surface $\Sigma_1 = \mathcal{D}^{(0)}_\infty \cup \left(S^1_\infty \times I\right)$



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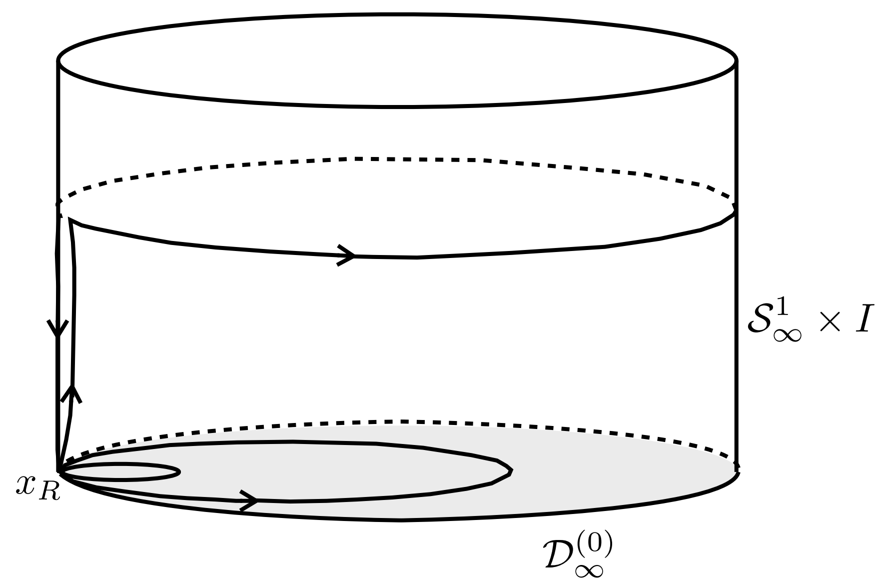

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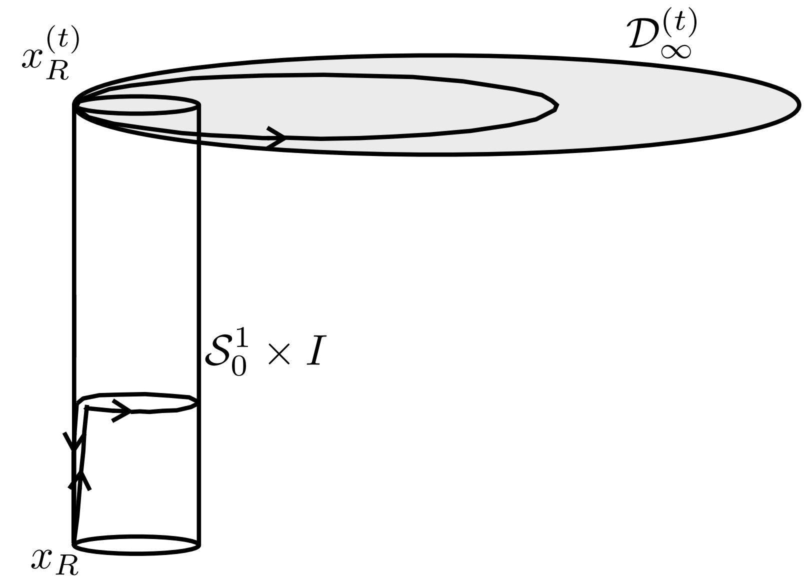

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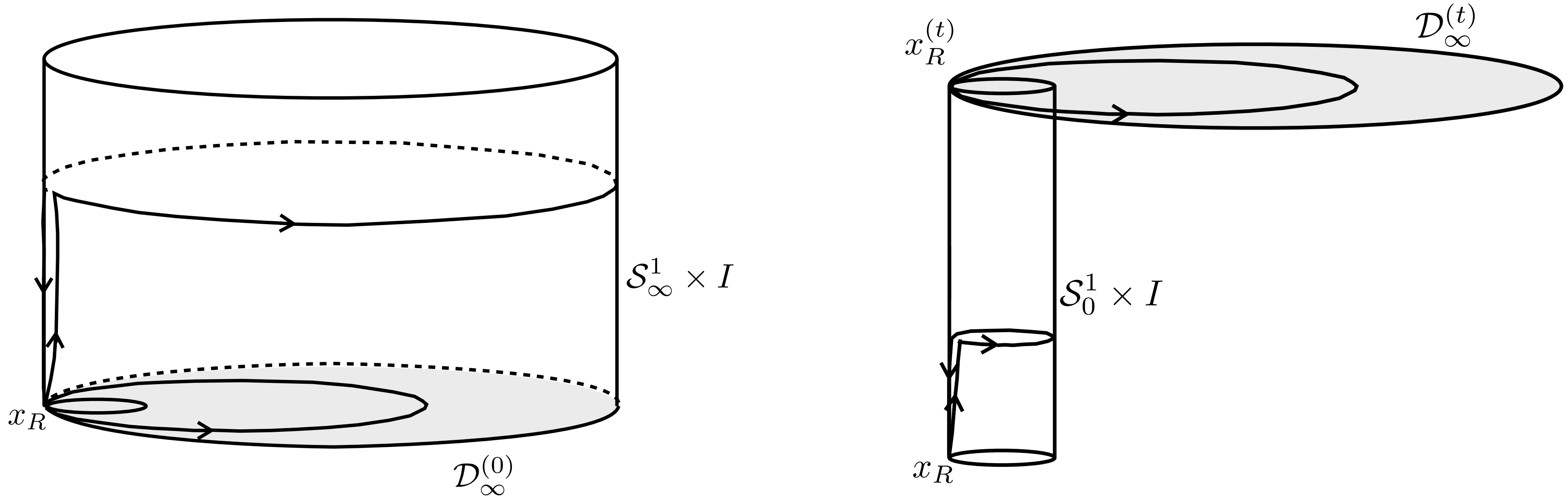

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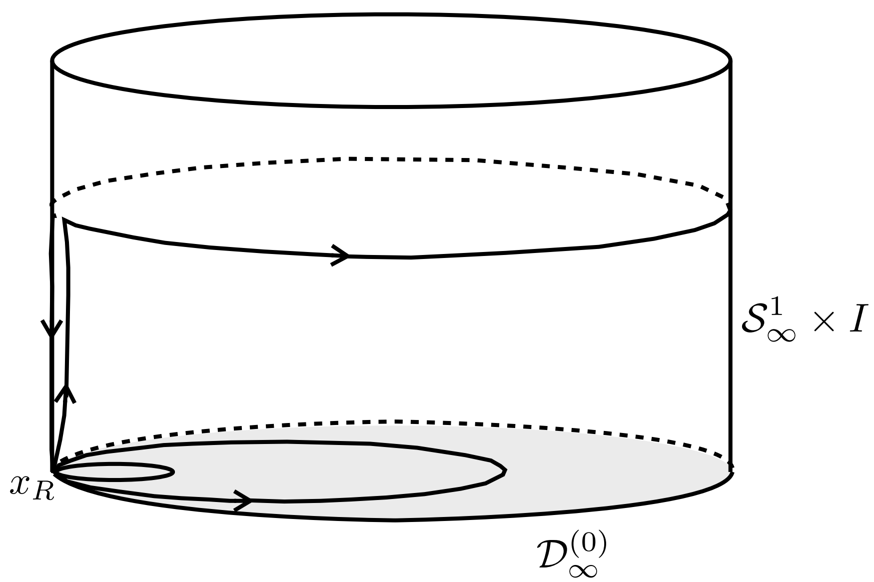

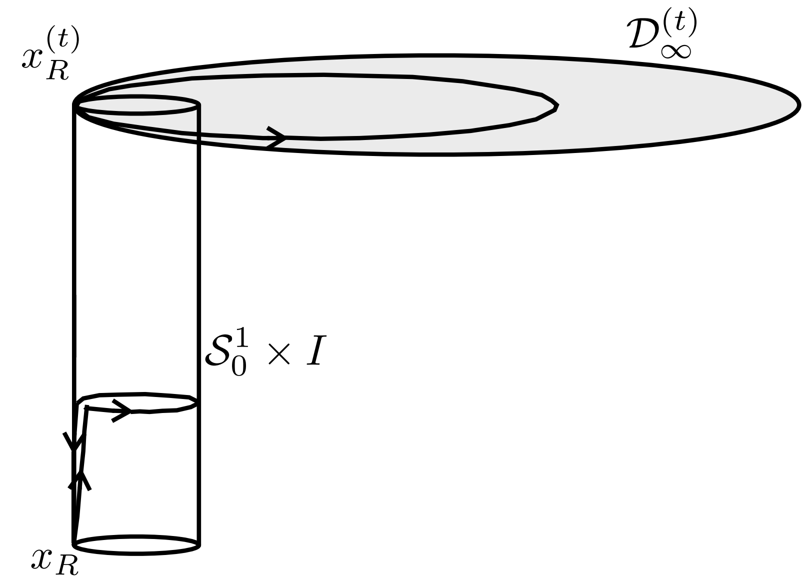

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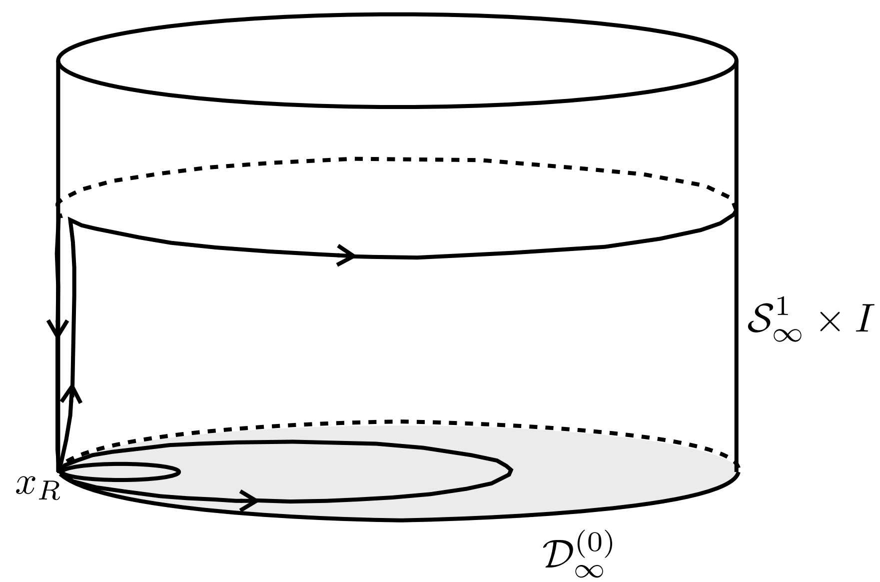

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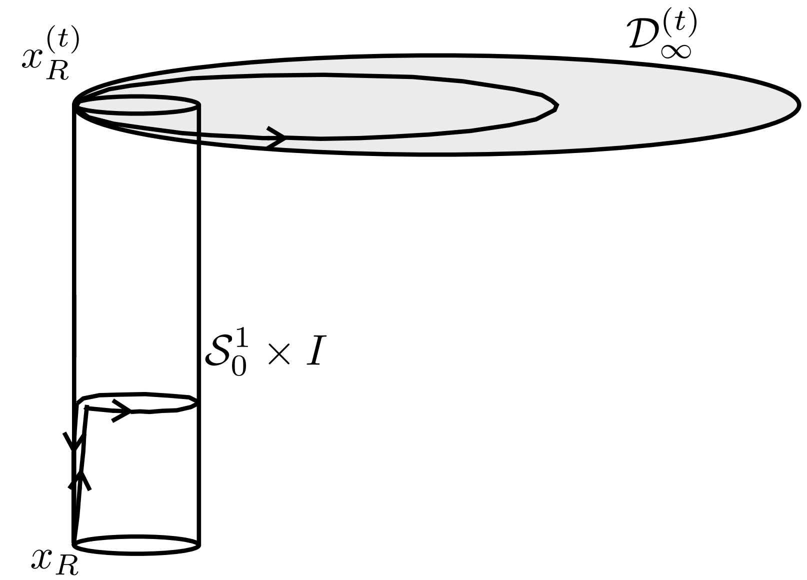

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Conserved charges → eigenvalues of the operator:

$$\boxed{V_{x_R^{(t)}}\left(\mathcal{D}^{(t)}_\infty\right) = P_2\, e^{\frac{ie}{\kappa}\int_{\mathcal{D}^{(t)}_\infty} d\tau\, d\sigma\, W^{-1}\widetilde{J}_{\mu\nu} W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_1 e^{-ie\oint_{S^1_\infty} d\sigma\, A_\mu \frac{dx^\mu}{d\sigma}}}$$

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Conserved charges are obtained the same way as for Chern-Simons

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Examples:
1) Integrable theories in $1+1$ dimensions (soliton theories)
2) Chern-Simons theories in $2+1$ dimensions
3) Yang-Mills in $2+1$ and $3+1$ dimensions