Studies in Certain Planar Field Theories

Thesis Submitted for the degree of
Doctor of Philosophy (Science)
of
JADAVPUR UNIVERSITY
2003

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Dedicated

to

My Parents

CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled **"Studies in Certain Planar Field Theories"** submitted by **Sri Tomy Scaria**, who got his name registered on **13.08.2001** for the award of **Ph.D.(Science) degree** of **Jadavpur University**, is absolutely based upon his own work under the supervision of **Dr. Rabin Banerjee** and **Dr. Biswajit Chakraborty** and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before .

2 APR 2003

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Chapter 1

Introduction and Overview

Field theories defined in 2+1 dimensional space-time (planar field theories) are of importance in theoretical physics in many ways [1, 2, 3, 4, 5, 6, 7]. One of the chief reasons for the usefulness of planar theories is that they serve as simplified versions of more complicated 3+1 dimensional ones that involve formidable mathematical and conceptual difficulties. In such instances, a study of planar theories may provide valuable insights and informations for developing possible methods of dealing with their more realistic 3+1 dimensional counterparts. A famous example for such a theory is the quantum gravity which is soluble in 2+1 dimensions [1, 2]. Another important reason for the widespread interest in the study of planar models is that they can describe the physics of 2+1 dimensional systems like surfaces and thin films in condensed matter physics. Elementary particles confined to two spatial dimensions behave in distinctly different ways compared to the their usual behavior in the familiar 3+1 dimensions [7]. Such difference in behavior gives rise to interesting observable phenomenon thus making the study of planar theories highly interesting both from theoretical and experimental points of view. For example, one such important application of planar theories is the well known fractional statistics and fractional quantum Hall effect [7, 8].

In this thesis we deal with certain planar field theoretical models uncovering various aspects of these theories and their interrelationships. A major part of the work is devoted to the study topologically massive gauge theories which possess the interesting property of gauge invariance co-existing with mass. The Maxwell-Chern-Simons (MCS) and Einstein-Chern-Simons (ECS) theories in 2+1 dimensions are typical examples of such theories and have evinced tremendous interest among theoretical physicists in recent times [9]. While MCS theory is a vector field theory, the ECS theory has a symmetric second rank tensor as its basic field. We have investigated certain aspects of MCS theory and the linearized version of ECS theory. The massive nongauge vector theories, namely Proca and Maxwell-Chern-Simons-Proca (MCSP) theories, are known to be equivalent to doublets of MCS theories [9, 10, 11]. Also, MCS theories are equivalent to self and anti-self dual models in 2+1 dimensions [12, 13, 14]. Similarly, certain studies have suggested the existence of a similar connection between linearized versions of ECS theory and the Einstein-Pauli-Fierz (EPF) theory [9]. We attempt to provide some fresh insight into the interrelationship between these various theories through the maximally reduced form of the polarization vectors/tensors of these theories. The polarization vectors and tensors are of importance in phenomenological calculations [15, 16, 17]. Further usefulness of these polarization vectors/tensors become more transparent when we consider the gauge transformations generated by the translational subgroup of Wigner's little group for massless particles [18]. It is quite well known that the polarization tensor of free Maxwell theory in 3+1 dimensions when acted upon by this translational subgroup undergoes gauge transformations [19, 20, 21, 22, 23, 24]. It has also been recently extended to linearized gravity which has a second rank symmetric tensor as the underlying gauge field [25, 26] and Kalb-Ramond (KR) theory involving 2-form gauge field [27]. It has also been shown that the translational group $T(3)$, group of translations in 3-dimensional space, in a particular representation acts as generator of gauge transformations in $B \wedge F$ theory [27] which is obtained by topologically coupling Maxwell field to the KR field. Analogous to MCS and ECS theories, $B \wedge F$ model is a topologically massive gauge theory, but in 3+1 dimensions [28, 29, 30, 31]. It was later discovered that one can in fact systematically derive this representation of $T(3)$ which generate gauge transformations in a massive gauge theory in 3+1 dimensions, using the gauge transformation properties of 4+1 dimensional Maxwell and massless KR theories by a method known as 'dimensional descent' [32]. In the present work, we not only unravel many new and interesting points pertaining to the gauge generation by translational groups in 3+1 dimensional theories, but extend them to planar gauge theories also. We show that a particular representation of the 1-dimensional translational group $T(1)$ generate gauge transformations in the topologically massive MCS and ECS theories [26, 32, 33]. Connection of this representation of $T(1)$ with the representation of $T(2)$ (group of translations in 2-dimensions) that generate gauge transformations in 3+1 dimensional massless theories is demonstrated explicitly by dimensional descent from 3+1 to 2+1 dimensions [26, 32].

We considered only Abelian theories in this work and the calculation of the polarization vector (or tensor) of these theories is of crucial importance in our context. In general, polarization vectors/tensors capture the Lorentz transformation property of the basic fields when expressed as a mode expansion. In the usual field theoretical models such as Maxwell theory or Proca theory, the components of the polarization vectors can be independent of each other and are real [34, 35]. However, the components of the polarization vector (tensor) of the topologically massive MCS (ECS) theory are necessarily complex and cannot be chosen independently, manifesting the internal structure of the theory [26, 36]. This is true also for MCSP theory and a comparison of the polarization vectors of MCSP theory with those of a pair of MCS theories with opposite helicities explicitly shows that the former can be considered to be equivalent to the MCS doublet [36]. This equivalence between a pair of MCS theories and MCSP theories was earlier studied at the level of the basic fields of the models [10, 11]. As we shall see later in this report, the polarization tensor of ECS theory is the tensor product of the polarization vectors of a pair of MCS theories with the same helicity. We take recourse to explicit expressions of the polarization vector or tensor of the theory under consideration in order to show how the appropriate translational group generate gauge transformation in the theory. For this purpose we consider only a single mode in the Fourier expansion of the basic field of the theory and restrict ourselves to a particular reference frame. We then use the Euler-Lagrange equation to derive the maximally reduced form of the polarization vector or tensor which is devoid of any spurious degrees of freedom and represent only the physical sector of the theory [35]. This procedure is named 'plane wave method' and it can be used, with a great economy of effort, to study the gauge generating nature of translational groups in various theories. Another important feature of this 'plane wave method' is that it yields the mass of the quanta of the theory under consideration rather effortlessly. Apart from the usual massive nongauge theories, this method of extracting the masses (zero or nonzero) of the quanta can be applied to ordinary gauge theories having massless excitations, topologically massive theories having massive quanta and nongauge theories elevated to gauge theories by Stückelberg mechanism [26, 27, 36, 37].

In chapter 2, we establish the equivalence of the MCSP model to a doublet of MCS models defined in a variety of covariant gauges. This equivalence is shown to hold at the level of polarization vectors of the basic fields. The analysis is done in both Lagrangian and Hamiltonian formalisms and compatible results are obtained [36]. A similar equivalence with a doublet of self and anti self dual models is briefly discussed.

In chapter 3, we review the role of translational subgroup $T(2)$ of Wigner's little group for massless particle as a generator of gauge transformations in 3+1 dimensional theories. First it is shown, following Kim et.al. [22], how the gauge transformation in Maxwell theory is generated by the defining representation of $T(2)$. It is then shown that the same representation of $T(2)$ generates gauge transformations in 3+1 dimensional linearized gravity [25, 26] and massless KR theory [27]. The gauge transformations generated by translational group form only a subset of the full range of gauge transformations available to linearized gravity and KR theories [37]. We also see in chapter 3 that the reducibility of gauge transformations in KR theory is clearly manifested in the gauge transformations generated by $T(2)$. Furthermore, in the case of the topologically massive $B \wedge F$ theory, the gauge transformations are generated by $T(3)$ [37].

We discuss in chapter 4, the role of $T(1)$ as generator of gauge transformations the topologically massive MCS theory as well the linearized ECS theory in 2+1 dimensions. Using plane wave method, we drive the maximally reduced form of the polarization vector and tensor of these theories and show that a suitable representation of $T(1)$ generates gauge transformation in these theories [26, 33]. Polarization vector of the ECS theory is clearly shown to be the tensor product of the polarization vector of MCS theory with itself [26].

The method of dimensional descent [32] is reviewed in chapter 5 and the representation of $T(1)$ that acts as gauge generator for MCS and ECS theories are derived using this method by starting from gauge transformation properties of massless gauge theories (Maxwell and linearized gravity) in 3+1 dimensions [26]. Finally, the polarization tensor of EPF theory in 2+1 dimensions is shown to split into the polarization tensors of a pair of ECS theories with opposite helicities suggesting a doublet structure for EPF theory [26].

One may also construct massive gauge theories by converting second class constrained systems (in the language of Dirac's theory of constraint dynamics [38, 39]) to first class (gauge) systems using the generalized embedding prescription of Batalin, Fradkin and Tyutin [40, 41, 42]. On the other hand, in the Lagrangian framework, one can convert the massive nongauge theories to gauge theories by the generalized Stückelberg extension mechanism [43, 44]. It is pointed out in [45] that there exists a one to one correspondence between this Hamiltonian embedding prescription and the Stückelberg extension mechanism based on Lagrangian formalism. By such embedding prescriptions, one may elevate gauge noninvariant Proca, massive KR [46] and EPF theories to gauge theories and obtain the corresponding Stückelberg extended versions of these models. Chapter 6 is devoted to the study of the connection between gauge generation and translational groups in such embedded massive gauge theories. Though the models considered in chapter 6 belong to 3+1 dimensional space-time, with suitable modifications these methods and results can be easily applied also to planar theories. We show that the same representation of $T(3)$ that generate gauge transformation in $B \wedge F$ theory also acts as gauge generator in the above mentioned Stückelberg extended models [37].

A brief description of the major results and conclusions are given in chapter 7.

Notation: We use subscripts/superscripts in Greek letters for denoting indices in 2+1 dimensional space-time. The letters a, b, c etc. from the beginning of Latin alphabet are used for indices in 3+1 dimensional space-time whereas x, y, z etc. from the end of the alphabet denote 4+1 dimensions. Letters like i, j, k from the middle of Latin alphabet represent spatial components of vectors/tensors in any dimension. Signature of the metrics used are mostly negative.

We adopt the following nomenclature for gauge theories having massive excitations. We discuss two types of gauge theories where the gauge fields are massive. When a massless gauge theory is coupled to a topological term, the theory acquires mass while retaining the gauge symmetry. The MCS, ECS [9] and $B \wedge F$ [29, 30] theories are examples of such gauge theories where the origin of mass is due to the presence of topological terms in their actions. These theories are clearly referred to as 'topologically massive gauge theories'. The term 'massive gauge theories' are used for those gauge theories obtained elevating the massive second class theories to first class (gauge) theories by the prevously mentioned embedding prescription given by Batalin, Fradkin and Tyutin [40, 41, 42]. Such massive gauge theories are also called Stückelberg extended theories of the corresponding massive nongauge theories because they can also be obtained by generalized Stückelberg extension mechanism [43, 44, 45].

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1. Polarization vectors and doublet structure in planar field theory [36]
   R. Banerjee, B. Chakraborty and Tomy Scaria
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2. On the role of Wigner's little group as a generator of gauge transformation in Maxwell-Chern-Simons theory [33]
   R. Banerjee, B. Chakraborty and Tomy Scaria

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3. Wigner's little group as a gauge generator in linearized gravity theories [26]
   Tomy Scaria and B. Chakraborty
   Class. Quant. Grav. **19** (2002) 4445.

4. Translational groups as generators of gauge transformations [37]
   archive report hep-th/0302130
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Chapter 2

Polarization Vectors and Doublet Structure in Planar Theories

The polarization vectors or tensors contained in the mode expansions of the basic fields of field theoretical models usually capture the Lorentz transformation properties of the fields under consideration. In some theories the polarization vectors/tensors may carry more information on the structure of the theory itself. In such cases, a simple analysis of the polarization vectors of the theories provide valuable information regarding these theories. In the present chapter, we study two such models, namely Maxwell-Chern-Simons theory and Maxwell-Chern-Simons-Proca theory. The topic of this chapter is the relationship between these models and we also consider the self and antiself dual model which have a close connection with these theories. As mentioned before MCS theory is topologically massive gauge theory where the gauge invariant mass occurring already at the classical (tree) level. Now, it is intriguing to note that the MCSP theory can be regarded as the embedding of a doublet of topologically massive gauge theories [10, 11]. Earlier, this was studied at the level of the basic fields in the two theories [10, 11]. Here we shall pursue this mapping at the more fundamental level involving polarization vectors associated with different modes of these fields. This is all the more important since proper evaluation of these vectors is crucial for reduction formulae and the study of the massless limit of the MCSP theory [15]. Besides, as stated before, the transformation properties of the polarization vectors of various gauge theories under the action of the translational subgroup of Wigner's little group display the precise gauge symmetry of the theory. Therefore, it is obvious that polarization vectors play a fundamentally important role in the momentum space analysis of these theories. This motivates us to compute the polarization vectors of these theories in the Lagrangian and Hamiltonian formulations and make a comparison between them. A difference in the Hamiltonian approach, in contrast to the Lagrangian approach, is the need to introduce a "new set" of polarization vectors for canonically conjugate momentum variables (π^μ) along with those of the basic vector fields (A^μ).

2.1 Polarization vectors in Lagrangian formalism

2.1.1 Maxwell-Chern-Simons theory

We first review the calculation of the polarization vectors in the Maxwell-Chern-Simons theory pointing out the differences from the corresponding analysis for the Maxwell theory. Apart from reviewing the standard analysis [34, 50] where imposition of Lorentz gauge is required, an alternative analysis depending only on the symmetries of the theory will also be discussed. The Lorentz gauge condition emerges naturally in the latter method.

The MCS Lagrangian in 2+1 dimensions is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \tag{2.1}$$

This is the well known topological gauge theory with a single mode of mass $|\vartheta|$ and spin $\frac{\vartheta}{|\vartheta|}$ [9]. The corresponding equation of motion is given by,

$$-\partial^\nu F_{\mu\nu} + \vartheta\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda = 0. \tag{2.2}$$

Imposing the Lorentz gauge,

$$\partial_\mu A^\mu = 0 \tag{2.3}$$

the above equation reduces to

$$\left(\Box g^{\mu\nu} + \vartheta\epsilon^{\mu\lambda\nu}\partial_\lambda\right)A_\nu = 0. \tag{2.4}$$

Substituting the solution

$$A^\mu(x) = \xi^\mu(k)\exp(ik.x) \tag{2.5}$$

for the negative energy component[1] in terms of the polarization vector $\xi^\mu(k)$ in the above two equations, gives, respectively,

$$k_\mu\xi^\mu = 0 \tag{2.6}$$

and

$$\Sigma^{\mu\nu}_{(MCS)}\xi_\nu(k) = 0 \tag{2.7}$$

where,

$$\Sigma^{\mu\nu}_{(MCS)} = -k^2 g^{\mu\nu} + i\vartheta\epsilon^{\mu\lambda\nu}k_\lambda. \tag{2.8}$$

[1]Here we simply suppressed the positive energy component which is just the complex conjugate of the negative energy component appearing in (2.5). Its presence is required to make $A^\mu(x)$ real. However, this suppression of the positive frequency part is of no consequence to our analysis.

For a non trivial solution to exist, we must have

$$\det \Sigma_{(MCS)} = -k^4(k^2 - \vartheta^2) = 0. \tag{2.9}$$

It follows therefore that, we must have either $k^2 = 0$ or $k^2 = \vartheta^2$. When $k^2 = 0$, the solution is,

$$\xi^\mu(k) = k^\mu f(k) \tag{2.10}$$

where $f(k)$ is an arbitrary function. Therefore massless excitations are pure gauge artefacts, which may be ignored.

Now consider the case $k^2 = \vartheta^2$, which implies that the quanta has mass[2] $|\vartheta|$. This enables a passage to the rest frame with $k^\mu = (|\vartheta|, 0, 0)$. Then the equation of motion (2.7) yields,

$$-\vartheta^2\xi_0(\mathbf{0}) = 0 \tag{2.11}$$

$$\vartheta^2\xi_1(\mathbf{0}) + i\vartheta(-|\vartheta|)\xi_2(\mathbf{0}) = 0 \tag{2.12}$$

$$\vartheta^2\xi_2(\mathbf{0}) + i\vartheta|\vartheta|\xi_1(\mathbf{0}) = 0 \tag{2.13}$$

where $\xi^\mu(\mathbf{0})$ stands for the MCS polarization in the rest frame. The above set yields,

$$\xi^0(\mathbf{0}) = 0 \tag{2.14}$$

$$\xi^2(\mathbf{0}) = -i\frac{\vartheta}{|\vartheta|}\xi^1(\mathbf{0}). \tag{2.15}$$

[2]This can also be seen by rewriting the motion (2.2) of MCS theory in terms of the gauge invariant pseudo vector dual field $\tilde{F}^\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}$:

$$(\Box + \vartheta^2)\tilde{F}^\mu = 0.$$

This equation clearly shows that MCS theory has massive excitations. Yet another way to arrive this result is by calculating the gauge field propagator whose pole gives the mass at $k^2 = \vartheta^2$ [9].

Therefore in the rest frame the polarization vector is given by,

$$\xi^{\mu}(\mathbf{0}) = \left(0, \xi^{1}(\mathbf{0}), -i\frac{\vartheta}{|\vartheta|}\xi^{1}(\mathbf{0})\right) \tag{2.16}$$

and is the maximally reduced form of the polarization vector ξ^{μ} in the rest frame representing just the single physical degree of freedom of the MCS theory[3]. The above expressions are determined modulo a normalization factor. This can be fixed from the normalization condition,

$$\xi^{*\mu}(\mathbf{0})\xi_{\mu}(\mathbf{0}) = -1 \tag{2.17}$$

following essentially from the space-like nature of the vector ξ^{μ}, as follows from (2.6) using the fact that k^{μ} is time-like. An important point of distinction from the Maxwell case is that ξ^{μ} has complex entries while the x and y components of the polarization vectors bear a simple ratio between them in the rest frame (2.15), so that the number of degrees of freedom reduces to one. Furthermore, the normalization condition(2.17) reveals a $U(1)$ invariance in the expression for ξ^{μ}; i.e., if ξ^{μ} is a solution, then $e^{i\phi}\xi^{\mu}$ is also a solution. This observation will be used later on to show the equivalence among different forms for ξ^{μ}.

The normalization condition fixes $|\xi^{1}(\mathbf{0})|^{2} = \frac{1}{2}$. Hence,

$$\xi^{\mu}(\mathbf{0}) = \frac{1}{\sqrt{2}}\left(0, 1, -i\frac{\vartheta}{|\vartheta|}\right). \tag{2.18}$$

Now we present another derivation of this result where only the symmetries of the model are used. Consider again the equation (2.2) and assume solutions of the form (2.5). Substituting (2.5) in (2.2) yields,

$$\xi_{\nu}k^{\nu}k^{\mu} - k^{2}\xi^{\mu} + i\vartheta\epsilon^{\mu\nu\lambda}\xi_{\lambda}k_{\nu} = 0. \tag{2.19}$$

[3]This method of obtaining the maximally reduced form of the polarization vector (or tensor) of a theory by starting from a plane wave solution of the Euler-Lagrange equation for the basic field is named plane wave method and is used extensively in this thesis

The two possibilities for k^2, corresponding to massless or massive modes are $(i)k^2 = 0$ and $(ii)k^2 \neq 0$. We first take up the case case (i). Using $k^2 = 0$ in (2.19) gives,

$$k^\mu(\xi \cdot k) = -i\vartheta\epsilon^{\mu\nu\lambda}k_\nu\xi_\lambda \tag{2.20}$$

Multiplying both sides with $\epsilon_{\mu\alpha\beta}k^\alpha$ one arrives at,

$$0 = i\vartheta k_\beta(\xi \cdot k)$$

which implies that the momentum space Lorentz condition

$$\xi \cdot k = 0 \tag{2.21}$$

holds automatically. Using $k^2 = 0$ and (2.21) in (2.19), we get $\xi^\mu = f(k)k^\mu$ which, as mentioned earlier, shows that massless excitations are pure gauge artefacts.

Next we consider the case $k^2 \neq 0$, from (2.19) we have,

$$\xi_\mu = \frac{1}{k^2}\left[(\xi \cdot k)k_\mu + i\vartheta\epsilon_{\mu\nu\lambda}k^\nu\xi^\lambda\right]. \tag{2.22}$$

and we are allowed to go to a rest frame where $k^\mu = (m, 0, 0)$ and $k^2 = m^2$. Let ξ^μ in this frame be given by,

$$\xi^\mu(0) = \left(\xi^0(0), \xi^1(0), \xi^2(0)\right). \tag{2.23}$$

Then (2.22) gives,

$$\xi^1(0) = \frac{i\vartheta}{m}\xi^2(0) \tag{2.24}$$

and

$$\xi^2(0) = -\frac{i\vartheta}{m}\xi^1(0). \tag{2.25}$$

Substituting for $\xi^2(0)$ from (2.25) in (2.24) gives

$$\vartheta^2 = m^2$$

from which it follows,

$$m = |\vartheta|. \tag{2.26}$$

From the gauge invariance of the model it follows that $\xi^0(\mathbf{0})$ can be set equal to zero. Therefore from (2.24), (2.25) and (2.26), we reproduce the earlier result (2.16). It is important to note that the result is compatible with the covariance condition (2.6) although it was not used explicitly in the analysis. This is also true for Maxwell theory where the polarization vector automatically satisfies this condition; but there k^μ corresponds to the massless physical excitations($k^2 = 0$). On the other hand, the massive excitations in the Maxwell theory are pure gauge artefacts, as can be easily seen from (2.19) by setting $\vartheta = 0$. Thus the roles of massive and massless excitations in MCS theory is just the opposite of Maxwell theory. It is now straightforward to calculate the polarization vector in a moving frame by giving a Lorentz boost [64] to the result in the rest frame,

$$\begin{pmatrix} \xi^0(k) \\ \xi^1(k) \\ \xi^2(k) \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta^1 & \gamma\beta^2 \\ \gamma\beta^1 & 1+\frac{(\gamma-1)(\beta^1)^2}{(\vec{\beta})^2} & \frac{(\gamma-1)\beta^1\beta^2}{(\vec{\beta})^2} \\ \gamma\beta^2 & \frac{(\gamma-1)\beta^1\beta^2}{(\vec{\beta})^2} & 1+\frac{(\gamma-1)(\beta^2)^2}{(\vec{\beta})^2} \end{pmatrix} \begin{pmatrix} \xi^0(\mathbf{0}) \\ \xi^1(\mathbf{0}) \\ \xi^2(\mathbf{0}) \end{pmatrix} \tag{2.27}$$

where $\vec{\beta} = \frac{\mathbf{k}}{k^0}$ and $\gamma = \frac{k^0}{|\vartheta|}$. The ensuing polarization vector is given by,

$$\xi^\mu(k) = \left(\frac{\vec{\xi}(\mathbf{0})\cdot\mathbf{k}}{|\vartheta|}, \vec{\xi}(\mathbf{0}) + \frac{\vec{\xi}(\mathbf{0})\cdot\mathbf{k}}{(k^0+|\vartheta|)|\vartheta|}\mathbf{k} \right) \tag{2.28}$$

where $\vec{\xi}(\mathbf{0})$ stands for the space part of the vector in (2.18). Thus,

$$\xi^\mu(k) = \left(\frac{k^1 - i\frac{\vartheta}{|\vartheta|}k^2}{\sqrt{2}|\vartheta|}, \frac{1}{\sqrt{2}} + \frac{k^1 - i\frac{\vartheta}{|\vartheta|}k^2}{\sqrt{2}(k^0+|\vartheta|)|\vartheta|}k^1, -i\frac{\vartheta}{\sqrt{2}|\vartheta|} + \frac{k^1 - i\frac{\vartheta}{|\vartheta|}k^2}{\sqrt{2}(k^0+|\vartheta|)|\vartheta|}k^2 \right) \tag{2.29}$$

which agrees with the expression given in [50] calculated in the Lorentz gauge.

2.1.2 Maxwell-Chern-Simons-Proca theory

The Maxwell-Chern-Simons-Proca(MCSP) Lagrangian is given by,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{m^2}{2}A^\mu A_\mu. \tag{2.30}$$

The equation of motion is,

$$-\partial^\nu F_{\mu\nu} + \theta\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda + m^2 A_\mu = 0 \tag{2.31}$$

which automatically satisfies the transversality condition $\partial_\mu A^\mu = 0$. Using this, the equation of motion simplifies to,

$$\left[(\Box + m^2)g^{\mu\nu} + \theta\epsilon^{\mu\lambda\nu}\partial_\lambda\right]A_\nu = 0 \tag{2.32}$$

Substitution of the solution $A^\mu = \varepsilon^\mu(k)\exp(ik.x)$ yields,

$$\left[(-k^2 + m^2)g^{\mu\nu} + i\theta\epsilon^{\mu\lambda\nu}k_\lambda\right]\varepsilon_\nu = 0. \tag{2.33}$$

From the transversality relation we get $k_\mu\varepsilon^\mu = 0$. Let us define,

$$\Sigma^{\mu\nu}_{(MCSP)} = (-k^2 + m^2)g^{\mu\nu} + i\theta\epsilon^{\mu\lambda\nu}k_\lambda \tag{2.34}$$

Then the equation of motion can be written as,

$$\Sigma^{\mu\nu}_{(MCSP)}\varepsilon_\nu = 0. \tag{2.35}$$

For ε_ν to have a non trivial solution the determinant of $\Sigma_{(MCSP)}$ should vanish. That is,

$$(-k^2 + m^2)\left[(-k^2 + m^2)^2 - \theta^2k^2\right] = 0. \tag{2.36}$$

This implies, either,

$$-k^2 + m^2 = 0 \tag{2.37}$$

or

$$(-k^2 + m^2)^2 - \theta^2 k^2 = 0. \tag{2.38}$$

Using (2.37) in (2.33), it follows that the solution must have the form, $\varepsilon^\mu(k) = k^\mu f(k)$, which is however incompatible with the transversality relation and is therefore ignored. The second case leads to

$$k^2 = \theta_\pm^2 \tag{2.39}$$

where,

$$\theta_\pm = \sqrt{\frac{2m^2 + \theta^2 \pm \sqrt{\theta^4 + 4m^2\theta^2}}{2}} = \sqrt{\frac{\theta^2}{4} + m^2} \pm \frac{\theta}{2}. \tag{2.40}$$

Two useful relations follow from this identification,

$$\theta = \theta_+ - \theta_- \tag{2.41}$$

and

$$m^2 = \theta_+ \theta_-. \tag{2.42}$$

We use the notation $\varepsilon_\pm(k_\pm)$ for the polarization vectors corresponding to $k^2 = {\theta_\pm}^2$ and let $\varepsilon_\pm(\mathbf{0})$ denote the polarization vectors in the rest frame. Taking the rest frames to be the ones in which $k^\mu = (k_\pm^0, 0, 0)^T = (|\theta_\pm|, 0, 0)^T$ we have from the equation of motion (2.33),

$$(m^2 - {\theta_\pm}^2)\varepsilon_{\pm 0}(\mathbf{0}) = 0,$$

$$-(m^2 - {\theta_\pm}^2)\varepsilon_{\pm 1}(\mathbf{0}) - i\theta\theta_\pm \varepsilon_{\pm 2}(\mathbf{0}) = 0$$

$$-(m^2 - {\theta_\pm}^2)\varepsilon_{\pm 2}(\mathbf{0}) + i\theta\theta_\pm \varepsilon_{\pm 1}(\mathbf{0}) = 0$$

where $\varepsilon_\pm^\mu = (\varepsilon_\pm^0, \varepsilon_\pm^1, \varepsilon_\pm^2)$. From the above set of equations we arrive at,

$$\varepsilon_{\pm 0}(\mathbf{0}) = 0 \tag{2.43}$$

$$\varepsilon_{\pm 2}(\mathbf{0}) = \frac{i\theta\theta_\pm}{m^2 - \theta_\pm{}^2}\varepsilon_{\pm 1}(\mathbf{0}) = \mp i\varepsilon_{\pm 1}(\mathbf{0}) \tag{2.44}$$

where the connection among various parameters has been used. Using a normalization condition analogous to (2.17)

$$\varepsilon^{*\mu}_{\pm}(\mathbf{0})\varepsilon_{\pm\mu}(\mathbf{0}) = -1 \tag{2.45}$$

gives,

$$|\varepsilon_{\pm 1}(\mathbf{0})|^2 = \frac{1}{2}. \tag{2.46}$$

Hence,

$$\varepsilon^{\mu}_{\pm}(\mathbf{0}) = \frac{1}{\sqrt{2}}(0, 1, \mp i). \tag{2.47}$$

The transversality condition $k_\mu \varepsilon^\mu = 0$ is preserved which acts as a consistency check. The polarization vectors in a moving frame corresponding to the two massive modes

with masses $\theta_\pm$ are easily found, as before, by giving a Lorentz boost,

$$\varepsilon^{\mu}_{\pm}(k_\pm) = \left(\frac{k^1 \mp ik^2}{\sqrt{2}\theta_\pm}, \frac{1}{\sqrt{2}} + \frac{k^1 \mp ik^2}{\sqrt{2}(k^0_\pm + \theta_\pm)\theta_\pm}k^1, \mp\frac{i}{\sqrt{2}} + \frac{k^1 \mp ik^2}{\sqrt{2}(k^0_\pm + \theta_\pm)\theta_\pm}k^2\right) \tag{2.48}$$

The pair of polarization vectors are related by the parity transformation in two space dimensions $k^1 \to k^1, k^2 \to -k^2$ augmented by $k^0_+ \to k^0_-$(which also implies $\theta_+ \to \theta_-$),

$$\begin{aligned}
\varepsilon^0_+(k^0_+, k^1, k^2) &= \varepsilon^0_-(k^0_- \to k^0_+, k^1 \to k^1, k^2 \to -k^2) \\
\varepsilon^1_+(k^0_+, k^1, k^2) &= \varepsilon^1_-(k^0_- \to k^0_+, k^1 \to k^1, k^2 \to -k^2) \\
\varepsilon^2_+(k^0_+, k^1, k^2) &= -\varepsilon^2_-(k^0_- \to k^0_+, k^1 \to k^1, k^2 \to -k^2).
\end{aligned} \tag{2.49}$$

Also, the pair is related by complex conjugation,

$$\varepsilon^{\mu}_{+}(k_+) = \varepsilon^{*\mu}_{-}(k_-). \tag{2.50}$$

where it is implied that this operation flips the parameter $\theta_+ \to \theta_-$. Now it may be pointed out that the polarization vectors satisfy the conditions,

$$\varepsilon_\pm^\mu(\mathbf{0})\varepsilon_{\pm\mu}(\mathbf{0}) = 0.$$

These polarization vectors are therefore light-like. Here however, it is possible to interpret these conditions as a consequence of the usual orthogonality relations,

$$\varepsilon_+^{*\mu}(\mathbf{0})\varepsilon_{-\mu}(\mathbf{0}) = 0.$$

together with the parity law (2.50). These observations suggest an inbuilt doublet structure in the MCSP model. The embedded doublet structure, related by the augmented parity transformations, in the MCSP theory will be further explored in the next subsection.

2.2 Application of polarization vectors

2.2.1 $U(1)$ invariance and doublet structure

The above methods of calculating the polarization vectors depend on the existence of a rest frame. The results obtained in this frame are Lorentz boosted to an arbitrary moving frame. There is another approach which directly yields the polarization vectors from a solution of the free field equations of motion. We now discuss this and compare with the previous analysis.

Let us consider the MCS theory (with $\vartheta > 0$). Since it has a single physical mode of mass ϑ, it is possible to write a general expression for the polarization vector, satisfying the Lorentz gauge condition (2.6) and the equation of motion (2.7),

$$\tilde{\xi}^\mu(k) = N\left(k^\mu - g^{\mu 0}\frac{\vartheta^2}{\omega} - i\frac{\vartheta}{\omega}\epsilon^{\mu\alpha 0}k_\alpha\right) \tag{2.51}$$

with, $\omega = k_0 = \sqrt{\vartheta^2 + |\mathbf{k}|^2}$ and N is the normalization. This is fixed from the condition $(\tilde{\xi}^{*\mu}\tilde{\xi}_\mu = -1)$,

$$N = \frac{1}{\sqrt{2}}\frac{\omega}{\vartheta|\mathbf{k}|}$$

This expression for the polarization vector was given in [15]. Though (2.51) appears quite different from the previous result (2.29) they only differ by a $U(1)$ phase factor. To see this, we express (2.51) in component form as follows.

$$\tilde{\xi}^\mu(k) = \frac{1}{\sqrt{2}\vartheta}\left(|\mathbf{k}|, \frac{\omega}{|\mathbf{k}|}(k_1 + i\frac{\vartheta}{\omega}k_2), \frac{\omega}{|\mathbf{k}|}(k_2 - i\frac{\vartheta}{\omega}k_1)\right) \tag{2.52}$$

By introducing a phase angle ϕ we can write the spatial components of k^μ as

$$k^1 = |\mathbf{k}|\cos\phi \tag{2.53}$$

$$k^2 = |\mathbf{k}|\sin\phi \tag{2.54}$$

Using (2.53, 2.54), since $\frac{\omega}{\vartheta} = 1 + \frac{|\mathbf{k}|^2}{(\omega+\vartheta)\vartheta}$, (2.52) can be rewritten as

$$\begin{aligned}\tilde{\xi}^\mu(k) &= \frac{1}{\sqrt{2}}\left[\frac{|\mathbf{k}|}{\vartheta}, \left(1 + \frac{|\mathbf{k}|^2}{(\omega+\vartheta)\vartheta}\right)\cos\phi + i\sin\phi, \left(1 + \frac{|\mathbf{k}|^2}{(\omega+\vartheta)\vartheta}\right)\sin\phi - i\cos\phi\right] \\ &= \frac{1}{\sqrt{2}}e^{i\phi}\left[e^{-i\phi}\frac{|\mathbf{k}|}{\vartheta}, 1 + \frac{|\mathbf{k}|^2}{(\omega+\vartheta)\vartheta}e^{-i\phi}\cos\phi, -i + \frac{|\mathbf{k}|^2}{(\omega+\vartheta)\vartheta}e^{-i\phi}\sin\phi\right] \\ &= e^{i\phi}\xi^\mu(k).\end{aligned}$$

(We have used (2.53, 2.54) again in the last step.) Up to a $U(1)$ phase, this exactly coincides with (2.29) thereby proving the equivalence of the two results. However, one may notice that the representation (2.51) does not have smooth rest frame limit whereas for (2.29) this limit is a well behaved one.

Following identical techniques the polarization vectors for MCSP theory turn out as,

$$\tilde{\varepsilon}_{\pm\mu} = \frac{1}{\sqrt{2}} \frac{\omega_\pm}{|\mathbf{k}|\sqrt{\omega_\pm^2 - |\mathbf{k}|^2}} \left(k_{\pm\mu} - g_{\mu 0} \frac{\omega_\pm^2 - |\mathbf{k}|^2}{\omega_\pm} - i \frac{\omega_\pm^2 - |\mathbf{k}|^2 - m^2}{\theta \omega_\pm} \epsilon_{\mu\alpha 0} k^\alpha \right) \quad (2.55)$$

where, $\omega_\pm = \sqrt{\theta_\pm^2 + |\mathbf{k}|^2}$. Once again this does not have a smooth transition to the rest frame. But we can show its equivalence with the expressions (2.48) by adopting the previous procedure. Expressions similar to (2.51) and (2.55) were reported earlier in [15].

Different representations of the polarization vectors find uses in different contexts. For instance when MCS theory is coupled to fermions (MCS-QED), the infra-red singularities of the 2+1 dimensional model leads to gauge dependence of the one-loop fermion mass-shell [9]. The expressions (2.51) and (2.55) are used in [15] for analyzing this fermion mass variance in the MCS-QED mentioned above. On the other hand, the expressions given in (2.29) and (2.48) reveal the presence of a doublet structure in the MCSP model. Specifically, the pair of polarization vectors $\varepsilon_\pm^\mu$, corresponding to the distinct massive modes $\theta_\pm$, can be exactly identified with the polarization vectors for a doublet of MCS models,

$$\mathcal{L}_+ = -\frac{1}{4} F^{\mu\nu}(A) F_{\mu\nu}(A) + \frac{\theta_+}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \quad (2.56)$$

$$\mathcal{L}_- = -\frac{1}{4} F^{\mu\nu}(B) F_{\mu\nu}(B) - \frac{\theta_-}{2} \epsilon^{\mu\nu\lambda} B_\mu \partial_\nu B_\lambda . \quad (2.57)$$

The necessary symmetry features are preserved provided both $\theta_\pm > 0$ or $\theta_\pm < 0$. It then follows from (2.29) that the polarization vectors of the MCS doublet exactly match with (2.48). The two massive modes $\theta_\pm$ of the doublet are exactly identified with the pair found in the MCSP model.

Yet another way of understanding the doublet structure is to look at the $m^2 \to 0$ limit of the MCSP model (2.30), which then reduces to the MCS model. From (2.41) and (2.42) we see that this limit corresponds to two possibilities;

$$(i) \qquad \theta_+ \to 0; \theta \to -\theta_- \tag{2.58}$$

$$(ii) \qquad \theta_- \to 0; \theta \to \theta_+ \tag{2.59}$$

These two cases ($\theta \to \pm\theta_\pm$) exactly correspond to the MCS doublet (2.56) and (2.57). Likewise the polarization vectors(2.48) also map to the corresponding doublet structure. Note that this expression is divergent for $\theta_+ \to 0$ or $\theta_- \to 0$, but this mode does not corresponds to the physical scattering amplitude when fermions are coupled [15].

It is worthwhile to mention that the limit $m^2 \to 0$ takes a second class system to a first class one. From the view point of a constrained system, such a limit is generally not smooth. However, the polarization vector shows a perfectly smooth transition. We might also recall that the $m^2 \to 0$ limit in the second class Proca model, to pass to the Maxwell theory, is problematic due to the change in the nature of the constraints. This is also manifested in the structure of the polarization vectors. Setting the $m^2 \to 0$ limit in the relevant expressions for the Proca model does not yield the Maxwell theory polarization vector. In this way, therefore, the massless limit in the MCSP theory is quite distinctive. Since a pair of MCS theories get mapped to the MCSP theory, such a smooth transition exists.

It is interesting to compare the above discussed relation between the second class MCSP theory and a doublet of first class MCS theories to the previously (chapter 1) mentioned embedding prescription due to Batalin, Fradkin and Tyutin [40, 41, 42]. In their prescription, a second class system is embedded in a first class system. In contrast, in the present context a pair of first class (MCS) theories gets embedded in a second class (MCSP) theory. Therefore, the mapping between MCSP is model and the doublet of MCS models is, in some sense, the opposite of the embedding procedure described in [40, 41, 42].

Another point worth mentioning is that, if $\theta_+ = \theta_-$, then $\theta = 0$ from (2.41). This means that a doublet of MCS theories having the same topological mass but with opposite helicities, maps to a Proca model, i.e., massive Maxwell model, with mass $m^2 = \theta_+^2 = \theta_-^2$. In this case parity is a symmetry which is also seen from the generalized transformations (2.49). This mapping was discussed earlier [57, 59] in terms of the basic fields of the respective models.

2.2.2 Mapping with a doublet of self dual models

We have shown how a doublet of MCS theories can be mapped to a MCSP theory using the explicit expressions for the polarization vectors of the theories. This subsection is devoted to a discussion of this seemingly paradoxical mapping between a doublet of gauge (MCS) theories and a non-gauge (MCSP) theory. We provide the justification for such a mapping and elucidate its physical interpretation and other related issues.

In order to explain the various subtleties regarding the mapping mentioned above, we begin by noting the well established equivalence between a self-dual model and the MCS theory [12, 13, 14]. The Lagrangian of the self-dual model is given by

$$\mathcal{L}_{SD} = \frac{1}{2}f^\mu f_\mu - \frac{1}{2M}\epsilon^{\mu\nu\lambda} f_\mu \partial_\nu f_\lambda \tag{2.60}$$

An obvious difference between the two theories is that, whereas the MCS theory is manifestly a gauge theory, possessing only first class constraints, the self dual model is a non-gauge theory and has second class constraints. It has been shown [13], using both operator and path integral techniques, that the gauge invariant sector of the

MCS theory given by $\mathcal{L}_+$ (2.56) gets mapped to the self dual model[4]. Specifically, the fundamental field f^μ of the self dual model and the dual field $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$ of the MCS theory get identified, so that $f^\mu = \tilde{F}^\mu$. Likewise the mass parameters in the two theories are also equated ($\theta_+ = M$).

In the present context the connection between the self dual and MCS theories will be discussed in terms of the polarization vectors. Indeed it can be verified explicitly that the polarization vector of the self dual model matches with the physical polarization vector [(2.18) and (2.28)] of MCS theory. The equation of motion following from (2.60) is

$$f^\mu - \frac{1}{M}\epsilon^{\mu\nu\lambda}\partial_\nu f_\lambda = 0. \tag{2.61}$$

As was done before, we consider a solution of the form

$$f^\mu = \xi^\mu(k)e^{ik.x} \tag{2.62}$$

where $\xi^\mu(k)$ stands for the polarization vector of the self dual theory. Substitution of (2.62) in (2.61) yields the equation

$$\left[g^{\mu\lambda} - \frac{i}{M}\epsilon^{\mu\nu\lambda}k_\nu\right]\xi_\lambda(k) = 0 \tag{2.63}$$

which will have a nontrivial solution only if

$$det\left[g^{\mu\lambda} - \frac{i}{M}\epsilon^{\mu\nu\lambda}k_\nu\right] = 0 \tag{2.64}$$

[4]This situation is just analogous to the well known equivalence between gauge non-invariant nonlinear sigma model(NLSM) and CP^1 model which is a $U(1)$ gauge theory. Here the mapping between the NLSM fields n^a satisfying $n^a n^a = 1 (a = 1,2,3)$ and the gauge-variant CP^1 field doublet $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfying $Z^\dagger Z = 1$ is given by the Hopf map $n^a = Z^\dagger \sigma^a Z$ with σ^a being Pauli matrices.

Now, (2.64) leads to the condition $k^2 = M^2$ which in turn implies that the excitations are massive and that there exists a rest frame for the quanta. Proceeding exactly as was done in the case of MCS and MCSP theory before, one can find the rest frame polarization vector of the self dual model as

$$\xi^\mu(\mathbf{0}) = \frac{1}{\sqrt{2}}(0, 1, -i) \tag{2.65}$$

To make a comparison, we now calculate the polarization vector $\tilde{\xi}^\mu(k)$ for the dual field $\tilde{F}^\mu = \tilde{\xi}^\mu(k)e^{ik.x}$ in the MCS theory given by the Lagrangian $\mathcal{L}_+$ (2.56). This is done by using the structure (2.5) for the A_μ field and passing to the momentum frame.

$$\tilde{F}^0 = \epsilon_{ij}\partial_i A_j = i\mathbf{k} \times \vec{\xi} e^{ik.x}$$

$$\tilde{F}^1 = \epsilon^{1\nu\lambda}\partial_\nu A_\lambda = i(k_2\xi_0 - k_0\xi_2)e^{ik.x}$$

$$\tilde{F}^2 = \epsilon^{2\nu\lambda}\partial_\nu A_\lambda = i(k_0\xi_1 - k_1\xi_0)e^{ik.x}$$

In the rest frame (where $k^\mu = (\theta_+, 0, 0)$) the above set of equations reduces to

$$\tilde{F}^0 = 0, \tilde{F}^1 = \theta_+ e^{ik.x}, \tilde{F}^2 = -i\theta_+ e^{ik.x}$$

where use has been made of the explicit form for ξ given in (2.18). Therefore, the polarization vector $\tilde{\eta}^\mu(\mathbf{0})$ in the rest frame is given by

$$\tilde{\xi}^\mu(\mathbf{0}) = \theta_+(0, 1, -i) \tag{2.66}$$

We thus find that the polarization vector for f^μ matches (up to a normalization factor) with that of $\tilde{F}^\mu$ thereby providing an alternative interpretation of the equivalence between the self dual and MCS theories. Moreover since the polarization vector for $\tilde{F}^\mu$ matches with that of A^μ calculated in the covariant gauge, this shows the equivalence of f^μ with A^μ taken in the covariant gauge. Clearly, the polarization vectors of f^μ and $\tilde{F}^\mu$ will match in a moving frame also. Similarly one can show that the polarization vector of the anti self dual model whose Lagrangian is given by

$$\mathcal{L}_{ASD} = \frac{1}{2}f^\mu f_\mu + \frac{1}{2M}\epsilon^{\mu\nu\lambda}f_\mu\partial_\nu f_\lambda \tag{2.67}$$

corresponds to that of the dual field in the MCS model given by $\mathcal{L}_-$.

The expressions for the polarization vectors in the self and anti self dual models obviously agree with the doublet structure found in the MCSP model. This shows the mapping of the self dual and anti self dual models with the MCSP theory[5]. A pair of gauge noninvariant models is mapped to a composite gauge noninvariant model. Since the self (or anti self) dual model is equivalent to the gauge invariant sector of the MCS theory, this clarifies the equivalence of the MCS doublet with the MCSP model. This equivalence should be interpreted as the mapping of the gauge invariant sector of the MCS doublet with the MCSP theory. Furthermore,since the mapping involves only the gauge invariant sector, the purported equivalence will hold in any gauge. Here it has been explicitly shown for a covariant gauge. For a proof in a noncovariant gauge like the Coulomb gauge we take recourse to an indirect argument. The equivalence of the self dual model with the MCS theory in different gauges including the Coulomb gauge has been shown in [13]. Since the mapping of the self dual and anti self dual pair with the MCSP has been illustrated, this shows that an analogous mapping must also hold for the MCS doublet in the Coulomb gauge.

One may also note that it is not possible to consider the two excitations in MCSP theory as arising from two scalar bosons. It was explicitly shown in [11] that the spins of the two excitations of MCSP theory are ± 1. Furthermore, group theoretical

[5]Such a mapping was also analysed in [11].

analysis of 2+1 dimensional theories shows that massive excitations arising due to the presence of parity violating terms(eg. the Chern-Simons term) have spin ± 1 [53]. For MCS theory it was shown that the spin of the massive excitation in $\mathcal{L}_+$ is +1 while that in $\mathcal{L}_-$ the spin is -1 [9]. This is also consistent with the mapping analysed here.

2.3 Polarization vectors in Hamiltonian formalism

The analysis of the polarization vectors of MCS theory in the Lagrangian formalism presented above is restricted to a single covariant gauge - the Lorentz gauge. However it is possible to obtain the polarization vectors with other gauge choices also. For example, Devecchi et. al.[50] studied the case of polarization vector of MCS theory in the Coulomb gauge which turned out to be of a different structure in comparison to (2.29). The MCSP theory not being a gauge theory, the form (2.48) of the polarization vectors of the theory is unique and corresponds to the fact that the Lorentz condition $\partial_\mu A^\mu = 0$ is automatically satisfied here, unlike MCS theory where it is imposed by hand. In this context one might wonder if it is possible to establish the doublet structure of MCSP theory in any other Lorentz covariant gauge. Furthermore, Lagrangian framework is a manifestly covariant formalism while the Hamiltonian formulation does not possess this covariance because of its very nature of singling out time. Also, because of the presence of momenta $\pi^\mu(x)$ conjugate to the field variables $A_\mu(x)$ in the Hamiltonian formalism, one has to introduce additional polarization vectors $(\pi^\mu(x))$ for to implement the mode expansion of the momenta. Hence it is not clear if the results obtained in the Lagrangian and the

Hamiltonian formalisms are mutually compatible. To settle these issues we compute the polarization vectors in the Hamiltonian formalism based on Dirac's constrained algorithm. Different covariant gauge conditions will be considered.

2.3.1 Maxwell-Chern-Simons theory

In this subsection we again consider the MCS field in the Lorentz gauge, but with the difference that the Lorentz gauge condition is now imposed at the level of the Lagrangian of the model itself. Consider the Lagrangian,

$$\mathcal{L}_\alpha = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \frac{1}{2\alpha}(\partial\cdot A)^2 \tag{2.68}$$

which is obtained from (2.1) by adding an extra gauge fixing term $-\frac{1}{2\alpha}(\partial\cdot A)^2$. (In this subsection α represents the gauge parameter). For simplicity, the parameter ϑ is taken positive. If the vector field A^μ satisfies the Lorentz constraint (2.3), the Lagrangian $\mathcal{L}_\alpha$ is equivalent to the MCS Lagrangian given by (2.1). The value of the gauge parameter α being arbitrary, we make the choice $\alpha = 1$(Feynman gauge). With this choice, after an integration by parts in the action, $\mathcal{L}_1$ transforms to,

$$\mathcal{L}_1 = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu\left[A_\nu(\partial^\nu A^\mu) - (\partial_\nu A^\mu)A^\mu\right] + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda.$$

Ignoring the total divergence term we write,

$$\mathcal{L}_1 = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \tag{2.69}$$

The conjugate momenta are given by,

$$\pi^\mu = \frac{\partial\mathcal{L}_1}{\partial\dot{A}_\mu} = (-\dot{A}^0, -\dot{A}^i + \vartheta\epsilon^{ij}A_j) \tag{2.70}$$

The Hamiltonian corresponding to (2.69) is given by

$$H_1 = \int d^2x\left[-\frac{1}{2}\pi^\mu\pi_\mu + \frac{1}{2}\partial^k A^\nu\partial_k A_\nu\right]$$

$$+\int d^2x \left[-\frac{\vartheta}{2}\epsilon^{ij}\left(A_i\pi_j + A_0\partial_i A_j + A_i\partial_j A_0\right) + \frac{1}{8}\vartheta^2\mathbf{A}^2\right] \tag{2.71}$$

The Hamilton's equations following from

$$\dot{A}^\mu = \{A^\mu, H_1\}$$

and

$$\dot{\pi}^\mu = \{\pi^\mu, H_1\}$$

are explicitly given as follows.

$$\dot{A}^0 = -\pi^0 \tag{2.72}$$

$$\dot{A}^1 = -\pi^1 - \frac{\vartheta}{2}A^2 \tag{2.73}$$

$$\dot{A}^2 = -\pi^2 + \frac{\vartheta}{2}A^1 \tag{2.74}$$

$$\dot{\pi}^0 = -\nabla^2 A^0 + \vartheta\epsilon^{ij}\partial_i A_j \tag{2.75}$$

$$\dot{\pi}^1 = -\nabla^2 A^1 - \vartheta\partial^2 A^0 + \frac{\vartheta^2}{4}A^1 - \frac{\vartheta}{2}\pi^2 \tag{2.76}$$

$$\dot{\pi}^2 = -\nabla^2 A^2 + \vartheta\partial^1 A^0 + \frac{\vartheta^2}{4}A^2 + \frac{\vartheta}{2}\pi^1 \tag{2.77}$$

Since our aim is to obtain the explicit form of the polarization vectors of the field, we consider solutions of the form,

$$A^\mu = \sum_{\lambda=1}^{2} \xi^\mu(\lambda, \mathbf{k}) a_{\mathbf{k}\lambda} \exp[ik\cdot x] + c.c \tag{2.78}$$

$$\pi^\mu = \sum_{\lambda=1}^{2} \xi^\mu(\lambda, \mathbf{k}) b_{\mathbf{k}\lambda} \exp[ik\cdot x] + c.c \tag{2.79}$$

Note that the polarization vectors $\xi^\mu(k)$ used in the previous section have been expanded in terms of their basis vectors $\xi^\mu(\lambda, \mathbf{k})$. Since $\xi^\mu(k)$ is space-like there are utmost two linearly independent vectors $\xi^\mu(\lambda, \mathbf{k})$ characterized by λ, which are used in the expansion of A^μ. The justification for using only the two space-like basis vectors $\xi^\mu(\lambda, \mathbf{k})$ for the mode expansion of π^μ is that the defining relation (2.70) of π^μ

involves components of A^μ (which can be written in terms of space-like polarization vectors) and their time derivatives. Therefore, π^μ also can be written in terms of the same space-like polarization vectors. The above solutions when substituted in (2.72) and (2.75) give, in the rest frame $(k_0, 0, 0)$ of the quanta of excitations,

$$(ik_0 a_1 + b_1)\xi^0(1, \mathbf{0}) + (ik_0 a_2 + b_2)\xi^0(2, \mathbf{0}) = 0$$

and

$$(ik_0 b_1)\xi^0(1, \mathbf{0}) + (ik_0 b_2)\xi^0(2, \mathbf{0}) = 0.$$

In these equations, the symbol $\mathbf{k}$ in $a_{\mathbf{k}\lambda}$ and $b_{\mathbf{k}\lambda}$ has been suppressed. It can be seen that the determinant $(a_1 b_2 - a_2 b_1)$ of the coefficients does not vanish, since in that case A^μ would be proportional to π^μ. Hence the only solutions to the above set of two equations are the trivial ones. That is

$$\xi^0(\lambda, \mathbf{0}) = 0 \tag{2.80}$$

Similar substitution of (2.78,2.79) in (2.73), (2.74) (2.76) and (2.77) gives, in the rest frame,

$$\Sigma_{MCS(H)} \bar{\xi}(0) = 0 \tag{2.81}$$

where

$$\Sigma_{MCS(H)} = \begin{pmatrix} ik_0 a_1 + b_1 & \frac{\vartheta}{2} a_1 & ik_0 a_2 + b_2 & \frac{\vartheta}{2} a_2 \\ -\frac{\vartheta}{2} a_1 & ik_0 a_1 + b_1 & -\frac{\vartheta}{2} a_2 & ik_0 a_2 + b_2 \\ ik_0 b_1 - \frac{\vartheta^2}{4} a_1 & \frac{\vartheta}{2} b_1 & ik_0 b_2 - \frac{\vartheta^2}{4} a_2 & \frac{\vartheta}{2} b_2 \\ -\frac{\vartheta}{2} b_1 & ik_0 b_1 - \frac{\vartheta^2}{4} a_1 & -\frac{\vartheta}{2} b_2 & ik_0 b_2 - \frac{\vartheta^2}{4} a_2 \end{pmatrix} \tag{2.82}$$

and

$$\bar{\xi}(0) = \begin{pmatrix} \xi^1(1, \mathbf{0}) \\ \xi^2(1, \mathbf{0}) \\ \xi^1(2, \mathbf{0}) \\ \xi^2(2, \mathbf{0}) \end{pmatrix} \tag{2.83}$$

The solutions of (2.81) are given by,

$$\xi^2(\lambda, \mathbf{0}) = -i\xi^1(\lambda, \mathbf{0}) \tag{2.84}$$

For $\vartheta > 0$, (2.80) and (2.84) agree with the result (2.14) and (2.15) obtained in the Lagrangian approach. The agreement clearly will be preserved for the polarizations vectors in a moving frame also.

Now we show that the above result is a special case of a more general one in which we introduce a Nakanishi-Lautrup auxiliary field $\mathcal{B}$ in the MCS Lagrangian and linearize the gauge fixing term [51]. In this covariant gauge formalism the Lagrangian (2.68) is expressed as,

$$\mathcal{L}_\alpha = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \mathcal{B}\partial_\mu A^\mu + \frac{\alpha}{2}\mathcal{B}^2 \tag{2.85}$$

Notice that $\mathcal{L}_\alpha$ does not contain $\dot{\mathcal{B}}$ and that it is linear in $\dot{A}^0$. The Euler-Lagrange equations of motion which follow from (2.85) are,

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu + \mathcal{B}) + \vartheta\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = 0 \tag{2.86}$$

and

$$\mathcal{B} = -\frac{1}{\alpha}\partial_\nu A^\nu. \tag{2.87}$$

Note that with the choice $\alpha = 1$ and eliminating $\mathcal{B}$ using the equation of motion (2.87), one can get $\mathcal{L}_1$. The momenta conjugate to the fields A^0, A^i and $\mathcal{B}$ are, respectively, given by

$$\pi_0 = \mathcal{B}$$

$$\pi_i = \partial_i A_0 - \dot{A}_i + \frac{\vartheta}{2}\epsilon_{ij}A^j$$

$$\pi_{\mathcal{B}} = 0$$

The Hamiltonian obtained from $\mathcal{L}_\alpha$ is

$$H_\alpha = \frac{1}{2}\int d^2x \left[(\pi_i)^2 + \vartheta\epsilon^{ij}\pi^i A^j + \frac{\vartheta^2}{4} + \frac{1}{2}(F^{ij})^2\right]$$

$$+\int d^2x \left[A^0(\partial^i\pi^i - \frac{\vartheta}{2}\epsilon^{ij}\partial^i A^j) - \mathcal{B}(\partial^i A^i) - \frac{1}{\alpha}\mathcal{B}^2\right] \tag{2.88}$$

The constraints of the model are,

$$\Phi_1 = \pi_0 - \mathcal{B} \approx 0 \tag{2.89}$$

and

$$\Phi_2 = \pi_{\mathcal{B}} \approx 0 \tag{2.90}$$

which form a second class pair. Setting the second class constraints strongly equal to zero, one obtains $\pi_0 = \mathcal{B}$, using which one can eliminate the auxiliary field $\mathcal{B}$ from the Hamiltonian:

$$H_\alpha = \frac{1}{2}\int d^2x \left[(\pi_i)^2 + \vartheta\epsilon^{ij}\pi^i A^j + \frac{\vartheta^2}{4} + \frac{1}{2}(F^{ij})^2\right]$$

$$+\int d^2x \left[A^0(\partial^i\pi^i - \frac{\vartheta}{2}\epsilon^{ij}\partial^i A^j) - \pi^0(\partial^i A^i) - \frac{1}{\alpha}(\pi^0)^2\right] \tag{2.91}$$

The Poisson brackets between the fields A^μ and their conjugate momenta π_ν are given by,

$$\{A^\mu(\mathbf{x},t), \pi_\nu(\mathbf{y},t)\} = g^\mu{}_\nu\delta(\mathbf{x}-\mathbf{y}). \tag{2.92}$$

It should be mentioned that the Dirac brackets in the (A^μ, π_ν) sector are identical to their Poisson brackets. Hence the Hamilton's equations for A^μ and π^μ are obtained from

$$\dot{A}^\mu = \{A^\mu, H_\alpha\}, \quad \dot{\pi}^\mu = \{\pi^\mu, H_\alpha\}$$

and are the following.

$$\dot{A}^0 = \partial^i A^i - \alpha\pi^0 \tag{2.93}$$

$$\dot{\pi}^0 = \frac{\vartheta}{2}\epsilon^{ij}\pi^i A^j - \partial^i\pi^i \tag{2.94}$$

$$\dot{A}^i = -\pi^i - \frac{\vartheta}{2}\epsilon^{ij}A^j + \partial^i A^0 \tag{2.95}$$

$$\dot{\pi}^i = \frac{\vartheta^2}{4}A^i + \partial^j F^{ij} - \frac{\vartheta}{2}\epsilon^{ij}\left(\partial^i A^0 + \pi^j\right) - \partial^i\pi^0 \tag{2.96}$$

Substitution of the expansions (2.78) and (2.79) in (2.93) and (2.94), in the rest frame $(k_0, 0, 0)$ leads to,

$$\left[\xi^0(1,0)a_1 + \xi^0(2,0)a_2\right] ik_0 + \left[\xi^0(1,0)b_1 + \xi^0(2,0)b_2\right]\alpha = 0$$

and

$$\left[\xi^0(1,0)b_1 + \xi^0(2,0)b_2\right] ik_0 = 0$$

If $\alpha \neq \infty$ we can rewrite the above set as

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \xi^0(1,0) \\ \xi^0(2,0) \end{pmatrix} = 0$$

The above equation does not have any nontrivial solution as $a_1 b_2 - a_2 b_1 \neq 0$ for the same reason mentioned earlier in the case of $\alpha = 1$. Hence,

$$\xi^0(\lambda, 0) = 0$$

if α is finite. When $\alpha = \infty$ there is no unambiguous solution. This case corresponds to the fact that when $\alpha = \infty$, the gauge fixing term in $\mathcal{L}_\alpha$ (2.68) vanishes. The expansions (2.78), (2.79) when substituted in (2.95,2.96) yields,

$$\xi^2(\lambda, 0) = -i\xi^1(\lambda, 0)$$

These results are independent of the gauge parameter and naturally agree with the previous $\alpha = 1$ calculation and are identical to the results (2.14, 2.15) obtained in the Lagrangian framework. Clearly the results will agree in a boosted frame also. Therefore one can conclude that the form (2.29,) of the physical polarization vector of MCS theory holds for the different Lorentz covariant gauge conditions considered here.

2.3.2 Maxwell-Chern-Simons-Proca theory

Taking the MCSP Lagrangian (2.30) the canonical momenta are defined as,

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -\left(F^{0i} + \frac{\theta}{2}\epsilon^{ij}A^j\right) \tag{2.97}$$

and

$$\pi^0 \approx 0 \tag{2.98}$$

is the primary constraint. The canonical Hamiltonian is,

$$H_{MCSP} = \frac{1}{2}\int d^2x \left[{\pi_i}^2 + \frac{1}{2}{F_{ij}}^2 + \left(\frac{\theta^2}{4} + m^2\right){A_i}^2 - \theta\epsilon_{ij}A_i\pi_j + m^2{A_0}^2\right]$$
$$+ \int d^2x A_0 \Omega \tag{2.99}$$

where,

$$\Omega = \partial_i\pi_i - \frac{\theta}{2}\epsilon_{ij}\partial_i A_j - m^2 A_0 \approx 0 \tag{2.100}$$

is the secondary constraint. Using (2.100) to eliminate A_0 from (2.99), we obtain the reduced Hamiltonian,

$$H_R = \frac{1}{2}\int d^2x \left[{\pi_i}^2 + \left(\frac{1}{2} + \frac{\theta^2}{8m^2}\right){F_{ij}}^2 + (\frac{\theta^2}{4} + m^2){A_i}^2 - \theta\epsilon_{ij}A_i\pi_j\right]$$
$$+ \frac{1}{2m^2}\int d^2x \left[(\partial_i\pi_i)^2 - \theta\partial_i\pi_i\epsilon_{lm}\partial_l A_m\right]. \tag{2.101}$$

The only non vanishing bracket between the phase space variables is,

$$\{A^i(\mathbf{x},t),\pi^j(\mathbf{y},t)\} = -\delta^{ij}\delta(\mathbf{x}-\mathbf{y}) \tag{2.102}$$

Therefore the Hamilton's equations are given by

$$\dot{A}^i = \{A^i, H_R\} = -\pi^i + \frac{1}{m^2}\partial^i\partial^j\pi^j - \frac{\theta}{2}\epsilon^{ij}A^j - \frac{\theta}{2m^2}\epsilon^{lm}\partial^i\partial^l A^m \tag{2.103}$$

and

$$\dot{\pi}^i = \{\pi^i, H_R\} = \left(1+\frac{\theta^2}{4m^2}\right)\left[\partial^i\partial^j A^j - \partial^j\partial^j A^i + m^2 A^i\right] - \frac{\theta}{2}\epsilon^{ij}\left[\pi^j + \frac{1}{m^2}\partial^j\partial^k\pi^k\right] \tag{2.104}$$

We consider solutions (in terms of the polarization vectors $\varepsilon(\mathbf{k},\lambda)$) of the form,

$$A^i = \sum_{\lambda=1}^{2}\varepsilon^i(\lambda,\mathbf{k})a_{\mathbf{k}\lambda}\exp[ik\cdot x] + c.c \tag{2.105}$$

$$\pi^i = \sum_{\lambda=1}^{2}\varepsilon^i(\lambda,\mathbf{k})b_{\mathbf{k}\lambda}\exp[ik\cdot x] + c.c \tag{2.106}$$

Substitution of the above solutions in the Hamilton's equations (2.103) and (2.104) yields, respectively,

$$\sum_{\lambda=1}^{2} -ik_0\varepsilon^i(\lambda,\mathbf{k})a_{\mathbf{k}\lambda} = \sum_{\lambda=1}^{2}\{[\varepsilon^i(\lambda,\mathbf{k}) + \frac{1}{m^2}k^ik^j\varepsilon^j(\lambda,\mathbf{k})]b_{\mathbf{k}\lambda}$$
$$+\frac{\theta}{2}[\epsilon^{ij}\varepsilon^j(\lambda,\mathbf{k}) - \frac{1}{m^2}\epsilon_{lm}k^ik^l\varepsilon^m(\lambda,\mathbf{k})]a_{\mathbf{k}\lambda}\} \tag{2.107}$$

and

$$\sum_{\lambda=1}^{2} -ik_0\varepsilon^i(\lambda,\mathbf{k})b_{\mathbf{k}\lambda} = \sum_{\lambda=1}^{2}\{[k^ik^j\varepsilon^j(\lambda,\mathbf{k}) - k^jk^j\varepsilon^i(\lambda,\mathbf{k}) - m^2\varepsilon^i(\lambda,\mathbf{k})](1+\frac{\theta^2}{4m^2})a_{\mathbf{k}\lambda}$$
$$+\frac{\theta}{2}\epsilon^{ij}[\varepsilon^j(\lambda,\mathbf{k}) - \frac{1}{m^2}k^jk^k\varepsilon^k(\lambda,\mathbf{k})]b_{\mathbf{k}\lambda}\} \tag{2.108}$$

In the rest frame $(k^0, 0, 0)$, the above set of four equations can be written in the matrix form as,

$$\Sigma_{MCSP(H)}\bar{\varepsilon} = 0 \tag{2.109}$$

where

$$\Sigma_{MCSP(H)} =$$

$$\begin{pmatrix} b_1 + ik_0a_1 & \frac{\theta}{2}a_1 & b_2 + ik_0a_2 & \frac{\theta}{2}a_2 \\ -\frac{\theta}{2}a_1 & b_1 + ik_0a_1 & -\frac{\theta}{2}a_2 & b_2 + ik_0a_2 \\ -Dm^2a_1 + ik_0b_1 & \frac{\theta}{2}b_1 & -Dm^2a_2 + ik_0b_2 & \frac{\theta}{2}b_2 \\ -\frac{\theta}{2}b_1 & -Dm^2a_1 + ik_0b_1 & -\frac{\theta}{2}b_2 & -Dm^2a_2 + ik_0b_2 \end{pmatrix} \tag{2.110}$$

and

$$\bar{\varepsilon} = \begin{pmatrix} \varepsilon^1(1, \mathbf{0}) \\ \varepsilon^2(1, \mathbf{0}) \\ \varepsilon^1(2, \mathbf{0}) \\ \varepsilon^2(2, \mathbf{0}) \end{pmatrix} \tag{2.111}$$

with $D = (1 + \frac{\theta^2}{4m^2})$. For a nontrivial solution of (2.109), $\det \Sigma_{MCSP(H)} = 0$. This condition, after a straightforward algebra, reduces to

$$(a_1b_2 - a_2b_1)^2 \left[k_0^4 - 2k_0^2(\frac{\theta^2}{2} + m^2) + m^4\right] = 0 \tag{2.112}$$

from which it follows that

$$\left[k_0^4 - 2k_0^2(\frac{\theta^2}{2} + m^2) + m^4\right] = 0 \tag{2.113}$$

That is,

$$k_0 = \sqrt{\frac{\theta^2}{4} + m^2} \pm \frac{\theta}{2} = \theta_\pm \tag{2.114}$$

Replacing k_0 with θ_+ in (2.109) we obtain, after suitable manipulations, the following relationship between the components of $\vec{\varepsilon}(\lambda, \mathbf{0})$;

$$\varepsilon^2(\lambda, \mathbf{0}) = -i\varepsilon^1(\lambda, \mathbf{0})$$

For $k_0 = \theta_-$ the corresponding relationship is given by

$$\varepsilon^2(\lambda, \mathbf{0}) = +i\varepsilon^1(\lambda, \mathbf{0}).$$

Denoting the rest frame polarization vectors corresponding to $k_0 = \theta_\pm$ by $\varepsilon_\pm(\mathbf{0})$, the above two expressions can be written as,

$$\varepsilon^2_\pm(\mathbf{0}) = \mp i\varepsilon^1_\pm(\mathbf{0}). \tag{2.115}$$

which agrees with the relationship obtained from the Lagrangian formalism. The polarization vectors in moving frames are obtained by boosting the rest frame vectors appropriately, and the result obviously agrees with that obtained earlier within the Lagrangian framework.

2.4 Summary

A detailed analysis of the polarization vectors in planar field theories involving both a topological mass and a usual mass has been done. The structure of these vectors is crucial for the reduction formulae, the study of unitarity in topologically massive gauge theories[15], as well as for considering the massless limit of these theories augmented by a normal mass term. Our general approach using either Lagrangian or Hamiltonian techniques, has shown a $U(1)$ invariance (in the k-space) in the form of the polarization vectors. This is quite distinct from the usual Abelian invariance associated with gauge theories. The $U(1)$ invariance reported here is connected with the presence of the Chern-Simons(CS) term and has nothing to do with the presence or absence of gauge freedom in the action. The CS term leads to complex entries in the polarization vector, thereby manifesting a $U(1)$ invariance. This can be contrasted with the pure Maxwell theory where all entries are real so that there is no $U(1)$ invariance of this type. It was seen that in MCS theories, the massive modes were physical while the massless ones could be gauged away and hence were unphysical. This is the exact counterpart of the Maxwell theory where the roles of the massive and massless modes are reversed. The structures found here naturally revealed a mapping between the Maxwell-Chern-Simons-Proca model and a doublet of Maxwell-Chern-Simons theories with opposite helicities. This was also helpful in studying the massless limit of the MCSP model. A mapping between the MCSP model and a pair of self and anti self dual models is shown, once again on the basis of the polarization vectors. The Hamiltonian analysis of MCS theory with various covariant gauge conditions revealed that the structure of the physical polarization vector for the theory is the same for all these gauge choices. For MCSP theory also both Lagrangian and Hamiltonian analyses produced identical results. This enables one to conclude that the mapping between MCSP theory to a doublet of MCS theories is manifested at the level of polarization vectors of the basic fields for arbitrary gauge parameter.

Chapter 3

Translational groups as generators of gauge transformations

Wigner's little group is quite familiar to physicists mainly because of its role in the classification of elementary particles. Wigner introduced the concept of little group in a seminal paper [52] published in 1939 and showed how particles can be classified on the basis of their spin/ helicity quantum numbers using the little group. Several decades later Wigner along with Kim showed that the little group relates the internal symmetries of massive and massless particles [54, 55, 56]. A comparatively less known facet of the little group, namely its role as a generator of gauge transformations in various Abelian gauge theories, was also unraveled in the mean time. Historically, the first attempts to study this aspect of little group were in the contexts of free Maxwell theory [19, 22, 23] and linearized Einstein gravity [25]. These studies revealed that the defining representation of 3+1 dimensional Wigner's little group for massless particles (which is isomorphic to $E(2)$ - Euclidean group in two dimensions)[1], or more precisely its translational subgroup $T(2)$, acts as generators of gauge transformations in 3+1 dimensional Maxwell theory and linearized gravity. Recently, this gauge generating property of the little group received more attention [27, 32, 33]. Moreover, it was shown that the same translational group was found to be generating gauge transformation in the 3+1 dimensional Kalb-Ramond (KR) theory involving a massless 2-form gauge field [27]. (A further study on the gauge generating property of Wigner's little group in the context of massless KR theory can be found in [58] where its connection with BRST cohomology is explored.) On the other hand, the same study [27] showed that for the $B \wedge F$ theory which is obtained by topologically coupling Maxwell field to a KR field, the generator of gauge transformation is a particular representation of the three dimensional translational group $T(3)$. Note that $B \wedge F$ theory is a topologically massive gauge theory where gauge invariance co-exist with mass [28, 29, 30]. Thus we see that different translational groups in their appropriate representations generate gauge transformations in various Abelian gauge theories in 3+1 dimensions. In this chapter we provide a review of the gauge generating property of translational group $T(2)$ and $T(3)$ with necessary details. All the theories considered in this chapter belong to 3+1 dimensional space-time. In later chapters, we borrow some of the techniques developed in the context of these 3+1 dimensional theories for similar investigations in planar theories.

[1] A brief review of the essential properties of this little group is given in appendix A.

3.1 Translational subgroup of Wigner's little group

A brief review of the essential aspects of Wigner's little group for massless particles [19, 22, 70] which are relevant to the present study is provided in appendix A. Wigner's little group is defined as the subgroup of homogeneous Lorentz group that preserves the energy-momentum vector of a particle. For a massive particle, it is trivial to see that the little group is given by $SO(3)$ as one makes the transition to rest frame. As discussed in appendix A, an element of the little group that preserves the four-momentum $k^a = (\omega, 0, 0, \omega)^T$ of a massless particle moving in the z-direction is given by

$$W(p,q;\phi) = W(p,q)R(\phi) \tag{3.1}$$

where

$$W(p,q) \equiv W(p,q;0) = \begin{pmatrix} 1+\frac{p^2+q^2}{2} & p & q & -\frac{p^2+q^2}{2} \\ p & 1 & 0 & -p \\ q & 0 & 1 & -q \\ \frac{p^2+q^2}{2} & p & q & 1-\frac{p^2+q^2}{2} \end{pmatrix} \tag{3.2}$$

is a particular representation of the translational subgroup $T(2)$ of the little group and $R(\phi)$ represents a $SO(2)$ rotation about the z-axis. Note that the representation $W(p,q)$ satisfies the relation $W(p,q)W(\bar{p},\bar{q}) = W(p+\bar{p},q+\bar{q})$.

3.2 Maxwell theory

We now discuss how this representation of $T(2)$ generates gauge transformation in Maxwell theory which has the well known Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab}. \tag{3.3}$$

Maxwell theory is invariant under the gauge transformation

$$A_a(x) \to A'_a(x) = A_a(x) + \partial_a \tilde{f}(x) \tag{3.4}$$

where $\tilde{f}(x)$ is an arbitrary scalar function. The Lagrangian (3.3) leads to following equation of motion:

$$\partial_a F^{ab} = 0. \tag{3.5}$$

Denoting the polarization vector of a photon by $\varepsilon^a(k)$, the gauge field $A^a(x)$ can be written as

$$A^a(x) = \varepsilon^a(k)e^{ik\cdot x}. \tag{3.6}$$

In terms of the polarization vector, the gauge transformation (3.4) is expressed as

$$\varepsilon_a(k) \to \varepsilon'_a(k) = \varepsilon_a(k) + if(k)k_a. \tag{3.7}$$

where $\tilde{f}(x)$ has been written as $\tilde{f}(x) = f(k)e^{ik\cdot x}$. The equation of motion, in terms of the polarization vector, will now be given by

$$k^2\varepsilon^a - k^a k_b \varepsilon^b = 0. \tag{3.8}$$

The massive excitations corresponding to $k^2 \neq 0$ leads to a solution $\varepsilon^a \propto k^a$ which can therefore be gauged away by a suitable choice of $f(k)$ in (3.7). For massless excitations ($k^2 = 0$), the Lorentz condition $k_a\varepsilon^a = 0$ follows immediately from (3.8). For a photon of energy ω propagating in the z-direction (i.e., $k^a = (\omega, 0, 0, \omega)^T$, it follows from (3.8) that the corresponding polarization tensor $\varepsilon^\mu(k)$ takes the form $(\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^0)$ which can be reduced to the maximally reduced form

$$\varepsilon^a(k) = (0, \varepsilon^1, \varepsilon^2, 0)^T \tag{3.9}$$

by a suitable gauge transformation (3.7) with $f(k) = \frac{i\varepsilon^0}{\omega}$. Note that the maximally reduced form displays just the two transverse physical degrees of freedom ε^1 and

ε^2. Under the action (3.2) of the translational group $T(2)$ this polarization vector transforms as follows:

$$\varepsilon^a \to \varepsilon'^a = W^a{}_b(p,q)\varepsilon^b = \varepsilon^a + \left(\frac{p\varepsilon^1 + q\varepsilon^2}{\omega}\right) k^a \,. \tag{3.10}$$

Clearly, this can be identified as a gauge transformation of the form (3.7) by choosing

$$f(k) = \frac{p\varepsilon^1 + q\varepsilon^2}{i\omega}. \tag{3.11}$$

Hence one says that the translational subgroup of Wigner's little group for massless particles acts as a gauge generator in Maxwell theory.

3.3 Linearized gravity

The pure Einstein-Hilbert action in 3+1 dimensions is given by

$$I^E = -\int d^4x \mathcal{L}^E, \qquad \mathcal{L}^E = \sqrt{g}R = \sqrt{g}g^{ab}R_{ab} \tag{3.12}$$

where $\mathcal{L}^E$ is the Einstein Lagrangian and R_{ab} is the Ricci tensor. In the linearized approximation the metric g_{ab} is assumed to be close to the flat background part η^{ab} and therefore

$$g_{ab} = \eta_{ab} + h_{ab} \tag{3.13}$$

where h_{ab} is the deviation such that $|h_{ab}| << 1$. When the deviation is small one considers only terms up to first order in h_{ab}. The raising and lowering of indices is done using η^{ab} and η_{ab} respectively.

The linearized version of Einstein-Hilbert Lagrangian is

$$\mathcal{L}_L^E = \frac{1}{2}h_{ab}\left[R_L^{ab} - \frac{1}{2}\eta^{ab}R_L\right]. \tag{3.14}$$

Here R_L^{ab} is the linearized Ricci tensor given by

$$R_L^{ab} = \frac{1}{2}(-\Box h^{ab} + \partial^a \partial_c h^{cb} + \partial^b \partial_c h^{ca} - \partial^a \partial^b h) \tag{3.15}$$

with $h = h^c{}_c$. Similarly $R_L = R_{Lc}^c$. As mentioned before, the translational subgroup of Wigner's little group for massless particles generates gauge transformations in linearized Einstein gravity also [25, 26]. However, the gauge transformations generated by the translational group constitute only a subset of the entire set of gauge transformations available in linearized gravity. Here we give a detailed review of this partial gauge generation in linearized gravity by the translational group $T(2)$.

Gravity (linearized) in d space-time dimensions is described by a symmetric second rank tensor gauge field and has $\frac{1}{2}d(d-3)$ degrees of freedom[2]. Therefore general relativity in 3+1 dimensions has two degrees of freedom. The field equations for h^{ab} following from the Lagrangian (3.14) is given by

$$-\Box h^{ab} + \partial^a \partial_c h^{cb} + \partial^b \partial_c h^{ca} - \partial^a \partial^b h + \eta^{ab}(\Box h - \partial_c \partial_d h^{cd}) = 0. \tag{3.16}$$

The above equation is invariant under the following gauge transformation:

$$h^{ab} \to h'^{ab} = h^{ab} + \partial^a \zeta^b(x) + \partial^b \zeta^a(x) \tag{3.17}$$

Here $\zeta^a(x)$ are completely arbitrary except that they are considered to be small.

[2]The degree of freedom counting can be done by following Weinberg [60]. To start with, note that a symmetric second rank tensor in d dimensions has $\frac{1}{2}d(d+1)$ independent components. Analogous to the Lorentz gauge condition ($\partial^\mu A_\mu = 0$) of Maxwell theory, in general relativity we have the harmonic gauge condition $g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0$ which amounts to d constraints on the components of $g_{\mu\nu}$. These along with the d independent components of the gauge parameter (which by itself is a d-vector now; see the ensuing discussion below, particularly (3.17)), in the linearized version of the theory, reduces the number of independent components of the tensor field to $\frac{1}{2}d(d+1) - 2d = \frac{1}{2}d(d-3)$.

Following the plane wave method, we now adopt the ansatz

$$h^{ab} = \chi^{ab}(k)e^{ik.x} + c.c. \tag{3.18}$$

where $\chi^{ab}(k)$ is the symmetric polarization tensor. With the choice

$$\zeta^a(x) = -i\zeta^a(k)e^{ik.x} + c.c. \tag{3.19}$$

the gauge transformation in h^{ab} can be written in terms of the polarization tensor as

$$\chi^{ab}(k) \rightarrow \chi'^{ab}(k) = \chi^{ab}(k) + k^a\zeta^b(k) + k^b\zeta^a(k). \tag{3.20}$$

Just as in the Maxwell case, hereafter we will consider only the negative frequency part for simplicity. Substituting the ansatz in the equation of motion yields

$$k^2\chi^{ab} - k^a k_c \chi^{cb} - k^b k_c \chi^{ca} + k^a k^b \chi + \eta^{ab}(-k^2\chi + k_c k_d \chi^{cd}) = 0. \tag{3.21}$$

As was done in the previous cases, we separately consider the two possibilities $k^2 \neq 0$ and $k^2 = 0$. Choosing the massive ($k^2 \neq 0$) case first, we contract the equation of motion (3.21) with η^{ab} to obtain

$$k^2\chi - k_c k_d \chi^{cd} = 0\ ; \qquad \chi = \chi^a{}_b. \tag{3.22}$$

A general solution to this equation is given by

$$\chi^{ab}(k) = k^a f^b(k) + k^b f^a(k) \tag{3.23}$$

with $f^a(k)$ being arbitrary functions of k. Therefore, it can be easily seen that this solution can be 'gauged away' by appropriate choice of the variables $\zeta^a(k)$ in (3.20) as this corresponds to pure gauge. Thus, analogous to Maxwell theory, the massive excitations of linearized Einstein gravity are gauge artefacts.

For massless ($k^2 = 0$) excitations, the equation of motion (3.21) reduces to

$$-k^a k_c \chi^{cb} - k^b k_c \chi^{ca} + k^a k^b \chi + \eta^{ab} k_c k_d \chi^{cd} = 0. \qquad (3.24)$$

In a frame of reference where $k^a = (\omega, 0, 0, \omega)^T$ the above equation can be written as

$$-\omega[k^a(\chi^{0b} - \chi^{3b}) + k^b(\chi^{0a} - \chi^{3a})] + k^a k^b \chi + \omega^2 \eta^{ab}(\chi^{00} + \chi^{33} - 2\chi^{03}) = 0. \quad (3.25)$$

Various components of the above equation together with the symmetricity of χ^{ab} leads to the reduction in the number of independent components of χ^{ab}. In (3.25), $a = b = 0$ leads to $\chi^{11} = -\chi^{22}$; $a = 0, b = i$ to $\chi^{0i} = \chi^{3i}$ and $a = b = i$ to $\chi^{00} = \chi^{33}$. Therefore, one can write the polarization tensor χ^{ab} in a reduced form as

$$\{\chi^{ab}\} = \begin{pmatrix} \chi^{00} & \chi^{01} & \chi^{02} & \chi^{00} \\ \chi^{01} & \chi^{11} & \chi^{12} & \chi^{01} \\ \chi^{02} & \chi^{12} & -\chi^{11} & \chi^{02} \\ \chi^{00} & \chi^{01} & \chi^{02} & \chi^{00} \end{pmatrix}. \qquad (3.26)$$

Now if we make a momentum space gauge transformation with the choice $\zeta^0 = \zeta^3 = -\frac{\chi^{00}}{2\omega}, \zeta^2 = -\frac{\chi^{01}}{\omega}, \zeta^2 = -\frac{\chi^{02}}{\omega}$, the polarization tensor $\{\chi^{ab}\}$ can be written in the maximally reduced form as follows:

$$\{\chi^{ab}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \chi^{11} & \chi^{12} & 0 \\ 0 & \chi^{12} & -\chi^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.27)$$

Here χ^{11} and χ^{12} represent the two physical degrees of freedom for the theory. (This form of the polarization tensor of linearized Einstein gravity in 3+1 dimensions is derived in [60] following a different approach.) In this form (3.27), the polarization tensor $\{\chi^{ab}\}$ satisfies the harmonic gauge condition,

$$k_a\chi^{ab} = \frac{1}{2}k^b\chi^a_a \tag{3.28}$$

in momentum space, automatically. Using the maximally reduced form (3.27) of the polarization tensor, it is now straightforward to show that the group $W(p,q)$ in (3.2) generate gauge transformations in linearized Einstein gravity also. For this purpose consider the action of $W(p,q)$ on $\{\chi^{ab}\}$ in (3.27),

$$\{\chi^{ab}\} \to \{\chi'^{ab}\} = W(p,q)\{\chi^{ab}\}W^T(p,q)$$

$$= \{\chi^{ab}\} + \begin{pmatrix} \begin{pmatrix} (p^2-q^2)\chi^{11} \\ +2pq\chi^{12}) \end{pmatrix} & (p\chi^{11}+q\chi^{12}) & (p\chi^{12}-q\chi^{11}) & \begin{pmatrix} (p^2-q^2)\chi^{11} \\ +2pq\chi^{12} \end{pmatrix} \\ (p\chi^{11}+q\chi^{12}) & 0 & 0 & (p\chi^{11}+q\chi^{12}) \\ (p\chi^{12}-q\chi^{11}) & 0 & 0 & (p\chi^{12}-q\chi^{11}) \\ \begin{pmatrix} (p^2-q^2)\chi^{11} \\ +2pq\chi^{12} \end{pmatrix} & (p\chi^{11}+q\chi^{12}) & (p\chi^{12}-q\chi^{11}) & \begin{pmatrix} (p^2-q^2)\chi^{11} \\ +2pq\chi^{12} \end{pmatrix} \end{pmatrix}. \tag{3.29}$$

The above transformation can be cast in the form of a gauge transformation (3.20) with the following choice for the arbitrary functions $\zeta^a(k)$:

$$\zeta^0 = \zeta^3 = \frac{(p^2-q^2)\chi^{11} + 2pq\chi^{12}}{\omega}, \quad \zeta^1 = \frac{p\chi^{11}+q\chi^{12}}{\omega}, \quad \zeta^2 = \frac{p\chi^{12}-q\chi^{11}}{\omega}. \tag{3.30}$$

However, since $k^a = (\omega, 0, 0, \omega)^T$, a general gauge transformation for this polarization tensor (3.27) has the form

$$\{\chi^{ab}\} \to \{\chi'^{ab}\} = \{\chi^{ab}\} + \{k^a\zeta^b\} + \{k^b\zeta^a\}$$

$$= \{\chi^{ab}\} + \omega \begin{pmatrix} 2\zeta^0 & \zeta^1 & \zeta^2 & (\zeta^0+\zeta^3) \\ \zeta^1 & 0 & 0 & \zeta^1 \\ \zeta^2 & 0 & 0 & \zeta^2 \\ (\zeta^0+\zeta^3) & \zeta^1 & \zeta^2 & 2\zeta^3 \end{pmatrix}. \tag{3.31}$$

Upon comparing the above form of general gauge transformation with the one generated by $W(p,q)$ given in (3.29), it becomes clear that the latter is only a special case of the former as the relation $\zeta^0 = \zeta^3$ in (3.30) restricts the number of independent components of the arbitrary vector ζ^a. Therefore, the translational subgroup $T(2)$ of Wigner's little group for massless particles generates only a subset of the full set of gauge transformations in linearized gravity. The reason for this partial gauge transformation is as follows. We must notice that our starting point of gauge generation by $W(p,q)$ in linearized gravity is the maximally reduced polarization tensor (3.27) which contains just the physical sector of the theory (in the reference frame where $k^\mu = (\omega, 0, 0, \omega)^T$) and is devoid of any arbitrary variables to begin with. Hence in the gauge generation by $W(p,q)$, we must rely entirely on the two parameters p and q of the translational group to manufacture the gauge equivalence class of the state corresponding the polarization tensor (3.27). However, the gauge freedom in linearized gravity is represented by the arbitrary vector variable ζ^a having four components. Naturally, in the gauge generation by $W(p,q)$ in linearized gravity, only two of the four components of ζ^a remain independent (as is evident from(3.30)) when expressed in terms of the two parameters (p,q) of the translational group and therefore the gauge generation is only partial. It was noted in [25] that the gauge generation by the little group in linearized gravity is subject to the 'Lorentz condition' $k_a\zeta^a(k) = 0$. This also can be seen from the first relation $\zeta^0 = \zeta^3$ in (3.30). Thus, we have unraveled all the constraints behind the partial gauge generation by Wigner's little group in linearized gravity. In contrast, we may note that the gauge freedom in free Maxwell theory is represented by a single arbitrary scalar variable $f(k)$ (3.11) which can be expressed (without any restrictions) in terms of the two parameters of $W(p,q)$ in the gauge generation by little group as is evident from (3.10). Hence translational subgroup of Wigner's little group generates the full set of gauge transformations in Maxwell theory.

3.4 Massless Kalb-Ramond theory

By similar methods it can be shown that gauge transformations are generated by $T(2)$ in the 3+1 dimensional Kalb-Ramond(KR) theory [27] which has a second rank antisymmetric tensor as its basic field. The KR theory is described by the Lagrangian

$$\mathcal{L} = \frac{1}{12} H_{abc} H^{abc}; \qquad H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} \tag{3.32}$$

where B_{ab} is a 2-form gauge field:

$$B_{ab} = -B_{ba}. \tag{3.33}$$

The equation of motion is

$$\partial_a H^{abc} = 0. \tag{3.34}$$

The KR theory is invariant under the gauge transformation

$$B_{ab}(x) \rightarrow B'_{ab}(x) = B_{ab}(x) + \partial_a F_b(x) - \partial_b F_a(x) \tag{3.35}$$

where $F_a(x)$ are arbitrary functions. However, these gauge transformations are not all independent. One can see that under the transformation

$$F_a(x) \rightarrow F'_a(x) = F_a(x) + \partial_a \beta(x) \tag{3.36}$$

(where $\beta(x)$ is an arbitrary scalar function) the gauge transformation (3.35) remains invariant. In particular, if $F_a = \partial_a \Lambda$, the gauge transformation vanish trivially. This is known as the 'gauge invariance of gauge transformations' and is a typical property of reducible gauge theories where the generators of gauge transformation are not all independent [49]. Since the components of the arbitrary field $F_a(x)$ (which represents the gauge freedom in the KR theory) are not all independent, there exists some superfluity in the gauge transformation (3.35).

In order to obtain the maximally reduced form of the antisymmetric polarization tensor $\varepsilon_{ab}(k)$ of the massless KR theory, as was done in the previous models, we employ the plane wave method using the ansatz

$$B_{ab}(x) = \varepsilon_{ab}(k)e^{ik\cdot x}. \tag{3.37}$$

In terms of the polarization tensor ε^{ab}, the gauge transformation (3.35) can be written in the momentum space as

$$\varepsilon_{ab}(k) \to \varepsilon'_{ab}(k) = \varepsilon_{ab}(k) + i(k_a f_b(k) - k_b f_a(k)) \tag{3.38}$$

while the counterpart of (3.36) is given by

$$f_a(k) \to f'_a(k) + ik_a\tilde{\beta} \tag{3.39}$$

(where we have written $F_a(x) = f_a(k)e^{ik\cdot x}$ and $\beta(x) = \tilde{\beta}(k)e^{ik.x}$) and the equation of motion (3.34) as

$$k_a[k^a\varepsilon^{bc} + k^b\varepsilon^{ca} + k^c\varepsilon^{ab}] = 0. \tag{3.40}$$

For massive excitations (i.e., when massive $k^2 \neq 0$), we have

$$\varepsilon^{bc}(k) = \frac{1}{k^2}[k^b(k_a\varepsilon^{ac}) - k^c(k_a\varepsilon^{ab})] \tag{3.41}$$

Using (3.38), this can be gauged away by choosing

$$f^c(k) = \frac{i}{k^2}k_a\varepsilon^{ac} \tag{3.42}$$

We thus find that massive excitations of KR theory are gauge artefacts.

For massless excitations ($k^2 = 0$), the momentum-space equation of motion (3.40) reduces to

$$k_a \varepsilon^{ab} = 0 \tag{3.43}$$

which is equivalent to the "Lorentz condition" $\partial_a B^{ab} = 0$. Using this condition, the six independent components of the antisymmetric polarization matrix $\{\varepsilon^{ab}\}$ can be reduced further. In the reference frame where the light-like vector k^a takes the form $k^a = (\omega, 0, 0, \omega)^T$, the condition (3.43) can be written as

$$\{\varepsilon^{ab}\} \cdot p = \begin{pmatrix} 0 & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{03} \\ -\varepsilon^{01} & 0 & \varepsilon^{12} & \varepsilon^{13} \\ -\varepsilon^{02} & -\varepsilon^{12} & 0 & \varepsilon^{23} \\ -\varepsilon^{03} & -\varepsilon^{13} & -\varepsilon^{23} & 0 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} = 0. \tag{3.44}$$

The above equation can easily be simplified to

$$\varepsilon^{03} = 0, \quad \varepsilon^{01} = \varepsilon^{13}, \quad \varepsilon^{02} = \varepsilon^{23} \tag{3.45}$$

so that the polarization tensor ε^{ab} can now be written as

$$\{\varepsilon^{ab}\} = \begin{pmatrix} 0 & \varepsilon^{01} & \varepsilon^{02} & 0 \\ -\varepsilon^{01} & 0 & \varepsilon^{12} & \varepsilon^{01} \\ -\varepsilon^{02} & -\varepsilon^{12} & 0 & \varepsilon^{02} \\ 0 & -\varepsilon^{01} & -\varepsilon^{02} & 0 \end{pmatrix}. \tag{3.46}$$

With the gauge choice $f^1 = \frac{i}{\omega}\varepsilon^{01}$ and $f^2 = \frac{i}{\omega}\varepsilon^{02}$, if we now make a gauge transformation (3.38), the above form of the polarization tensor yields the following maximally reduced form:

$$\{\varepsilon^{ab}\} = \varepsilon^{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.47}$$

The transformation of this maximally reduced polarization tensor (3.47) of the massless KR theory under the translational subgroup $W(p,q)$ (3.2) of Wigner's little group, can be written as

$$\{\varepsilon^{ab}\} \to \{\varepsilon'^{ab}\} = W(p,q)\{\varepsilon^{\mu\nu}\}W^T(p,q) = \{\varepsilon^{ab}\} + \varepsilon^{12}\begin{pmatrix} 0 & -q & p & 0 \\ q & 0 & 0 & q \\ -p & 0 & 0 & -p \\ 0 & -q & p & 0 \end{pmatrix} \quad (3.48)$$

This can be cast in the form of (3.38) with

$$f^1 = \frac{-q\varepsilon^{12}}{i\omega}, \quad f^2 = \frac{p\varepsilon^{12}}{i\omega}, \quad f^3 = f^0. \quad (3.49)$$

Hence we can say that defining representation $W(p,q)$ of the translational subgroup $T(2)$ of Wigner's little group for massless particles generate gauge transformations in massless KR theory also. However, as in the case of linearized gravity, on account of the requirement $f^3 = f^0$ the gauge transformations generated by the translational group $T(2)$ fails to include the entire set of gauge transformations in KR theory. The general form of gauge transformation (3.38) in the matrix form is

$$\{\varepsilon^{ab}\} \to \{\varepsilon'^{ab}\} = \{\varepsilon^{ab}\} + \omega\begin{pmatrix} 0 & f^1 & f^2 & f^0 - f^3 \\ -f^1 & 0 & 0 & -f^1 \\ -f^2 & 0 & 0 & -f^2 \\ f^3 - f^0 & f^1 & f^2 & 0 \end{pmatrix} \quad (3.50)$$

which makes it quite explicit that the transformation (3.48) does not exhaust (3.50), but is only a special case (where $f^0 = f^3$) of it. Here again, for the case of gauge transformation (3.48) generated by the translational group $W(p,q)$, the arbitrary vector function $f^a(k)$ which correspond to the gauge freedom of KR theory satisfy the 'Lorentz condition' $k_a f^a(k) = 0$ since $k^a = (\omega, 0, 0, \omega)^T$ corresponds to a KR quantum propagating in the z-direction.

Similar to the gauge generation (3.29) in linearized gravity, the transformation (3.48) is an attempt to generate the gauge equivalence class of the completely physical (maximally reduced) polarization tensor (3.47) of KR theory using only the two parameters of the translational group $W(p,q)$ while the full gauge freedom of the theory is represented by the four arbitrary components of the vector f^a. Here again, the components f^1 and f^2 of f^μ can be expressed in terms of the parameters p, q of the translational group $W(p,q)$ and they remain independent of each other as can be seen from (3.49). However, unlike in the case of linearized gravity, the other two components (f^0, f^3) are independent of the parameters (and of the components of maximally reduced polarization tensor) and are left completely undetermined subject only to the constraint $f^0 = f^3$. Thus, in the gauge generation by $W(p,q)$ in KR theory, corresponding to any given pair (f^1, f^2) there exists a continuum of allowed choices for $f^0 (= f^3)$ representative of the invariance of gauge transformations (3.38) under (3.39). Therefore, the partial gauge generation by $W(p,q)$ in massless KR theory clearly exhibits the reducibility of its gauge transformations. The reducibility of the gauge transformation (3.35) is manifested in the special choice (3.49) which makes the transformation (3.48) of the maximally reduced polarization tensor ε^{ab} effected by $W(p,q)$, a gauge transformation of the KR theory. This may be compared to the gauge generation (3.29) in linearized gravity by $W(p,q)$ where all the components of the arbitrary vector variable ζ^μ are expressed in terms of the parameters (p,q) (see (3.30)) hence indicating the absence of any reducibility in the gauge transformation of the theory.

Notice that the transformation (3.36) is of same form as the gauge transformation (3.4) of Maxwell theory where the generator of gauge transformations is $W(p,q)$. Hence, one may consider that the 'gauge transformation (3.36) of gauge transformations' in KR theory as being generated by a translational subgroup $W(p,q)$ of little group for massless particles. Therefore, in KR theory which is a 2-form gauge theory, two independent elements of the translational group $W(p,q)$ are involved in generating gauge transformations, one for the underlying 2-form field $B_{\mu\nu}$ and the other for the field F_μ. In the gauge generation for massless theories by the translational group $W(p,q)$, we therefore perceive an appealing hierarchical structure, namely in a n-form theory, n elements of the translational group $W(p,q)$ being involved in gauge generation.

3.5 $B \wedge F$ theory

$B \wedge F$ theory [28, 29, 30, 31] is obtained by topologically coupling the B_{ab} field of Kalb-Ramond theory (3.32) with the Maxwell field A_a and is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} + \frac{1}{12}H_{abc}H^{abc} - \frac{m}{6}\epsilon^{abcd}H_{abc}A_d. \tag{3.51}$$

The equations for motion for $A_a(x)$ and $B_{ab}(x)$ fields are given by

$$\partial_a F^{ad} - \frac{m}{6}\epsilon^{abcd}H_{abc} = 0 \tag{3.52}$$

and

$$\partial_a H^{abc} = \frac{1}{2}m\epsilon^{debc}F_{de}. \tag{3.53}$$

The gauge transformations of the fields $A_a(x)$ and $B_{ab}(x)$ are respectively of the same form as (3.4) and (3.35). Just like the massless KR theory discussed in section 3.4, the gauge transformation of the B_{ab} is reducible for $B \wedge F$ theory also. Substituting for A_a and B_{ab} respectively from (3.6) and (3.37), we obtain the momentum space version of the above equations of motion in terms of the polarization vector ε_a and

ε_{ab} as follows;

$$k^2\varepsilon^d - k^d k_a \varepsilon^a + \frac{i}{2} m \epsilon^{abcd} k_a \varepsilon_{bc} = 0 \tag{3.54}$$

$$k^2\varepsilon^{ab} + \varepsilon^{ca} k_c k^b - \varepsilon^{cb} k_c k^a + imk_d \varepsilon_e \epsilon^{deab} = 0. \tag{3.55}$$

The momentum space gauge transformations of the fields are obviously of the form (3.7) and (3.38). Considering the massless case $k^2 = 0$, we see using (3.54) that

$$\epsilon_{dabc} k^d k_e \varepsilon^e = im(k_a \varepsilon_{bc} - k_b \varepsilon_{ac} + k_c \varepsilon_{ab}) \tag{3.56}$$

Contracting with k^b on either side of (3.56) yields,

$$k_a k^b \varepsilon_{bc} + k_c k^b \varepsilon_{ab} = 0. \tag{3.57}$$

Using (3.57) and the masslessness condition ($k^2 = 0$), one can immediately see using (3.55) that

$$k_d \varepsilon_a \epsilon^{dabc} = 0 \tag{3.58}$$

so that any general solution of ε_a can now be written as,

$$\varepsilon_a = f(k) k_a \tag{3.59}$$

for some function $f(k)$. Therefore, using (3.7), one can easily see that massless excitations, if any, are gauge artefacts. This is in contrast with the Maxwell and KR models considered earlier, where the massive excitations are gauge artefacts. Next let us consider the massive case ($k^2 = M^2$). Going to the rest frame with $k^a = (M, 0, 0, 0)^T$, one can relate the spatial components of ε^a and ε^{ab} by making use of (3.54) and (3.55) to get the following coupled equations

$$\varepsilon^i = -\frac{im}{2M} \epsilon^{0ijk} \varepsilon_{jk} \tag{3.60}$$

$$\varepsilon^{ij} = -\frac{im}{M} \epsilon^{0ijk} \varepsilon_k \tag{3.61}$$

whereas ε^0 and ε^{0i} remain arbitrary which can be trivially gauged away by making use of the gauge transformations (3.7) and (3.38) and the form $k^a = (M, 0, 0, 0)^T$ for the four-momentum in the rest frame. On the other hand, the mutual compatibility of the pair of equations (3.60) and (3.61) implies that we must have

$$M^2 = m^2. \tag{3.62}$$

This indicates that the strength 'm' (taking m to be positive) of B$\wedge$F term in (3.51) can be identified as the mass of the quanta in B$\wedge$F model. With this, (3.60) and (3.61) simplify further and one can write ε^{ab} and ε^a in terms of the three independent parameters,

$$\{\varepsilon^{ab}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c & -b \\ 0 & -c & 0 & a \\ 0 & b & -a & 0 \end{pmatrix}, \qquad \varepsilon^a = -i \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}. \tag{3.63}$$

displaying a dual structure. Another way of understanding the degree of freedom count is to recall that the B$\wedge$F Lagrangian (3.51) can be regarded either as a massive Maxwell (i.e., Proca) theory or a massive KR theory [46, 47, 48]. This can be achieved by eliminating once the KR field or, alternatively, the vector field from the coupled set of equations (3.52, 3.53). Both these theories have three massive degrees of freedom. It is interesting to note that the polarization tensor/vector in (3.63), satisfy an orthogonality relation,

$$\varepsilon^{ab}\varepsilon_b = 0. \tag{3.64}$$

By a straightforward calculation involving the explicit forms of the polarization tensor ε^{ab} and vector ε^a (3.63), one can see that $W(p, q)$ fails to be a generator in $B \wedge F$ theory. Therefore, it appears that the translational $T(2)$ in the representation

$W(p,q)$, in contrast to the Maxwell and KR examples, is not a generator of gauge transformation in the B∧F theory. So, what would be generator of gauge transformations in the topologically massive B∧F theory? In order to answer this question consider the matrix,

$$D(p,q,r) = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.65}$$

involving three real parameters p, q, r. This generates gauge transformations (3.7) and (3.38) acting on the polarization tensor and polarization vector (3.63):

$$\varepsilon^a \to \varepsilon'^a = D^a{}_b(p,q,r)\varepsilon^b = \varepsilon^a - \frac{i}{m}(pa + qb + rc)k^\mu, \tag{3.66}$$

$$\{\varepsilon^{ab}\} \to \{\varepsilon'^{ab}\} = D(p,q,r)\{\varepsilon^{ab}\}D^T(p,q,r)$$

$$= \{\varepsilon^{ab}\} + \begin{pmatrix} 0 & (rb-qc) & (pc-ra) & (qa-pb) \\ -(rb-qc) & 0 & 0 & 0 \\ -(pc-ra) & 0 & 0 & 0 \\ -(qa-pb) & 0 & 0 & 0 \end{pmatrix} \tag{3.67}$$

as both (3.66) and (3.67) can be easily cast into the form (3.7) and (3.38) with the proper choices of $f(k)$ and $f^a(k)$ given by

$$f(k) = \frac{pa + qb + rc}{im} \tag{3.68}$$

$$f^1(k) = \frac{rb - qc}{im}, \quad f^2(k) = \frac{pc - ra}{im}, \quad f^3(k) = \frac{qa - pb}{im}. \tag{3.69}$$

while the component $f^0(k)$ of $f^a(k)$ remain completely undetermined manifesting the reducible nature of the gauge transformation of the B^{ab} field as explained in the previous section in the context of KR theory. However, unlike KR theory, the gauge transformations generated by $D(p,q,r)$ (3.65)exhaust the entire set of gauge transformations in $B \wedge F$ theory. This is because the components of $f^a(k)$ as given in (3.69) are independent of one another: the three spatial components f^1, f^2, f^3 are expressed in terms of the three parameters (p, q, r) while f^0 is left completely undetermined. This is unlike massless KR theory where the gauge transformations generated by $D(p, q, r)$ is restricted by the condition $f^0 = f^3$.

We can now identify the group in which $D(p, q, r)$ belongs. One can easily show that

$$D(p,q,r) \cdot D(p',q',r') = D(p+p', q+q', r+r') \tag{3.70}$$

and

$$[T_1, T_2] = [T_1, T_3] = [T_2, T_3] = 0 \tag{3.71}$$

where

$$T_1 = \frac{\partial D(p,0,0)}{\partial p}; \quad T_2 = \frac{\partial D(0,q,0)}{\partial q}; \quad T_3 = \frac{\partial D(0,0,r)}{\partial r} \tag{3.72}$$

can be thought of as three mutually commuting "translational" generators. The group can therefore be identified with $T(3)$ - the invariant subgroup of E(3) or ISO(3) [70]. Although this gauge generating representation of $D(p, q, r)$ has been obtained here in a somewhat empirical manner, it can be derived systematically from Wigner's little group in a space-time of one higher dimension, i.e. in 4+1 dimensions through dimensional descent. We shall elaborate on this in a subsequent chapter.

3.6 Summary

In this chapter we have reviewed the gauge generating nature of the translational subgroup $T(2)$ of Wigner's little group for massless particle. We have seen that the representation of $T(2)$ inherited from the defining representation of Wigner's little group generates gauge transformations in several massless Abelian gauge theories in 3+1 dimensions. Our illustrative examples consisted of free Maxwell theory, Kalb-Ramond theory and linearized gravity. In the case of Maxwell theory $T(2)$ is found to generate the entire spectrum of gauge transformations while in KR and linearized gravity, it generates only a subset of the whole range of available gauge transformations. In Kalb-Ramond theory theory, the reducibility of the gauge transformation is clearly manifested in the gauge generation by $T(2)$. When it comes to the topologically massive $B \wedge F$ theory, one has to go beyond the Wigner's little group and it is the translational group $T(3)$ that generates the full set of gauge transformation in this theory. The generation by $T(3)$ also explicitly manifest the reducibility of the gauge transformation in the antisymmetric tensor field in $B \wedge F$ theory.

Chapter 4

Translational groups as gauge generators in planar theories

We have seen in the previous chapter that the translational subgroup of Wigner's little group generate gauge transformations in 3+1 dimensional massless gauge theories. However, the 2+1 dimensional little group for massless particles has only one parameter and is isomorphic to the translational group $T(1)$ in 1-dimension. Nevertheless, this little group will generate gauge transformation in Maxwell theory in 2+1 dimensions[1]. As we know, there exist topologically massive gauge theories in 2+1 dimensions, namely the MCS and ECS theories. The question now is if the translational group $T(1)$ generate gauge transformations in these theories as well. The present chapter addresses this question. We first discuss the 2+1 dimensional little group briefly and then go on to study the relationship between the little group and gauge transformations in the topologically massive theories.

[1]The linearized gravity and massless KR theories do not have any propagating degree of freedom in 2+1 dimensions.

4.1 Wigner's little group in 2+1 dimensions

Following identical techniques as in 3+1 dimensions, one can derive the 2+1 dimensional Wigner's little group that preserves the momentum 3-vector $k^\mu = (\omega, 0, \omega)^T$ for a massless particle [33] as

$$\{W^\mu{}_\nu\}(p) = \begin{pmatrix} 1+\frac{p^2}{2} & ap & -\frac{p^2}{2} \\ p & a & -p \\ \frac{p^2}{2} & ap & 1-\frac{p^2}{2} \end{pmatrix} \tag{4.1}$$

where $a = \pm 1$. For $a = +1$, one can easily show that,

$$W(p) \cdot W(p') = W(p+p') \tag{4.2}$$

and therefore the little group represented by (4.1) is isomorphic to $\mathcal{R}$, the additive group of real numbers. It is well known that the Wigner's little group for massless particles (in 2+1 dimensions) is isomorphic to $\mathcal{R} \times \mathcal{Z}_2$ [53]. The $\mathcal{Z}_2$ factor is required to take into account of the fact that the value of a is restricted to ± 1.

The generator G in the representation $W(p)$ in (4.1) is clearly given (with $a = +1$) by,

$$G = \frac{\partial W}{\partial p} |_{p=0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.3}$$

satisfying,

$$G^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}; \quad G^3 = 0 \tag{4.4}$$

so that $W(p)$ can be re-expressed as,

$$W(p) = e^{pG} = 1 + pG + \frac{1}{2}p^2G^2 \tag{4.5}$$

One can thus construct various representations of Wigner's little group for massless particles in 2+1 dimensions just as in the usual 3+1 dimensions. The important point is that although the various representations are, by definition, isomorphic to each other, not all of them belong to the Lorentz group. In the following section we construct another representation of the little group in 2+1 dimensions, not inherited from $SO(1,2)$, which will be shown to act as a gauge generator in topologically massive MCS theory.

The little group for the massive particle, which in this case can be trivially seen to be $O(2)$, however, does not have any role as a generator of gauge transformation. This does not mean however that they are completely unrelated. In fact, one can write $W(p)$ for $|p| < 1$, as a product of three matrices;

$$W(p) = B_y^{-1}(p)R(p)B_x(p) \tag{4.6}$$

where

$$B_y^{-1}(p) = \begin{pmatrix} \frac{2-p^2}{2\gamma} & 0 & -\frac{p^2}{2\gamma} \\ 0 & 1 & 0 \\ -\frac{p^2}{2\gamma} & 0 & \frac{2-p^2}{2\gamma} \end{pmatrix} ; R(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & -p \\ 0 & p & \gamma \end{pmatrix} ; B_x(p) = \begin{pmatrix} \frac{1}{\gamma} & \frac{p}{\gamma} & 0 \\ \frac{p}{\gamma} & \frac{1}{\gamma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.7}$$

with $\gamma = \sqrt{1-p^2}$. These matrices are themselves the elements of the Lorentz group $SO(1,2)$; B_x represents a boost along the x-direction, R represents a spatial rotation in the $x-y$ plane and B_y^{-1} represents a boost along the negative y-direction. Appropriate transformations in this order can preserve the energy-momentum 3-vector of a massless particle moving in the y-direction. Here R clearly corresponds to the little group of a massive particle. Thus (4.6) relates the elements of the connected parts of identity element of the little group of massless particles with massive ones as long as $|p| < 1$. But this does not provide the natural homomorphism existing between $\mathcal{R}$ (the additive group of real numbers) with $SO(2)$.

4.2 Little group as gauge generator in Maxwell-Chern-Simons theory

It is easy to see that the little group $W(p)$ in (4.1) with $a = +1$, generate gauge transformations in 2+1 dimensional Maxwell theory. For that consider a photon of energy ω moving in the y-direction and polarized in the x-direction so that the potential 3-vector takes the form

$$A^{\mu}(x) = \xi^{\mu}_{x} \exp(-ik.x) = \xi^{\mu}_{x} \exp(-i\omega(t-y)), \tag{4.8}$$

where

$$\xi^{\mu}_{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{4.9}$$

is the polarization vector and the subscript denotes that this vector is in the x-direction. Under the action of $W(p)$ (4.1), ξ^{μ}_{x} undergoes the transformation

$$\xi^{\mu}_{x} \to \xi'^{\mu}_{x} \equiv W^{\mu}{}_{\nu}\xi^{\nu}_{x} = \xi^{\mu}_{x} + \frac{p}{\omega}k^{\mu}. \tag{4.10}$$

This can be identified as the gauge transformation as the corresponding gauge field undergoes the transformation

$$A^{\mu}(x) \to A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu}\left(\frac{ip}{\omega}e^{-i\omega(t-y)}\right). \tag{4.11}$$

In contrast to Maxwell theory, MCS excitations are massive as we have seen in section 2.1.1 and the rest frame polarization vector $\xi^{\mu}(\mathbf{0})$ takes the simple form

(2.18) in the rest frame. That is,

$$\xi^\mu(\mathbf{0}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i\frac{\vartheta}{|\vartheta|} \end{pmatrix}. \tag{4.12}$$

where ϑ is the CS parameter. Note that it has complex entries having both x and y components unlike the Maxwell photon polarization ξ^μ_x. In fact in their Coulomb gauge analysis Devecchi et. al.[50] have pointed out that the spin $(\frac{\vartheta}{|\vartheta|})$ of the MCS quanta stems from this particular complex structure of the polarization vector.

We shall now investigate whether this same little group can generate similar gauge transformation on the MCS polarization vector (4.12). To that end, let us apply $W(p)$ on $\xi^\mu(\mathbf{0})$ (4.12). Without loss of generality, henceforth we shall consider $\vartheta < 0$ case only. We find that it undergoes the following transformation,

$$\xi^\mu(\mathbf{0}) \rightarrow \xi'^\mu(\mathbf{0}) \equiv W^\mu{}_\nu(p)\xi^\nu(\mathbf{0}) = \frac{1}{\sqrt{2}} \begin{pmatrix} p - \frac{i}{2}p^2 \\ 1 - ip \\ p + i(1 - \frac{p^2}{2}) \end{pmatrix} \tag{4.13}$$

Clearly this cannot be cast in the form of (4.10). One cannot therefore interpret this transformation as a gauge transformation. However, taking advantage of the fact that this little group involves a single parameter only, we can easily construct a (non-unique) representation which does the required job. This is given by,

$$D(p) = \begin{pmatrix} 1 & p & -ip \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad p \in \mathcal{R} \tag{4.14}$$

so that in place of (4.13) one has the desirable form in the sense that it can now be put in the form of (4.10);

$$\xi^\mu(\mathbf{0}) \rightarrow \xi'^\mu(\mathbf{0}) \equiv D^\mu{}_\nu\xi^\nu(\mathbf{0}) = \xi^\mu(\mathbf{0}) + \frac{\sqrt{2}p}{|\vartheta|}k^\mu \tag{4.15}$$

where $k^\mu = (|\vartheta|, 0, 0)^T$ is the energy-momentum 3-vector of a MCS particle in the rest frame. This shows that $D(p)$ acts a generator of gauge transformation in the MCS theory.

By denoting the rest frame polarization vectors of a doublet $\mathcal{L}_\pm$ ((2.56)and (2.57)) of MCS theories with helicities ± 1 by $\xi^\mu_\pm$ and the corresponding elements of 2+1 dimensional little group that generate their gauge transformation by $D_\pm(p_\pm)$ respectively, we may write

$$\xi^\mu_\pm \to D_\pm(p_\pm)\xi^\mu_\pm = \xi^\mu_\pm + \frac{\sqrt{2}p_\pm}{|\vartheta|}k^\mu \tag{4.16}$$

where

$$D_\pm(p_\pm) = \begin{pmatrix} 1 & p_\pm & \pm i p_\pm \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \xi^\mu_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \mp i \end{pmatrix}. \tag{4.17}$$

with $p_\pm$ representing the parameters of the little group elements $D_\pm$. In the next chapter we show how we can derive these representations of Wigner's little group in 2+1 dimensions from the gauge generating representation $W(p,q)$ of $T(2)$ for Maxwell theory in 3+1 dimensions by a method called dimensional descent.

Now, certain comments on some subtle points regarding representation $D(p)$ (4.14) are in order. Although $D(p)$ is not an element of the Lorentz group, it is perfectly admissible to regard it as a representation[2] of the little group for a massless particle in 2+1 dimensions. This is because it satisfies $D(p) \cdot D(p') = D(p+p')$, so that there exists a natural isomorphism between $W(p)$ (4.1) and $D(p)$ (4.14). This is analogous to 3+1 dimensional case of Wigner's little group (3.1) for massless particles. The Lie algebra of this little group (which acts as gauge

[2]Note that this representation is different from the defining representation, the latter can only be obtained as a subgroup of the Lorentz group $SO(1,2)$.

generators [22, 23, 24, 27, 33, 26]) is isomorphic to the algebra of the Euclidean group $E(2)$ as explained in appendix A. However, notice that the algebra of the little group, being a combination of boost and rotation generators (A.24), is a subalgebra of the homogeneous Lorentz algebra whereas the defining representation of $E(2)$ algebra (comprising of two translational generators and a rotational generator in a plane), is a subalgebra of Poincare algebra but not of the homogeneous Lorentz algebra.

Coming back to the issue of similarities and dissimilarities between the polarization vectors of pure Maxwell theory and that of MCS theory, note that a photon state is entirely characterized by (4.9) where both the "spatial" transversality condition, $\mathbf{k}.\vec{\xi}_{\mathbf{x}} = 0$ and the temporal gauge condition $\xi^0_x = 0$ are trivially satisfied. Therefore the gauge field configuration (4.8) corresponds to the radiation gauge. Clearly the same gauge condition will no longer be valid under a Lorentz boost. However, we shall show now that the radiation gauge condition can still be satisfied, provided the gauge field undergoes an appropriate gauge transformation preceding the Lorentz boost. Considering the Maxwell case first, the gauge transformed field configuration $A'^{\mu}(x)$ corresponding to a photon polarized along the x-direction and propagating along the y-direction can be written as,

$$A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu}\tilde{p}(x) = A^{\mu}(x) + pk^{\mu}e^{-ik\cdot x} \tag{4.18}$$

where the scalar function $\tilde{p}(x)$is taken to be of the form $ip(k)$. A Lorentz boost of velocity $v = \tanh\phi$ for example, in the x-direction yields,

$$\tilde{A}'^{\mu} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \\ p \end{pmatrix} e^{-ik'\cdot x'} = \begin{pmatrix} p\cosh\phi + \sinh\phi \\ p\sinh\phi + \cosh\phi \\ p \end{pmatrix} e^{-ik'\cdot x'} \tag{4.19}$$

where k'^{μ} is the appropriate energy-momentum 3-vector in the new boosted co-ordinate frame x'^{μ} and is given by,

$$k'^{\mu} = \omega \begin{pmatrix} \cosh\phi \\ \sinh\phi \\ 1 \end{pmatrix} \tag{4.20}$$

Preservation of spatial transversality condition implies that we must have,

$$\tilde{\mathbf{A}}'(x') \cdot \mathbf{k}' = \omega \begin{pmatrix} \cosh\phi + p\sinh\phi \\ p \end{pmatrix}^T \begin{pmatrix} \sinh\phi \\ 1 \end{pmatrix} = 0 \tag{4.21}$$

Solving for p, one gets,

$$p = -\tanh\phi = -v \tag{4.22}$$

This solution, when substituted back in (4.19) yields,

$$\tilde{A}'^{0}(x') = 0 \tag{4.23}$$

which is nothing but the temporal gauge condition. Thus with an appropriate gauge transformation preceding a Lorentz boost, the radiation gauge condition can be satisfied. But, as we shall see now, the same is not true for MCS Theory. Upon a gauge transformation, the polarization vector (4.12), in the rest frame, becomes

$$\xi^{\mu}(x) \to \tilde{\xi}^{\mu}(x) = D(p)\xi^{\mu}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & p & -ip \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2p \\ 1 \\ i \end{pmatrix} \tag{4.24}$$

Then a Lorentz boost like (4.19) along x-axis, for example transforms this to,

$$\tilde{\xi}^{\mu} \to \tilde{\xi}'^{\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sinh\phi + 2p\cosh\phi \\ \cosh\phi + 2p\sinh\phi \\ i \end{pmatrix} \tag{4.25}$$

Simultaneously, the $k^\mu = (1,0,0)^T$, associated to the rest frame, transforms to,

$$k^\mu \to k'^\mu = \begin{pmatrix} \cosh\phi \\ \sinh\phi \\ 0 \end{pmatrix} \tag{4.26}$$

Demanding that the spatial transverality condition is satisfied in an arbitrary boosted frame (i.e., $\phi \neq 0$) and using (4.25) and (4.26) we get,

$$\begin{pmatrix} \sinh\phi \\ 0 \end{pmatrix}^T \begin{pmatrix} \cosh\phi + 2p\sinh\phi \\ i \end{pmatrix} = 0, \tag{4.27}$$

which when solved for the gauge transformation parameter p in terms of the boost parameter ϕ, yields,

$$p = -\frac{1}{2\tanh\phi} = -\frac{1}{2v}. \tag{4.28}$$

So just like in the Maxwell case the spatial transversality condition can be maintained in any boosted frame, provided the boost is preceded by a suitable gauge transformation. However, in contrast to Maxwell case (4.23), the temporal gauge condition ($A^0 = 0$) is not satisfied simultaneously since for p satisfying (4.28)

$$\tilde{\xi}'^0 = -\frac{1}{\sqrt{2}\sinh\phi}. \tag{4.29}$$

Nevertheless, $\tilde{\xi}'^0$ (4.29) can be made to vanish in the infinite momentum frame (in the limit $\phi \to \infty$), i.e., when $p \to -\frac{1}{2}$ and $\tanh\phi \to 1$.

For a boost along the x-direction, one can write

$$\tanh\phi = \frac{k^1}{k^0}. \tag{4.30}$$

Using (4.28), (4.30) and the mass-shell condition $k^2 = \vartheta^2$, one can simplify (4.25) to get the polarization vector as,

$$\tilde{\xi}'^{\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\vartheta}{k^1} \\ 0 \\ i \end{pmatrix} \tag{4.31}$$

At this stage, we can see clearly that although the spatial transversality ($\mathbf{k}.\vec{\xi} = 0$) holds trivially in the rest frame, the temporal gauge condition is not well defined (since $\tilde{\xi}'^0 \to \infty$ as $k^1 \to 0$). This is expected from the simple consideration that a Lorentz transformation (to the rest frame) alone should not also lead to a complete gauge fixing. Note that, unlike MCS theory, a rest frame is not available in Maxwell theory.

4.3 Linearized Einstein-Chern-Simons theory

As explained in section 3.3, pure gravity in 2+1 dimensions is a null theory in the sense that it does not have a propagating degree of freedom. However, 2+1 dimensional gravity coupled to a non-Abelian Chern-Simons topological term, with gauge group being the Lorentz group itself, possesses a single propagating massive degree of freedom [9]. Just like the MCS theory, the gauge invariance coexists with mass in the linearized version of this theory too where the gauge group reduces to Abelian group $T(1)$. In this section we study the role of translational group in generating gauge transformations in the linearized version of gravity coupled to Chern-Simons term 2+1 dimensions [26].

The full action of the topologically massive gravity in 2+1 dimensions is

$$I^{ECS} = I^E + I^{CS} \tag{4.32}$$

where the 2+1 dimensional Einstein action here is

$$I^E = \int d^3x\sqrt{g}R \tag{4.33}$$

and the Chern-Simons action I^{CS} is given by

$$I^{CS} = -\frac{1}{4\mu}\int d^3x\epsilon^{\mu\nu\lambda}[R_{\mu\nu\alpha\beta}\omega_\lambda{}^{\alpha\beta} + \frac{2}{3}\omega_{\mu\beta}{}^\gamma\omega_{\nu\gamma}{}^\alpha\omega_{\lambda\alpha}{}^\beta]. \tag{4.34}$$

The $\omega_{\mu\alpha\beta}$ are the components of the spin connection one-form and are related to the curvature two-form by the second Cartan's equation of structure ($R^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$). Note that the sign in front of the Einstein action is now opposite to the conventional one (3.12) and is required to make the full theory free of ghosts [9]. The linearization, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, of the ECS theory (4.32) results in the Abelian theory [9, 61] given by

$$I_L^{ECS} = \int d^3x\mathcal{L}_L^{ECS} \tag{4.35}$$

where

$$\mathcal{L}_L^{ECS} = \mathcal{L}_L^E + \mathcal{L}_L^{CS}. \tag{4.36}$$

Here

$$\mathcal{L}_L^E = -\frac{1}{2}h_{\mu\nu}\left[R_L^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R_L\right] \tag{4.37}$$

now is the Lagrangian for linearized version of pure gravity in 2+1 dimensions and is the same as (3.14) except that it has the opposite sign and the indices, in this case, vary over $0, 1, 2$. Similarly,

$$\mathcal{L}_L^{CS} = -\frac{1}{2\mu}\epsilon_{\alpha\beta\gamma}\left[R_L^{\beta\delta} - \frac{1}{2}\eta^{\beta\delta}R_L\right]\partial^\alpha h_\delta^\gamma \tag{4.38}$$

is the linearized Chern-Simons term with the Chern-Simons parameter μ. Under the gauge transformation $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\tilde{\zeta}_\nu(x) + \partial_\nu\tilde{\zeta}_\mu(x)$, the Chern-Simons part $\mathcal{L}_L^{CS}$ changes by a total derivative:

$$\delta\mathcal{L}_L^{CS} = \frac{1}{\mu}\epsilon_{\mu\nu\lambda}\partial_\delta\left(R_L^{\nu\delta}\partial^\mu\tilde{\zeta}^\lambda\right) \tag{4.39}$$

The equation of motion corresponding to $\mathcal{L}_L^{ECS}$ is given by [9, 66]

$$\Box h^{\mu\nu} - \partial^\mu \partial_\gamma h^{\gamma\nu} - \partial^\nu \partial_\gamma h^{\gamma\mu} + \partial^\mu \partial^\nu h - \eta^{\mu\nu}(\Box h - \partial_\gamma \partial_\delta h^{\gamma\delta})$$

$$-\frac{1}{2\mu}\epsilon^{\mu\gamma\delta}\partial_\gamma(\Box h^\nu_\delta - \partial_\tau \partial^\mu h^\tau_\delta) - \frac{1}{2\mu}\epsilon^{\nu\gamma\delta}\partial_\gamma(\Box h^\mu_\delta - \partial_\tau \partial^\mu h^\tau_\delta) = 0. \tag{4.40}$$

With the ansatz $h^{\mu\nu} = \chi^{\mu\nu}(k)e^{ik.x}$, the above equation of motion can be written in terms of the symmetric polarization tensor $\chi_{\mu\nu}(k)$ and the 3-momentum k^μ as follows

$$-k^2\chi^{\mu\nu} + k^\mu k_\gamma \chi^{\gamma\nu} + k^\nu k_\gamma \chi^{\gamma\mu} - k^\mu k^\nu \chi - \eta^{\mu\nu}(-k^2\chi + k_\gamma k_\delta \chi^{\gamma\delta})$$

$$-\frac{i}{2\mu}\left[\epsilon^{\mu\gamma\delta}k_\gamma(-k^2\chi^\nu_\delta + k_\tau k^\nu \chi^\tau_\delta) \; + \; \epsilon^{\nu\gamma\delta}k_\gamma(-k^2\chi^\mu_\delta + k_\tau k^\mu \chi^\tau_\delta)\right] = 0. \tag{4.41}$$

Analogous to (3.20), the expression for the gauge transformation for ECS theory in terms of its polarization tensors $\chi^{\mu\nu}(k)$ is given by

$$\chi^{\mu\nu}(k) \to \chi'^{\mu\nu}(k) = \chi^{\mu\nu}(k) + k^\mu \zeta^\nu(k) + k^\nu \zeta^\mu(k) \tag{4.42}$$

where $\zeta_\mu(k)$ are small arbitrary functions of k. Depending on whether the excitations are massless or massive, we have two options for k^2:

$$\text{(i) } k^2 = 0 \text{ or (ii) } k^2 \neq 0.$$

case (i): $k^2 = 0$

Contracting the (4.41) with $\eta_{\mu\nu}$ gives

$$k_\mu k_\nu \chi^{\mu\nu} = 0. \tag{4.43}$$

A general solution to this equation consistent with the equation of motion (4.41) is

$$\chi^{\mu\nu} = k^\mu f^\nu(k) + k^\nu f^\mu(k) \tag{4.44}$$

where $f^\mu(k)$ are arbitrary functions of k. However, with $f^\mu = -\zeta^\mu$ we can 'gauge away' these solutions. Therefore, massless excitations of ECS theory are pure gauge artefacts. We now proceed to the other option:

case (ii) $k^2 \neq 0$

Let $k^2 = m^2$. On contraction with $\eta_{\mu\nu}$ (4.41) gives

$$k_\mu k_\nu \chi^{\mu\nu} = m^2 \chi \tag{4.45}$$

where $\chi = \chi^\mu{}_\mu$. With $k^\alpha = (m, 0, 0)^T$, this yields

$$\chi_{11} + \chi_{22} = 0. \tag{4.46}$$

By considering the spatial part of (4.41) one can show that the mass m of the excitations can be identified with the Chern-Simons parameter μ as follows. The spatial part of (4.41) is

$$-m^2\chi^{ij} + k^i k_\mu \chi^{\mu j} + k^j k_\mu \chi^{\mu i} - k^i k^j \chi - \eta^{ij}(-k^2\chi + k_\mu k_\nu \chi^{\mu\nu})$$

$$-\frac{i}{2\mu}\left[\epsilon^{i\gamma\delta}k_\gamma(-k^2\chi^j_\delta + k_\tau k^j \chi^\tau_\delta) \ + \ \epsilon^{j\gamma\delta}k_\gamma(-k^2\chi^i_\delta + k_\tau k^i \chi^\tau_\delta)\right] = 0. \tag{4.47}$$

In this equation i, j takes values 1 and 2. On passing to the rest frame the above equation simplifies to

$$-\chi^{ij} + \eta^{ij}\chi^k{}_k - \frac{im}{2\mu}\left[\epsilon^{ik}\chi^j_k + \epsilon^{jk}\chi^i_k\right] = 0 \tag{4.48}$$

from which we obtain (for $i = j = 1$ and $i = j = 2$ respectively)

$$\chi^{11} = +\frac{im}{\mu}\chi^{12}\,; \qquad \chi^{22} = -\frac{im}{\mu}\chi^{12}. \tag{4.49}$$

With $i = 1$ and $j = 2$ we have

$$\chi^{12} = +\frac{im}{\mu}\chi^{22}. \tag{4.50}$$

This relation together with (4.49) implies that $m^2 = \mu^2$. The remaining components can be made to vanish by a suitable gauge choice. Finally, for the Chern-Simons parameter $\mu > 0$, the polarization tensor χ_+ of the gravity coupled to Chern-Simons theory in 2+1 dimensions in the rest frame can be written as

$$\chi_+ \equiv \{\chi_+^{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & -1 \end{pmatrix} \tau \tag{4.51}$$

where τ is an arbitrary real parameter. Notice that the ECS theory has only a single degree of freedom corresponding to the parameter τ. Similarly, the rest frame polarization tensor for an ECS theory having the Chern-Simons parameter $\mu < 0$ is

$$\chi_- \equiv \{\chi_-^{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix} \tau. \tag{4.52}$$

It is important to note that these rest frame polarization tensors $\chi_\pm$ of ECS theories (with $\tau = \frac{1}{2}$) can be obtained as direct products of the rest frame polarization vectors $\xi_\pm^\mu$ (4.17) of MCS theories. i.e.,

$$\chi_\pm^{\mu\nu} = \xi_\pm^\mu \xi_\pm^\nu. \tag{4.53}$$

This suggests that we adopt orthonormality conditions for $\chi_\pm$ which are similar to the ones (2.17) used for ξ^μ. Hence we require

$$tr\left((\chi_+)^\dagger(\chi_-)\right) = 0\ ; \qquad tr\left((\chi_\pm)^\dagger(\chi_\pm)\right) = 1. \tag{4.54}$$

Therefore, we have the following maximally reduced form for the polarization tensors of a pair of ECS theories with opposite helicities

$$\chi_{\pm} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \mp i \\ 0 & \mp i & -1 \end{pmatrix}. \tag{4.55}$$

Note that these polarization matrices of ECS theories are traceless and singular. We are now equipped to study the role played by the translational group in generating the gauge transformation in this theory. The representations of $T(1)$ that generates gauge transformation in pair of MCS theories with opposite helicities is given by $D_{\pm}(p)$ (4.17). On account of the relation (4.53) it is expected that the same representations will generate gauge transformations in ECS theories also. Indeed one can easily see that $D_{\pm}(p_{\pm})$ are the gauge generators in ECS theories:

$$\chi_{\pm} \to \chi'_{\pm} = D_{\pm}(p_{\pm})\chi_{\pm}D^T_{\pm}(p_{\pm}) = \chi_{\pm} + \begin{pmatrix} 2p^2_{\pm} & p_{\pm} & \mp ip_{\pm} \\ p_{\pm} & 0 & 0 \\ \mp ip_{\pm} & 0 & 0 \end{pmatrix}. \tag{4.56}$$

This transformation can be cast in the form of the gauge transformation (4.42) with the following choice of ζ's;

$$\zeta_0 = \frac{p^2_{\pm}}{|\mu|}, \quad \zeta_1 = \frac{p_{\pm}}{|\mu|}, \quad \zeta_2 = \frac{\mp ip_{\pm}}{|\mu|}. \tag{4.57}$$

One can obtain the moving frame expression for polarization tensors $\chi_{\pm}(k)$ from the above rest frame results by applying appropriate Lorentz boost as follows:

$$\chi_{\pm}(k) = \Lambda^T(k)\chi_{\pm}(0)\Lambda(k)$$

$$= \frac{1}{2\mu^2}\begin{pmatrix} k_0^2 - \mu^2 & k^0k^1 \mp i|\mu|k^2 & k^0k^2 \pm i|\mu|k^1 \\ k^0k^1 \mp i|\mu|k^2 & \frac{(k^0k^1 \mp i|\mu|k^2)^2}{k_0^2-\mu^2} & \frac{(k^0k^1 \mp i|\mu|k^2)(k^0k^2 \pm i|\mu|k^1)}{k_0^2-\mu^2} \\ k^0k^2 \pm i|\mu|k^1 & \frac{(k^0k^1 \mp i|\mu|k^2)(k^0k^2 \pm i|\mu|k^1)}{k_0^2-\mu^2} & \frac{(k^0k^2 \pm i|\mu|k^1)^2}{k_0^2-\mu^2} \end{pmatrix} e^{\pm 2i\phi(k)} \tag{4.58}$$

where $\phi(k) = \arctan(\frac{k^2}{k^1})$ and the momentum space boost matrix [64]

$$\Lambda(k) = \begin{pmatrix} \gamma & \gamma\beta^1 & \gamma\beta^2 \\ \gamma\beta^1 & 1+\frac{(\gamma-1)(\beta^1)^2}{(\vec{\beta})^2} & \frac{(\gamma-1)\beta^1\beta^2}{(\vec{\beta})^2} \\ \gamma\beta^2 & \frac{(\gamma-1)\beta^1\beta^2}{(\vec{\beta})^2} & 1+\frac{(\gamma-1)(\beta^2)^2}{(\vec{\beta})^2} \end{pmatrix} \tag{4.59}$$

with $\vec{\beta} = \frac{\mathbf{k}}{k^0}$ and $\gamma = \frac{k^0}{|\mu|}$ as given in (2.27). Results identical to (4.53) and (4.58) were obtained in other contexts [66] by different methods. Here, we have shown how these results can be obtained in a simpler and straightforward manner just by considering the momentum space expression of the equation of motion in the rest frame using the plane wave method with a subsequent boost transformation. Obviously, the relation (4.53) holds true in the moving frame also.

4.4 Summary

Though, the defining representation of Wigner's little group for massless particles in 2+1 dimension generate gauge transformations for Maxwell theory, the same representation does not generate gauge transformations in the topologically massive Maxwell-Chern-Simons and Einstein-Chern-Simons theories. However, using the fact that this little group has only one parameter, we have obtained a different representation of the little group that generate gauge transformations in these theories. The similarities and dissimilarities between the Maxwell and Maxwell-Chern-Simons theories in the context of gauge fixing (spatial transversality and temporal gauge) are also analyzed. Detailed analysis of the polarization tensor of Einstein-Chern-Simons theory is carried out and the polarization tensor is found to be a tensor product of a pair of polarization vectors of Maxwell-Chern-Simons theory with the same helicity. This is quite natural since Maxwell-Chern-Simons (spin ± 1) and Einstein-Chern-

Simons (spin ± 2) theories correspond to different spin representations of the 2+1 dimensional Poincare algebra[3] [65] and these spin representations are related by a tensor product [9, 66].

[3] Note that spin is a scalar in 2+1 dimensions.

Chapter 5

Dimensional descent

As introduced in [32], dimensional descent is a method by which one can obtain the energy-momentum vector, polarization vector/tensor and the gauge generating representation of the translational subgroup of Wigner's little group etc, in a topologically massive gauge theory living in a certain space-time dimension from similar results for massless gauge theories inhabiting a space-time of one higher dimension. In this sense, dimensional descent is a unification scheme for the results presented in the previous chapters regarding various gauge theories. In the present chapter, we discuss dimensional descent as applied to topologically massive gauge theories [26, 32]. Dimensional descent is also applicable to massive gauge theories [37] obtained by the previously mentioned embedding procedure of Batalin, Fradkin and Tyutin and we provide a short account of dimensional descent for such theories in the following chapter. It may be noted in this connection that the relationship between massless gauge theories in a given space-time dimension and lower dimensional gauge theories having massive excitations can be studied using other methods also (for example, see [68, 69]).

5.1 Review of dimensional descent: 4+1 to 3+1 dimensions

We begin our discussion of dimensional descent by noting that, the translational group $T(3)$ which generates gauge transformation in 3+1 dimensional $B \wedge F$ theory is an invariant subgroup of $E(3)$. Now, just as $E(2)$ is the generator of gauge transformation in 4-dimensional Maxwell and massless KR theories, $E(3)$ generates gauge transformation in the 5-dimensional versions of these massless theories. This indicates that the generators of gauge transformations in $B \wedge F$ theory and 5-dimensional massless gauge theories are related. This relationship is explicitly demonstrated through the method of dimensional descent [32] which we will describe below.

An element of Wigner's little group in 5 dimensions can be written as

$$W_5(p,q,r;\psi,\phi,\eta) = \begin{pmatrix} 1+\frac{p^2+q^2+r^2}{2} & p & q & r & -\frac{p^2+q^2+r^2}{2} \\ p & & & & -p \\ q & & R(\psi,\phi,\eta) & & -q \\ r & & & & -r \\ \frac{p^2+q^2+r^2}{2} & p & q & r & 1-\frac{p^2+q^2+r^2}{2} \end{pmatrix} \qquad (5.1)$$

where p, q, r are any real numbers, while $R(\psi,\phi,\eta) \in SO(3)$, with (ψ,ϕ,η) being a triplet of Euler angles. The above result can be derived by following the standard treatment (see, for example [19]) adopted in appendix A for 3+1 dimensional case. The element of the translational subgroup $T(3)$ of $W_5(p,q,r;\psi,\phi,\eta)$ can be trivially obtained by setting $R(\psi,\phi,\eta)$ to be the identity matrix and will be denoted by

$W(p,q,r)$;

$$W(p,q,r) \equiv W_5(p,q,r;0) = \begin{pmatrix} 1+\frac{p^2+q^2+r^2}{2} & p & q & r & -\frac{p^2+q^2+r^2}{2} \\ p & 1 & 0 & 0 & -p \\ q & 0 & 1 & 0 & -q \\ r & 0 & 0 & 1 & -r \\ \frac{p^2+q^2+r^2}{2} & p & q & r & 1-\frac{p^2+q^2+r^2}{2} \end{pmatrix}. \qquad (5.2)$$

Let us now consider free Maxwell theory in 5-dimensions,

$$\mathcal{L} = -\frac{1}{4}F^{xy}F_{xy}; \qquad x,y = 0,1,2,3,4. \qquad (5.3)$$

For a photon of energy ω (in 5-dimensional space-time) propagating in the $x = 4$ direction, the momentum 5-vector is given by

$$k^x = (\omega, 0, 0, 0, \omega)^T. \qquad (5.4)$$

By following the plane wave method and proceeding exactly as in section 3.2, one can show that the maximally reduced form of the polarization vector of the photon is

$$\varepsilon^x = (0, a, b, c, 0)^T \qquad (5.5)$$

where a, b, c represent the three transverse degrees of freedom (since the polarization vector satisfies the 'Lorentz gauge' $\varepsilon^x k_x = 0$). If we now suppress the last rows of the column matrices k^x (5.4) and ε^x (5.5), we end up respectively with the energy-momentum 4-vector and the polarization vector of 3+1 dimensional Proca theory in the rest frame of the quanta if we make the identification $\omega = m$ (m being the mass of the Proca particle), or equivalently, of $B \wedge F$ theory since the gauge invariant sector of the latter is equivalent to the former. This is equivalent to applying the projection operator given by the matrix

$$\mathcal{P} = diag(1,1,1,1,0) \qquad (5.6)$$

to the momentum 5-vector (5.4) and the polarization vector (5.5).

Similarly, to reproduce the polarization tensor of $B \wedge F$ or equivalently, massive Kalb-Ramond theory in 3+1 dimensions let us consider free massless KR model in 5-dimensions,

$$\mathcal{L} = \frac{1}{12} H^{xyz} H_{xyz} \tag{5.7}$$

Analogous to (3.47) (for massless KR theory in 3+1 dimensions), the maximally reduced form of the polarization tensor ε^{xy} of the 4+1 dimensional massless KR model can be obtained by plane wave method as,

$$\{\varepsilon^{xy}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{12} & \varepsilon^{13} & 0 \\ 0 & -\varepsilon^{12} & 0 & \varepsilon^{23} & 0 \\ 0 & -\varepsilon^{13} & -\varepsilon^{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.8}$$

Again deleting the last row and column, one gets the polarization tensor(ε^{ab}) in (3.63) of the $B \wedge F$ model or of massive KR theory. This is equivalent to applying the projection operator as $\mathcal{P}\varepsilon\mathcal{P}$. Thus the polarization vector and tensor of the Proca and massive KR models respectively, or $B \wedge F$ theory have been reproduced. This is quite natural since, as mentioned in section 3.5, the gauge invariant physical sector of $B \wedge F$ theory can be considered equivalent either to Proca theory or to massive KR theory in 3+1 dimensions.

Now coming to the gauge transformation properties of polarization vector (5.5) and polarization tensor (5.8) under the translational subgroup $T(3)$, let $W(p,q,r)$ act on these objects one by one. First, acting on ε^x(5.5), one gets

$$\varepsilon^x \to \varepsilon'^x = W(p,q,r)^x{}_y \varepsilon^y = \varepsilon^x + \delta\varepsilon^x = \varepsilon^x + (pa + qb + rc)\frac{k^x}{\omega} \tag{5.9}$$

which is indeed a gauge transformation in (4+1) dimensional Maxwell theory. Applying the projection operator $\mathcal{P}$ (5.6) on (5.9) yields

$$\delta\varepsilon^a = \mathcal{P}\delta\varepsilon^x = \frac{1}{\omega}(pa + qb + rc)k^a \tag{5.10}$$

where $\varepsilon^a = (0, a, b, c)^T$ and, modulus the i-factor, corresponds to the expression in (3.63) of $B \wedge F$ theory. This is precisely how the polarization vector in $B \wedge F$ theory transforms under gauge transformation [27]. In fact we can write

$$\delta\varepsilon^a = D^a{}_b(p, q, r)\varepsilon^b - \varepsilon^a = \frac{1}{\omega}(pa + qb + rc)k^a \tag{5.11}$$

where $D(p, q, r)$ is a representation of the translational group $T(3)$ given in (3.65),

$$D(p, q, r) = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.12}$$

In terms of the generators (3.72) of $T(3)$, the change in the polarization vector (5.11) can be expressed as the action of a Lie algebra element:

$$\delta\varepsilon^a = (pT_1 + qT_2 + rT_3)\frac{\varepsilon^a}{\omega}. \tag{5.13}$$

Coming next to the polarization matrix $\{\varepsilon^{xy}\}$, its transformation law is given by,

$$\{\varepsilon^{xy}\} \to \{\varepsilon'^{xy}\} = W(p, q, r)\{\varepsilon^{xy}\}W^T(p, q, r) = \{\varepsilon^{xy}\} + \{\delta\varepsilon^{xy}\}$$

where,

$$\{\delta\varepsilon^{xy}\} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & 0 & 0 & -\alpha_1 \\ -\alpha_2 & 0 & 0 & 0 & -\alpha_2 \\ -\alpha_3 & 0 & 0 & 0 & -\alpha_3 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{pmatrix} \tag{5.14}$$

and $\alpha_1 = -(qc + rb)$, $\alpha_2 = (pc - ra))$, $\alpha_3 = (pb + qa)$. Again this can be easily recognized as a gauge transformation in (4+1) dimensional KR theory involving massless quanta, as $\delta\varepsilon^{xy}$ can be expressed as $\approx (k^x f^y(k) - k^y f^x(k))$ with suitable choice for $f^x(k)$, where k^x is of the form (5.4). Now applying the projection operator $\mathcal{P}$ (5.6) on (5.14), we get the change in the 3+1 dimensional polarization matrix $\{\varepsilon^{ab}\}$, by the formula, $\{\delta\varepsilon^{ab}\} = \mathcal{P}\{\delta\varepsilon^{xy}\}\mathcal{P}^T$. This simply amounts to a deletion of the last row and column of $\{\delta\varepsilon^{xy}\}$. The result can be expressed more compactly as

$$\{\delta\varepsilon^{ab}\} = (D(p,q,r)\{\varepsilon^{ab}\}D^T((p,q,r) - \{\varepsilon^{ab}\}). \tag{5.15}$$

Again this has the precise form of gauge transformation of the polarization matrix of $B \wedge F$ model, since it can be cast in the form

$$\{\delta\varepsilon^{ab}\} = i(k^a f^b(k) - k^b f^a(k)) \tag{5.16}$$

for a suitable $f^a(k)$, where $k^a = (m, 0, 0, 0)^T$.

Thus we have shown that the gauge generation representation $D(p,q,r)$ of $T(3)$ for topologically massive $B \wedge F$ gauge theory in 3+1 dimensions can be connected to the Wigner's little group for massless particle in 4+1 dimensions through dimensional descent. This involved appropriate projections in the intermediate steps, where the massless particles moving in 4+1 dimensions can be associated with a massive particle at rest in 3+1 dimensions. Similarly the polarization vector and tensor respectively of $B \wedge F$ theory in 3+1 dimensions can be associated with polarization vector and polarization tensor of free Maxwell and KR theories in 4+1 dimensions.

5.2 Dimensional descent: 3+1 to 2+1 dimensions

We have seen from the previous discussion that using dimensional descent one can obtain several properties regarding the gauge transformation in 3+1 dimensional topologically massive gauge theory (the $B \wedge F$ theory) by starting from higher dimensional massless gauge theories. We have also seen earlier that the MCS and ECS theories are topologically massive theories in 2+1 dimensions. Therefore one is naturally led to the question as to what would be the role of dimensional descent in these 2+1 dimensional topologically massive gauge theories with respect to the massless gauge theories (Maxwell and linearized gravity) in 3+1 dimensions. This section is devoted to a detailed discussion of this issue and the application of dimensional descent from 3+1 dimensions to 2+1 dimensions [26, 32].

5.2.1 Proca theory and doublet of Maxwell-Chern-Simons theories

Here we describe the method of dimensional descent from 3+1 to 2+1 dimensions for vector theories [32]. Let us recapitulate certain properties of free Maxwell theory in 3+1 dimensions from section 3.2, which are essential in the present context. The 3+1 dimensional Maxwell theory has two transverse degrees of freedom. Correspondingly the polarization vector ε^a takes the maximally reduced form $\varepsilon^a = (0,,\varepsilon^1,\varepsilon^2,0)^T$ (3.9), if the 4-momentum is $k^a = (\omega,0,0,\omega)^T$. We have seen earlier (section 3.2, (3.10)) that the generator of gauge transformation in this case is $T(2)$, which is a subgroup of $E(2)$:

$$\delta\varepsilon^a = W^a{}_b(p,q)\varepsilon^b - \varepsilon^a = \left(p\varepsilon^1 + q\varepsilon^2\right)\frac{k^a}{\omega}. \tag{5.17}$$

We now proceed forth to discuss dimensional descent from 3+1 dimensions to 2+1 dimensions. By applying the projection operator $\mathcal{P} = diag(1,1,1,0)$ on ε^a and k^a, and then suppressing the last rows of ε^a and k^a, we obtain the descended objects $\varepsilon^\mu = (0, a, b)^T$ (with $\varepsilon^1 = a$ and $\varepsilon^2 = b$) and $k^\mu = (\omega, 0, 0)^T$. These can be considered to be the polarization vector and momentum 3-vector in the rest frame of 2+1 dimensional Proca theory described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu \tag{5.18}$$

provided we make the identification $\omega = m$, the mass of Proca excitations. Analogous to (5.10), the projection operator $\mathcal{P} = diag(1,1,1,0)$ when applied on the gauge transformation (5.17) yields

$$\delta\varepsilon^\mu = \mathcal{P}\delta\varepsilon^a = \frac{1}{\omega}(pa + qb)k^\mu \tag{5.19}$$

Since Proca theory is not a gauge theory, (5.19) cannot be considered as a gauge transformation. We have, however, seen earlier that Proca theory in 2+1 dimensions is actually a doublet of Maxwell-Chern-Simons theories [36, 57, 59]

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \tag{5.20}$$

where

$$\mathcal{L}_\pm = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \pm \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{5.21}$$

with $\vartheta > 0$ and each of $\mathcal{L}_+$ or $\mathcal{L}_-$ being a topologically massive gauge theory. The mass of the MCS quanta is $m = \vartheta$, where m is the parameter entering in (5.18). We can therefore study the gauge transformation generated in this doublet. For this purpose, analogous to (5.12), it is essential to provide a 3×3 representation of $T(2)$

(denoted by $D(p,q)$),

$$D(p,q) = \begin{pmatrix} 1 & p & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.22}$$

The corresponding generators are given by,

$$T_1 = \frac{\partial D(p,q)}{\partial p} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad T_2 = \frac{\partial D(p,q)}{\partial q} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.23}$$

In analogy with (5.13), here also one can rewrite (5.19) as,

$$\delta\varepsilon^\mu = (pT_1 + qT_2)^\mu{}_\nu \varepsilon^\nu \tag{5.24}$$

so that the change in the polarization vector ε^μ is obtained as the actoin of a Lie algebra element. Proca polarization vector ε^μ is just a linear combination of the two real orthonormal canonical vectors ε^1 and ε^2 where,

$$\varepsilon^\mu = a\varepsilon^{(1)} + b\varepsilon^{(2)}; \qquad \varepsilon^{(1)} = (0,1,0)^T, \varepsilon^{(2)} = (0,0,1)^T. \tag{5.25}$$

Correspondingly the generators T_1 and T_2 (5.23), form an orthonormal basis as they satisfy $tr(T_i^\dagger T_j) = \delta_{ij}$. Furthermore,

$$T_1\varepsilon^{(1)} = T_2\varepsilon^{(2)} = (1,0,0)^T = \frac{k^\mu}{m}, \qquad T_1\varepsilon^{(2)} = T_2\varepsilon^{(1)} = 0 \tag{5.26}$$

On the other hand, the rest frame polarization vectors for $\mathcal{L}_\pm$, with only one degree of freedom for each of $\mathcal{L}_+$ and $\mathcal{L}_-$ (section 2.1.1), as given by

$$\xi^\mu_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \mp i \end{pmatrix} \tag{5.27}$$

also provide an orthonormal basis (complex) in the plane as

$$(\xi_+^\mu)^\dagger(\xi_-^\mu) = 0; \qquad (\xi_+^\mu)^\dagger(\xi_+^\mu) = (\xi_-^\mu)^\dagger(\xi_-^\mu) = 1. \tag{5.28}$$

Here we note that spatial part $\vec{\xi}_\pm$ of $\xi_\pm$ can be obtained from the space part of the above mentioned canonical ones by appropriate $SU(2)$ transformation. That is, $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in \mathrm{SU}(2)$ when acts on $\vec{\varepsilon}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{\varepsilon}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ yields respectively the vectors $\vec{\xi}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\vec{\xi}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (up to an irrelevant factor of i)[1]:

$$i\vec{\xi}_+ = U\vec{\varepsilon}^{(2)}, \qquad \vec{\xi}_- = U\vec{\varepsilon}^{(1)}. \tag{5.29}$$

This suggests that we consider the following orthonormal basis for the Lie algebra of $T(2)$:

$$T_\pm = \frac{1}{\sqrt{2}}(T_1 \pm iT_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & \pm i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.30}$$

instead of T_1 and T_2. Note that they also satisfy relations similar to the $(T_1 - T_2)$ basis,

$$tr(T_+^\dagger T_+) = tr(T_-^\dagger T_-) = 1; \quad tr(T_+^\dagger T_-) = 0 \tag{5.31}$$

One can now easily see that

$$T_+\xi_+ = T_-\xi_- = \frac{k^\mu}{m}, \qquad T_+\xi_- = T_-\xi_+ = 0 \tag{5.32}$$

analogous to (5.26). Furthermore,

$$\delta\xi_\pm^\mu = p_\pm T_\pm \xi_\pm^\mu = \frac{p_\pm}{m}k^\mu. \tag{5.33}$$

[1]This ambiguity of i factor is related to the U(1) phase arbitrariness of the polarization vector discussed in section 2.1.1.

This indicates that $T_\pm$ - the generators of the Lie algebra of $T(2)$ in the rotated (complex) basis - generate independent gauge transformations in $\mathcal{L}_\pm$ respectively. One therefore can understand how the appropriate representation of the generator of gauge transformation in the doublet of MCS theory can be obtained from higher 3+1 dimensional Wigner's group through dimensional descent. A finite gauge transformation is obtained by integrating (5.33) i.e., exponentiating the corresponding Lie algebra element. This gives two representations of Wigner's little group for massless particles in $2+1$ dimensions, which is isomorphic to $\mathcal{R} \times \mathcal{Z}_2$, although here we are just considering the component which is connected to the identity,

$$D_\pm(p_\pm) = e^{p_\pm T_\pm} = 1 + p_\pm T_\pm = \begin{pmatrix} 1 & \frac{p_\pm}{\sqrt{2}} & \pm i\frac{p_\pm}{\sqrt{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.34}$$

Note that $D_\pm(p_\pm)$ generates gauge transformation in the doublet $\mathcal{L}_\pm$,

$$D^\mu_{\pm\nu}\xi^\nu_\pm = \xi^\mu_\pm + \frac{p_\pm}{|\vartheta|}k^\mu \tag{5.35}$$

and are related by complex conjugation. This complex conjugation is also a symmetry of the doublet as we saw in section 2.2.

Therefore, it is clear that the gauge generating representation of little group for MCS and ECS theories can be obtained by the method of dimensional descent.

5.2.2 EPF theory and doublet of Einstein-Chern-Simons theories

Now we discuss dimensional descent from 3+1 dimensions to 2+1 dimensions for theories with symmetric second rank tensor fields [26]. For this, it is essential to discuss the relevant aspects of 2+1 dimensional Einstein-Pauli-Fierz (EPF) theory [9, 62, 63] whose action is given by

$$I^{EPF} = \int d^3x \left(-\sqrt{g}R - \frac{\mu^2}{4}(h^2_{\mu\nu} - h^2) \right). \tag{5.36}$$

Note that the usual sign in front of Einstein action has been restored to avoid ghosts and tachyons, as has been observed recently by Deser and Tekin [67]. As noted in [67], both the relative and overall signs of the two terms in (5.36) have to be of conventional Einstein and Pauli-Feirz mass terms in order to have a physically meaningful theory. On the other hand in the ECS theory, sign of the Einstein term has to be opposite to that of the conventional one for the theory to be viable. Therefore, if one attempts to couple the ECS theory with a Pauli-Fierz term, one is faced with an unavoidable conflict of signs. Upon linearization, (5.36) reduces to

$$\mathcal{L}_L^{EPF} = \frac{1}{2}h_{\mu\nu}\left[R_L^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R_L\right] - \frac{\mu^2}{2}(h^2_{\mu\nu} - h^2). \tag{5.37}$$

Analogous to the doublet structure of Proca theory discussed above, the EPF theory is a doublet, as was suggested in [9], comprising of a pair of ECS theories having opposite helicities. And just like the Proca theory, EPF theory does not possess any gauge symmetry. The equation of motion following from the EPF Lagrangian is given by

$$-\Box h^{\mu\nu}+\partial^\mu\partial_\gamma h^{\gamma\nu}+\partial^\nu\partial_\gamma h^{\gamma\mu}-\partial^\mu\partial^\nu h+\eta^{\mu\nu}(\Box h-\partial_\gamma\partial_\delta h^{\gamma\delta})-\mu^2(h^{\mu\nu}-\eta^{\mu\nu}h) = 0. \tag{5.38}$$

With the ansatz $h^{\mu\nu} = \chi^{\mu\nu}e^{ik.x}$, where $\chi^{\mu\nu}$ is the polarization tensor in this case, this equation can be written as

$$k^2\chi^{\mu\nu}-k^\mu k_\gamma\chi^{\gamma\nu}-k^\nu k_\gamma\chi^{\gamma\mu}+k^\mu k^\nu\chi+\eta^{\mu\nu}(-k^2\chi+k_\gamma k_\delta\chi^{\gamma\delta})-\mu^2(\chi^{\mu\nu}-\eta^{\mu\nu}\chi) = 0 \tag{5.39}$$

We now proceed along the same lines as was done in the previous cases to arrive at the physical polarization tensor of EPF theory. By a heuristic argument we can easily see that the EPF theory does not possess any massless excitations. (It can also be seen from the propagator of the $h^{\mu\nu}$ field that the EPF theory has massive excitations [62, 63].) If we choose $k^2 = 0$, the equation of motion (5.39) leads, upon contraction with k_μ, to the condition

$$k_\mu \chi^{\mu\nu} = k^\nu \chi. \tag{5.40}$$

On the other hand, the contraction of (5.40) with k_ν leads to

$$k_\mu k_\nu \chi^{\mu\nu} = 0. \tag{5.41}$$

A solution of the above pair of equations (5.40, 5.41) is given by $\chi^{\mu\nu} = k^a f^b(k) + k^b f^a(k)$ where f's are arbitrary functions of k which satisfies the condition $k.f = 0$. However, such a solution is compatible with the equation of motion (5.39) if and only if the f's vanish identically thus demonstrating that EPF theory does not have massless excitations.

Now for $k^2 \neq 0$, in the rest frame $k^\mu = (m, 0, 0)^T$ and so the (00) component of the equation of motion yields $\mu^2(\chi^{00} - \chi) = 0$ which in turn gives

$$\chi^{11} + \chi^{22} = 0. \tag{5.42}$$

Therefore, one is free to arbitrarily choose either χ^{11} or χ^{22}. Similarly, the $(0i)$ component in the rest frame becomes $\mu^2 \chi^{0i} = 0$ implying

$$\chi^{0i} = 0. \tag{5.43}$$

The space part ((ij)-components) of the equation of motion with $i = j = 1$ and $i = j = 2$, respectively yields in the rest frame,

$$-\mu^2({\chi^0}_0 + {\chi^2}_2) + m^2 \chi^{11} = 0 \tag{5.44}$$

$$-\mu^2(\chi^0{}_0 + \chi^1{}_1) + m^2\chi^{22} = 0. \tag{5.45}$$

Adding the above two equation gives,

$$\chi^{00} = 0. \tag{5.46}$$

Substituting (5.46) back in (5.44) and using (5.42) one arrives at

$$(m^2 - \mu^2)\chi^1{}_1 = 0. \tag{5.47}$$

Finally, the spatial component ($\mu = i,\ \nu = j$) of (5.39) for $i \neq j$ in the rest frame becomes

$$(m^2 - \mu^2)\chi^1{}_2 = 0. \tag{5.48}$$

The equations (5.47) and (5.48) can be satisfied if either $(m^2 - \mu^2) = 0$ or $\chi^1{}_1 = \chi^1{}_2 = 0$. On account of (5.42), (5.43) and (5.46), the latter choice will lead to a null theory and can be ruled out. Therefore, we must have

$$m^2 = \mu^2 \tag{5.49}$$

thus establishing that the mass of the EPF excitation to be $|\mu|$. This choice leaves the two components χ^{11} and χ^{12} arbitrary representing the two physical degrees of freedom in the theory.

On the other hand, if the Einstein term in (5.36) were +ve (i.e., the same as the Einstein term in (4.32)), instead of (5.48) we would have had

$$(m^2 + \mu^2)\chi^1{}_2 = 0. \tag{5.50}$$

This in term would have meant $m^2 = -\mu^2$ leading to an unphysical tachyonic mode for theory. Thus, in EPF theory, the sign of the Einstein term must necessarily be negative, i.e., the conventional one. Therefore, our analysis also shows the unviability of coupling both Chern-Simons and Pauli-Fierz terms to Einstein gravity in

2+1 dimensions since such a coupling puts conflicting demands on the sign on the Einstein term.

Therefore, it is obvious that we can write the polarization tensor of EPF model in the rest frame as

$$\{\chi^{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a \end{pmatrix}. \tag{5.51}$$

where $\chi^{11} = -\chi^{22} = a$ and $\chi^{12} = \chi^{21} = b$. With the aid of the expressions (5.51) for $\chi^{\mu\nu}$ of EPF theory and (4.55) for $\{\chi_{\pm}\}$ of a pair of ECS theories, we now embark on a discussion of dimensional descent for the case of second rank symmetric tensor gauge fields emphasizing the near exact parallel with the case of vector gauge fields discussed in the previous subsection. One can obtain the momentum 3-vector k^{μ} and polarization tensor $\{\chi^{\mu\nu}\}$ (5.51) of EPF model in 2+1 dimensions from those of linearized gravity in 3+1 dimensions as follows. By applying the projection operator $\mathcal{P}$ =diag$(1,1,1,0)$ on momentum 4-vector $k^a = (\omega, 0, 0, \omega)^T$ of a massless graviton moving in the z-direction of 3+1 dimensional linearized Einstein gravity and subsequently deleting the last row of the resulting vector, one get momentum 3-vector k^{μ} of 2+1 dimensional EPF quanta at rest. By a similar application of $\mathcal{P}$ on $\{\chi^{ab}\}$ (3.27) and deleting the last row and column, one gets the polarization tensor $\{\chi^{\mu\nu}\}$ (5.51) in the rest frame of the EPF quanta. Next we notice that, just like the way Proca polarization vector $\tilde{\epsilon}^{\mu}$ is written as a linear combination of two orthonormal canonical vectors (5.25), one can write the EPF polarization tensor $\{\chi^{\mu\nu}\}$ as

$$\{\chi^{\mu\nu}\} = a\mathcal{X}_1 + b\mathcal{X}_2 = a\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.52}$$

We may consider the above equation to be the EPF analogue of (5.25) in the case Proca theory. Notice that the space parts of the matrices appearing in the above linear combination are nothing but the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.53}$$

Clearly, the space part $\{\chi_\pm{}^{ij}\}$ of the ECS polarization tensors $\{\chi_\pm^{\mu\nu}\}$ (4.55) can be expressed in terms of σ_1 and σ_3 as follows:

$$\{\chi_\pm{}^{ij}\} = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}\sigma_3 \mp \frac{i}{\sqrt{2}}\sigma_1) \, . \tag{5.54}$$

This amounts to an SU(2) transformation in the 2-dimensional subspace (of the SU(2) Lie algebra in an orthonormal basis) spanned by $\Sigma_1 = \frac{1}{\sqrt{2}}\sigma_1$ and $\Sigma_2 = \frac{1}{\sqrt{2}}\sigma_3$ as can be seen from the following[2]:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} = \begin{pmatrix} i\{\chi_+{}^{ij}\} \\ \{\chi_-{}^{ij}\} \end{pmatrix}. \tag{5.55}$$

It should be noticed that (5.54) is the analogue of (5.29) in the case of Proca and MCS theories. In the case of vector (Proca and MCS) field theories, the basis vectors $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ (5.25) of the Proca polarization vector, when transformed by a suitable SU(2) transformation yield the polarization vectors $\xi_\pm$ of a pair of MCS theories. We can see an exact analogy of this in the EPF theory as follows. Similarly, in the case of tensor (EPF and ECS) field theories, the same SU(2) transformation when acted on the $\mathcal{X}_1$ and $\mathcal{X}_2$ provides the polarization tensors $\{\chi_\pm^{\mu\nu}\}$ of a doublet of ECS theories with opposite helicities just as Proca theory is a doublet of MCS theories having opposite spins. This corroborates the proposition that EPF theory

[2]Note that the $\{\chi_-^{ij}\}$ obtained in (5.54) differs from the one obtained by SU(2) rotation by an irrelevant i factor just as in the vector case.

is consisted of a doublet of ECS theories with opposite spins at least at the level of polarization tensor. Moreover, as we have discussed earlier, the polarization tensor and momentum vector of (2+1 dimensional) EPF theory can be obtained from those of linearized Einstein gravity (in 3+1 dimensions) by applying suitable projection operator. This relationship between EPF and ECS theories resemble the one between Proca and MCS theories. Therefore we expect that the procedure of dimensional descent to be valid here as well. As described in section 5.2.1, the generator of the representations of $T(1)$, obtained by dimensional descent, that generate gauge transformation in a pair of MCS theories with opposite helicities are given by $T_{\pm}$ (5.30). Also, the ECS polarization tensors $\chi_{\pm}$ can be made to satisfy the orthonormality relations (4.54) similar to (5.28) for MCS case owing to the fact that the former is a tensor product of MCS polarization vectors. Hence it is natural to expect that the $T(1)$ group representation $D_{\pm}(p_{\pm})$ (5.34) obtained by exponentiation of $T_{\pm}$ generates gauge transformation in ECS doublet, which in fact it does, as we have shown in (4.56). Therefore, it is evident that by a dimensional descent from 3+1 dimensional linearized gravity one could obtain the representations of $T(1)$ that generate gauge transformations in the doublet of topologically massive ECS theories in 2+1 dimensions. This is similar to the dimensional descent from 3+1 dimensional Maxwell theory to 2+1 dimensional MCS theory discussed in the subsection 5.2.1.

5.3 Summary

We presented a review of how the gauge transformations in the 3+1 dimensional $B \wedge F$ theory is related through the method of dimensional descent to the gauge transformations in Maxwell and massless Kalb-Ramond theories in 4+1 dimensions.

In the same fashion, dimensional descent relates the gauge transformations in 3+1 dimensional Maxwell theory to those in the doublet of Maxwell-Chern-Simons theories which is the equivalent to a Proca theory in 2+1 dimensions. There exists an analogous relationship between the gauge transformation in 3+1 dimensional linearized gravity and in a doublet of Einstein-Chern-Simons theories in 2+1 dimensions. Analysis of the polarization tensors of the Einstein-Pauli-Fierz theory and of a doublet of Einstein-Chern-Simons theories with opposite helicities suggests that the EPF theory is the doublet of ECS theories just like the Proca theory is a doublet of MCS theories. However, the analogy between EPF and Proca theories with their respective doublet structures breaks down if one considers the fact that sign of the Einstein term flips from EPF to ECS theories in contrast to Proca theory where the sign of the Maxwell term remains unchanged irrespective of whether it is coupled to a Chern-Simons term or a usual mass term. Therefore, further investigations are necessary in order to rigorously establish the doublet structure, if any, of EPF theory beyond the level of polarization tensors.

Chapter 6

Massive Gauge theories

In the preceding chapters we have seen that various translational groups in their appropriate representations act as generators of gauge transformations in a variety of Abelian gauge theories. We have demonstrated this for theories which have vector, symmetric and antisymmetric second rank tensors as the underlying gauge fields. Such theories considered so far are either massless or topologically massive gauge theories. However, there are other types of gauge theories which are obtained by converting second class constrained systems(in the language of Dirac's theory of constraint dynamics) to first class (gauge) systems using the generalized prescription of Batalin, Fradkin and Tyutin [40, 41, 42, 43, 45]. By such a prescription, one can obtain from the massive Proca theory, the Stückelberg model for vector field that is massive while possessing a gauge invariance [46]. Similarly one can obtain the Stückelberg extended versions of massive KR and EPF theories. The discussion in this chapter is confined to 3+1 dimensions since the Stückelberg embedded theories are usually studied in that space-time. Nevertheless, with suitable modifications, all the methods as well as results of this chapter are equally valid in 2+1 dimensions also. This chapter consists of the study of the gauge transformations in such theories and their relationship with translational groups. Here we show that the same representation of the translational group $T(3)$ that generates gauge transformation in the topologically massive $B \wedge F$ theory also generates gauge transformation in the Stückelberg extended first class version of Proca, massive KR and EPF theories in 3+1 dimensions.

6.1 Massive vector gauge theory

One can convert the 4-dimensional Proca theory (which does not possess any gauge symmetry) into a gauge theory by Stückelberg mechanism with the introduction of a new scalar field $\alpha(x)$ as follows;

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} + \frac{m^2}{2}(A_a + \partial_a \alpha)(A^a + \partial^a \alpha) \tag{6.1}$$

The Lagrangian remains invariant under the transformations

$$A_a(x) \to A'_a(x) = A_a(x) + \partial_a \Lambda(x), \qquad \alpha(x) \to \alpha'(x) = \alpha(x) - \Lambda(x) \tag{6.2}$$

where $\Lambda(x)$ is an arbitrary scalar function. The equations of motion for the theory are

$$-\partial_b F^{ab} + m^2(A^a + \partial^a \alpha) = 0, \qquad \partial^a(A_a + \partial_a \alpha) = 0. \tag{6.3}$$

One must notice that by operating ∂_a on the first equation in (6.3) one yields the second. Hence the latter is consequence of the former. This implies that the gauge transformation of the α-field can be deduced by knowing that of the A^a-field. Similar to (3.6), here we adopt the ansatz $A^a(x) = \varepsilon^a \exp(ik.x)$ and $\alpha(x) = \tilde{\alpha}(k)\exp(ik.x)$. As before, $\varepsilon^a(k)$ is the polarization vector of the field $A^a(x)$ and $\tilde{\alpha}(k)$ is a particular

Fourier component of $\alpha(x)$. In terms of the polarization vector $\varepsilon^a(k)$, the equations of motion become respectively,

$$k_b(k^a\varepsilon^b - k^b\varepsilon^a) + m^2(\varepsilon^a + ik^a\tilde{\alpha}) = 0, \quad ik^b(\varepsilon_b + ik_b\tilde{\alpha}) = 0. \tag{6.4}$$

Analogous to (6.3), the second equation in (6.4) is a consequence of the first one. For massless excitations $k^2 = 0$, the second equation of the above pair of equations gives the Lorentz condition $k_b\varepsilon^b = 0$ which when substituted in the first gives,

$$\varepsilon^a = -ik^a\tilde{\alpha}. \tag{6.5}$$

Since this is a solution proportional to the 4-momentum k^a, it can be gauged away by an appropriate choice of the gauge. Thus, massless excitations are gauge artefacts. For $k^2 = M^2$ (massive excitations), the equations of motion (6.4) becomes,

$$(m^2 - M^2)\varepsilon^a + k^a k_b\varepsilon^b + im^2k^a\tilde{\alpha} = 0, \quad \tilde{\alpha} = \frac{ik_b\varepsilon^b}{M^2}. \tag{6.6}$$

Substituting the second equation in (6.6) in the first yields,

$$(m^2 - M^2)\left[\eta^{ab} - \frac{k^a k^b}{M^2}\right]\varepsilon_b = 0 \tag{6.7}$$

In this equation, the expression inside the parenthesis is a projection operator which projects out the transverse component of ε_b. Since longitudinal component can be gauged away, transverse part of ε_b should be nonvanishing in order to avoid having no physical excitations. Therefore we must have $m^2 - M^2 = 0$. Therefore, one can conclude that the mass of the excitation is given by m itself and the rest frame momentum 4-vector of the theory can be written as $k^b = (m, 0, 0, 0)$. In the rest frame, the second equation in (6.4) gives

$$\varepsilon^0 = -im\tilde{\alpha}. \tag{6.8}$$

Therefore, the polarization vector of $A^a(x)$ field of (6.1) can be written as

$$\varepsilon^a = (-im\tilde{\alpha}, \varepsilon^1, \varepsilon^2, \varepsilon^3)^T \tag{6.9}$$

The maximally reduced form of the polarization vector can be obtained from (6.9), by a gauge transformation with the choice $\Lambda(x) = \alpha(x)$ and is given by

$$\varepsilon^a = (0, a, b, c)^T \tag{6.10}$$

with the free components $a = \varepsilon^1, b = \varepsilon^2, c = \varepsilon^3$ representing the three physical degrees of freedom of the theory. One must note that (6.10) is of the same form as that of the $B \wedge F$ theory polarization vector (section 3.5). Therefore, just as in the case of $B \wedge F$ theory, the action of representation $D(p, q, r)$ (3.65) of $T(3)$ on the polarization vector (6.10) amounts to a gauge transformation in Stückelberg extended Proca theory:

$$\varepsilon^a \to \varepsilon'^a = D^a{}_b(p, q, r)\varepsilon^b = \varepsilon^a + \frac{i}{m}(pa + qb + rc)k^a \tag{6.11}$$

The above transformation can be cast in the form of the momentum space gauge transformation

$$\varepsilon^a \to \varepsilon^a + ik^a\lambda(k) \tag{6.12}$$

(where $\Lambda(x) = \lambda(k)e^{ik.x}$) corresponding to the field $A(x)$, by choosing the field $\Lambda(x)$ such that

$$\lambda(k) = \frac{(pa + qb + rc)}{m}. \tag{6.13}$$

As mentioned before, it is possible to obtain the gauge transformation property of α-field from that of the A^a-field for which we now proceed as follows.

Consider the second relation in (6.6);

$$\tilde{\alpha} = \frac{ik_a\varepsilon^a}{m^2}. \tag{6.14}$$

and let ε^a undergo the gauge transformation (6.12) which has the effect of making a corresponding transformation in α-field as

$$\tilde{\alpha} \to \tilde{\alpha}' = \frac{ik_a(\varepsilon^a + ik^a\lambda)}{m^2} = \frac{ik_a\varepsilon^a}{m^2} - \lambda = \tilde{\alpha} - \lambda. \tag{6.15}$$

Here λ is given by (6.13) corresponding to the gauge transformation generated by the translational group $T(3)$ in the $A^a(x)$ field. Notice that the above equation (6.15) corresponds to the second equation in (6.2). We have thus obtained the gauge transformation generated in the α field by $T(3)$ from that in the $A^a(x)$-field. It follows therefore that α-field can be gauged away completely by a suitable gauge fixing condition (unitary gauge) and it does not appear in the physical spectrum of the theory. Hence it is obvious that the representation $D(p, q, r)$ of $T(3)$ generates gauge transformation in the massive vector gauge theory governed by (6.1).

We noticed that the maximally reduced polarization vector ε^a (6.10) of the Stückelberg embedded Proca model takes the same form as that of $B \wedge F$ theory (3.63). Also both are massive gauge theories and the gauge transformations in both cases are generated by the representation $D(p, q, r)$ of $T(3)$. Therefore, by starting from 4+1 dimensional Maxwell theory, the dimensional descent can be employed to study gauge transformations in Stückelberg embedded Proca model just as the gauge transformation of the vector field in $B \wedge F$ theory is studied using dimensional descent in section 5.1.

6.2 Massive symmetric tensor gauge theory

Consider the massive and non-gauge Einstein-Pauli-Fierz (EPF) theory in 3+1 dimension as given by the Lagrangian,

$$\mathcal{L}_L^{EPF} = \frac{1}{2}h_{ab}\left[R_L^{ab} - \frac{1}{2}\eta^{ab}R_L\right] - \frac{\mu^2}{2}\left((h_{ab})^2 - h^2\right). \tag{6.16}$$

The EPF theory does not posses any gauge invariance. Just as the Proca theory was elevated to a gauge theory (section 6.1) by Stückelberg mechanism, the linearized EPF theory also can be provided with a gauge symmetry by introducing the an additional vector field A^a as follows:

$$\mathcal{L}_L^{EPF} = \frac{1}{2}h_{ab}\left[R_L^{ab} - \frac{1}{2}\eta^{ab}R_L\right] - \frac{\mu^2}{2}\left((h_{ab} + \partial_a A_b + \partial_b A_a)^2 - (h + 2\partial\cdot A)^2\right). \tag{6.17}$$

The transformations

$$h_{ab} \to h'_{ab} = h_{ab} + \partial_a\Lambda_b + \partial_b\Lambda_a \tag{6.18}$$

$$A_a(x) \to A'_a(x) = A_a(x) - \Lambda_a(x) \tag{6.19}$$

represent the gauge symmetry of the theory described by (6.17). The equation of motion for h_{ab} is

$$\begin{aligned} -\Box h^{ab} + \partial^a\partial_c h^{cb} + \partial^b\partial_c h^{ca} - \partial^a\partial^b h + \eta^{ab}(\Box h - \partial_c\partial_d h^{cd}) \\ -\mu^2\left[(h^{ab} + \partial^a A^b + \partial^b A^a) - \eta^{ab}(h + 2\partial\cdot A)\right] = 0 \end{aligned} \tag{6.20}$$

and that for A_a is

$$\Box A^a + \partial_b h^{ba} - \partial^a h - \partial^a(\partial\cdot A) = 0. \tag{6.21}$$

Analogous to the case of massive vector gauge theory discussed before, the equation of motion (6.21) for A_a can be obtained from (6.20) by applying the operator ∂_b. Therefore, gauge transformation of A^a is obtainable by knowing the gauge transformation of the h^{ab} field. As in the previous cases, we employ the plane wave method to obtain the maximally reduced polarization tensor χ_{ab} and vector ε_a involved in h_{ab} and A_a respectively. Considering only the negative frequency part of a single mode in the corresponding mode expansions, we write,

$$h_{ab}(x) = \chi_{ab}(k)e^{ik.x} \tag{6.22}$$

$$A_a(x) = \varepsilon_a(k)e^{ik.x}. \tag{6.23}$$

In terms of the polarization tensor χ_{ab} and vector ε_a the gauge transformations (6.18) and (6.19) respectively can be written as

$$\chi_{ab} \rightarrow \chi'_{ab} = \chi_{ab} + ik_a\zeta_b + ik_b\zeta_a \tag{6.24}$$

$$\varepsilon_a \rightarrow \varepsilon'_a = \varepsilon_a - \zeta_a \tag{6.25}$$

where $\Lambda_a(x) = \zeta_a(k)\exp(ik.x)$. Substituting (6.22) and (6.23) in (6.20), one gets

$$k^2\chi^{ab} - k^a k_c \chi^{cb} - k^b k_c \chi^{ca} + k^a k^b \chi + \eta^{ab}(-k^2\chi + k_c k_d \chi^{cd})$$
$$-\mu^2\left[\chi^{ab} + ik^a\varepsilon^b + ik^b\varepsilon^a - \eta^{ab}(\chi + 2ik_c\varepsilon^c)\right] = 0. \tag{6.26}$$

Contracting with η_{ab} and considering only massless ($k^2 = 0$) excitations the above equation reduces to

$$2k_a k_b \chi^{ab} + \mu^2\left[3(\chi + 2ik_a\varepsilon^a)\right] = 0. \tag{6.27}$$

The solution of the above equation is

$$\chi^{ab} = -i(k^a\varepsilon^b + k^b\varepsilon^a). \tag{6.28}$$

Hence it is also the solution of (6.26) with $k^2 = 0$. It is obvious that this solution is a gauge artefact since one can choose the arbitrary vector field $\Lambda_a = A_a$ in (6.19) so as to make this solution vanish.

Next we consider the massive case ($k^2 = M^2, M \neq 0$) and consider the (00) component of the equation of motion (6.26) which, by a straightforward algebra, can be reduced to

$$\chi^1{}_1 + \chi^2{}_2 + \chi^3{}_3 = 0 \tag{6.29}$$

Similarly the ($0i$) component of (6.26) gives

$$\chi_{0i} = -iM\varepsilon_i \tag{6.30}$$

Now, the (ij) component of (6.26) is given by

$$k^2\chi_{ij} - \eta_{ij}M^2(\chi - \chi^{00}) - \mu^2[\chi_{ij} - \eta_{ij}(\chi + 2iM\varepsilon^0)] = 0 \tag{6.31}$$

Using (6.29), the above equation can be reduced to

$$M^2\chi_{ij} - \mu^2[\chi_{ij} - \eta_{ij}(\chi + 2iM\varepsilon^0)] = 0 \tag{6.32}$$

For $i = j = 1, 2, 3$ respectively in (6.32), we have the following set of equations;

$$M^2\chi_{11} - \mu^2[\chi_{00} - \chi_{22} - \chi_{33}] - 2iM\varepsilon_0 = 0,$$

$$M^2\chi_{22} - \mu^2[\chi_{00} - \chi_{11} - \chi_{33}] - 2iM\varepsilon_0 = 0,$$

$$M^2\chi_{33} - \mu^2[\chi_{00} - \chi_{11} - \chi_{22}] - 2iM\varepsilon_0 = 0.$$

Adding the above three equations together and subsequently using (6.29), we arrive at

$$\chi_{00} = -2iM\varepsilon_0 \tag{6.33}$$

When $i \neq j$, the equation (6.32) reduces to

$$(\mu^2 - M^2)\chi_{ij} = 0. \tag{6.34}$$

At this juncture notice that only two of the three components $\chi_{ii}, i = 1, 2, 3$ are independent on account of the equation (6.29). Also, the χ_{00} and χ_{0i} components can be set equal to zero by choosing the arbitrary field Λ_a to be A_a. Therefore, if $\chi_{ij} = 0$ (for $i \neq j$) in the above equation (6.34), the number of independent components of χ_{ab} will be only two. Since this is not the case, we can satisfy the equation (6.34) only if $\mu^2 = M^2$. Thus we see that the parameter μ represents the mass of the physical excitations of the field h_{ab} and that its polarization tensor is

$$\{\chi_{ab}\} = \begin{pmatrix} -2i\mu\varepsilon_0 & -i\mu\varepsilon_1 & -i\mu\varepsilon_2 & -i\mu\varepsilon_3 \\ -i\mu\varepsilon_1 & \chi_{11} & \chi_{12} & \chi_{13} \\ -i\mu\varepsilon_2 & \chi_{12} & \chi_{22} & \chi_{23} \\ -i\mu\varepsilon_3 & \chi_{13} & \chi_{23} & \chi_{33} \end{pmatrix} \tag{6.35}$$

where $\chi_{11} + \chi_{22} + \chi_{33} = 0$ (see (6.29)). As mentioned before, by choosing the field Λ_a to be A_a and making a gauge transformation, the above form of the polarization tensor can be converted to its maximally reduced form given by

$$\{\chi_{ab}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \chi_{11} & \chi_{12} & \chi_{13} \\ 0 & \chi_{12} & \chi_{22} & \chi_{23} \\ 0 & \chi_{13} & \chi_{23} & \chi_{33} \end{pmatrix}; \quad \chi_{11} + \chi_{22} + \chi_{33} = 0. \tag{6.36}$$

Our next task is to show explicitly that it is possible to obtain the gauge transformation of A^a from that of h^{ab}. For this purpose we consider now the equation of motion (6.21) corresponding to the vector field A_a and the associated polarization tensor ε_a. Substituting (6.22) and (6.23) in (6.21)(or by contracting (6.26) with k_b) we get,

$$k^2\varepsilon^a - k^a k_b \varepsilon^b - ik_b\chi^{ba} + ik^a\chi = 0. \tag{6.37}$$

On making a gauge transformation (6.24) in the polarization tensor χ_{ab}, the polarization vector ε_a in (6.37) automatically undergoes a gauge transformation.

$$k^2\varepsilon'_a - k_a k_b \varepsilon'^b = ik_b(\chi^b{}_a + ik_a\zeta^b + ik^b\zeta_a) - ik_a(\chi + 2ik^b\zeta_b). \tag{6.38}$$

This implies

$$k^2\varepsilon'_a - k_a k_b \varepsilon'^b = [ik_b\chi^b{}_a - ik_a\chi] - k^2\zeta_a + k_a(k.\zeta) \tag{6.39}$$

From equation (6.37), substitute for the expression inside the square bracket in (6.39) to obtain

$$[k^2\varepsilon'_a - k_a k_b \varepsilon'^b] - [k^2\varepsilon_a - k_a k_b \varepsilon^b] = -k^2\zeta_a + k_a(k.\zeta) \tag{6.40}$$

It is now easy to see that this relation can be satisfied only if $\varepsilon'_a - \varepsilon_a = -\zeta_a$ in agreement with the previous relation (6.25). Therefore a knowledge of the gauge transformation of h_{ab} is enough to deduce the gauge transformation property of A_a-field. Like the α-field in Stückelberg extended Proca theory, this A_a-field too disappears from the physical spectrum.

Now we study the gauge transformation properties of the field h_{ab} under the action of translational group $T(3)$. It is easy to see that, similar to the case of $B\wedge F$ theory, the representation $D(p,q,r)$ (3.65) of $T(3)$ generates gauge transformation of the massive field (h_{ab}). The action of $D(p,q,r)$ on the polarization tensor $\{\chi_{ab}\}$ (6.36) is given by,

$$\{\chi_{ab}\} \to \{\chi_a\}' = D(p,q,r)\{\chi_{ab}\}D^T(p,q,r) = \{\chi_{ab}\}$$

$$+\begin{pmatrix} \begin{pmatrix} p(p\chi_{11}+q\chi_{12}+r\chi_{13}) \\ +q(p\chi_{12}+q\chi_{22}+r\chi_{23}) \\ +r(p\chi_{13}+q\chi_{23}+r\chi_{33}) \end{pmatrix} & \begin{pmatrix} p\chi_{11}+q\chi_{12} \\ +r\chi_{13} \end{pmatrix} & \begin{pmatrix} p\chi_{12}+q\chi_{22} \\ +r\chi_{23} \end{pmatrix} & \begin{pmatrix} p\chi_{13}+q\chi_{23} \\ +r\chi_{33} \end{pmatrix} \\ p\chi_{11}+q\chi_{12}+r\chi_{13} & 0 & 0 & 0 \\ p\chi_{12}+q\chi_{22}+r\chi_{23} & 0 & 0 & 0 \\ p\chi_{13}+q\chi_{23}+r\chi_{33} & 0 & 0 & 0 \end{pmatrix} \tag{6.41}$$

By choosing

$$\zeta_0 = \frac{1}{2}(p\zeta_1 + q\zeta_2 + r\zeta_3)$$

$$\zeta_1 = \frac{1}{\mu}(p\chi_{11} + q\chi_{12} + r\chi_{13})$$

$$\zeta_2 = \frac{1}{\mu}(p\chi_{12} + q\chi_{22} + r\chi_{23})$$

$$\zeta_3 = \frac{1}{\mu}(p\chi_{13} + q\chi_{23} + r\chi_{33}); \qquad \chi_{11} + \chi_{22} + \chi_{33} = 0$$

it is straightforward to see that (6.41) is of the form (6.24) which is the gauge transformation of χ_{ab}. Notice that when one makes the above choices for the components $\zeta_1, \zeta_2, \zeta_3$ in terms of the parameters p, q, r of the translational group $T(3)$, the component ζ_0 gets automatically fixed. Therefore, in the gauge transformation (6.41) generated by the representation $D(p, q, r)$ of $T(3)$ only the three space components of the field ζ_a remain arbitrary. However, in the complete set of gauge transformations (6.24) all the four components of ζ_a should be chosen independent of one another. Hence, the above gauge transformations (6.41) generated by the translational group $D(p, q, r)$ does not exhaust the complete set of gauge transformations available to the massive symmetric tensor gauge theory (6.17). This is because of the fact that in order to generate the entire gauge equivalence class of the maximally reduced polarization tensor (6.36) we require four independent variables (corresponding to the four components of the arbitrary vector function $\zeta_a(k)$ which represents the gauge freedom) whereas the translational group $T(3)$ provides only three independent parameters.

Therefore, in the present case (6.17) of massive tensor gauge theory, a partial set of gauge transformations are generated the representation $D(p, q, r)$ (3.65) of the translational group $T(3)$. The gauge transformation of the A_a-field can be obtained from that of the h_{ab}-field, though the former one does not appear in the physical spectrum of the theory.

By considering the linearized gravity in 4+1 dimensions, one may obtain through the method of dimensional descent, the above discussed results concerning the gauge transformation properties of Stückelberg extended version of EPF theory. The maximally reduced form of the polarization tensor for linearized gravity in 4+1 dimension can be obtained using plane wave method by proceeding exactly as in the 3+1 dimensional case (section 3.3) and is given by

$$\{\chi^{xy}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \chi^{11} & \chi^{12} & \chi^{13} & 0 \\ 0 & \chi^{12} & \chi^{22} & \chi^{23} & 0 \\ 0 & \chi^{13} & \chi^{23} & \chi^{33} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \chi^{11} + \chi^{22} + \chi^{33} = 0. \tag{6.42}$$

Notice now that by suitably applying the projection operator $\mathcal{P}$ =diagonal$(1, 1, 1, 1, 0)$ on this polarization tensor, one can get the maximally reduced polarization tensor (6.36). Also by a similar application of this $\mathcal{P}$ on the energy-momentum vector $k^x = (\omega, 0, 0, 0, \omega)$ of a five dimensional particle propagating along the $x = 4$ direction yields the rest frame momentum 4-vector of particle belonging to Stückelberg extended version of EPF theory. Therefore it is clear that the dimensional descent connects these two theories also.

6.3 Massive antisymmetric tensor gauge theory

Here we show that the translational group $T(3)$ generates the full range of gauge transformations in the Stückelberg extended massive Kalb-Ramond theory. Similar to the massless KR theory discussed in section 3.4, the gauge transformation in Stückelberg extended massive KR theory is also reducible. Though the analysis in this case closely resembles that of Stückelberg extended EPF theory detailed before, here the reducibility of the gauge transformation is manifested in the gauge generation by $T(3)$.

The Lagrangian of the Stückelberg extended massive KR theory is

$$\mathcal{L} = \frac{1}{12} H_{abc} H^{abc} - \frac{m^2}{4} (B_{ab} + \partial_a A_b - \partial_b A_a)(B^{ab} + \partial^a A^b - \partial^b A^a) \tag{6.43}$$

with $B_{ab} = -B_{ba}$ and $H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$. It can be easily verified that the above Lagrangian is invariant under the joint gauge transformations

$$B_{ab}(x) \to B_{ab}(x) + \partial_a F_b(x) - \partial_b F_a(x) \tag{6.44}$$

and

$$A_a(x) \to A_a(x) - F_a(x). \tag{6.45}$$

Here we must notice that the transformation (6.44) is reducible exactly as in the case of massless KR theory; i.e., the transformation (6.44) remains invariant if we make the change $F_a(x) \to F_a(x) + \partial_a \beta(x)$. Therefore, there exist a 'gauge invariance of gauge transformation' in the theory described by (6.43) also. The equation of motion corresponding to $B^{bc}(x)$ is given by

$$\partial_a H^{abc} + m^2 (B^{bc} + \partial^b A^c - \partial^c A^b) = 0 \tag{6.46}$$

and that corresponding to A^b is

$$\partial_c (B^{cb} + \partial^c A^b - \partial^b A^c) = 0. \tag{6.47}$$

As in the case of the Stückelberg extended massive theories considered previously in sections 6.1 and 6.2, the equation of motion (6.47) for A^b can be obtained from the equation (6.46) by the application of the operator ∂_c upon the latter equation. Hence, one can obtain the gauge transformation property (6.45) of the A^b-field from that of the B^{bc}-field. (This can be easily achieved by a straightforward procedure similar to the ones adopted before in the cases of Stückelberg extended Proca and EPF theories for the same purpose and hence is not elaborated here again.)

We now proceed to obtain the maximally reduced polarization tensor $\varepsilon^{ab}(k)$ corresponding to the antisymmetric field $B^{ab}(x)$ so that the role of $T(3)$ as a generator of gauge transformations in (6.43) can be studied. For this purpose, as usual we use the ansatz

$$B^{ab}(x) = \varepsilon^{ab}(k)e^{ik\cdot x}, \quad A^a(x) = \varepsilon^a(k)e^{ik\cdot x}, \quad F^a(x) = f^a(k)e^{ik\cdot x} \tag{6.48}$$

and employ the plane wave method. The momentum space gauge transformation corresponding to (6.44) now has the same form as (3.38);

$$\varepsilon_{ab}(k) \to \varepsilon_{ab}(k) + i(k_a f_b(k) - k_b f_a(k)) \tag{6.49}$$

Then the equation of motion (6.46) can be written (in the momentum space) as

$$-k^2\varepsilon^{bc} - k^b k_a \varepsilon^{ca} - k^c k_a \varepsilon^{ab} + m^2(\varepsilon^{bc} + ik^b\varepsilon^c - ik^c\varepsilon^b) = 0. \tag{6.50}$$

If $k^2 = 0$ (massless excitations), the above equations reduces to

$$-k^b k_a \varepsilon^{ca} - k^c k_a \varepsilon^{ab} + m^2(\varepsilon^{bc} + ik^b\varepsilon^c - ik^c\varepsilon^b) = 0 \tag{6.51}$$

whose solution must be of the form

$$\varepsilon^{bc}(k) = C(ik^b\varepsilon^c - ik^c\varepsilon^b) + D(\epsilon^{bcde}k_d\varepsilon_e) \tag{6.52}$$

where C and D are constants to be fixed. Substituting (6.52) in (6.51), we can easily see that $C = -1$ and $D = 0$. Therefore, the solution to (6.50) corresponding to massless excitations is

$$\varepsilon^{bc}(k) = -ik^b\varepsilon^c + ik^c\varepsilon^b. \tag{6.53}$$

However, such solutions can be gauged away by choosing the arbitrary field $F^a(x) = A^a(x)$ in (6.45), which shows that massless excitations are gauge artefacts.

Next we consider the massive case, $k^2 = M^2$, $(M \neq 0)$ where it is possible to go to the rest frame and one has the momentum 4-vector $k^a = (M, 0, 0, 0)^T$. In the rest frame, the equation of motion (6.50) reduces to

$$(m^2 - M^2)\varepsilon^{bc} - M(k^b\varepsilon^{c0} + k^c\varepsilon^{0b}) + m^2(ik^b\varepsilon^c - ik^c\varepsilon^b) = 0. \tag{6.54}$$

Note that, since the polarization tensor ε^{bc} is antisymmetric, all its diagonal entries are automatically zero. Considering the components of (6.54) for which $(b = 0, c = i)$, we have

$$\varepsilon^{i0} = iM\varepsilon^i \tag{6.55}$$

For $(b = i, c = j)$ with $i \neq j$, the equation (6.54) gives

$$(m^2 - M^2)\varepsilon^{ij} = 0. \tag{6.56}$$

This leads to two possibilities; either $\varepsilon^{ij} = 0$ or $M^2 = m^2$. The former possibility can be ruled out by the following reasoning. Since (6.43) is the first class version (obtained by a Stückelberg extension mechanism) of massive KR theory possessing three physical degrees of freedom, the former too must inherit the same number of degrees of freedom. However, the ε^{i0} elements can all be made to vanish by the gauge choice $F_a = A_a$. Therefore the possibility $\varepsilon^{ij} = 0$ leads to a null theory and hence should be discounted. Therefore, we have $M^2 = m^2$ which is also consistent with the degrees of freedom counting. Finally, analogous to (6.36), the maximally reduced form of the polarization tensor corresponding to (6.43) is given by,

$$\{\varepsilon^{ab}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{12} & \varepsilon^{13} \\ 0 & -\varepsilon^{12} & 0 & \varepsilon^{23} \\ 0 & -\varepsilon^{13} & -\varepsilon^{23} & 0 \end{pmatrix}. \tag{6.57}$$

As in the case of Stückelberg extended EPF theory, the A_a-field disappears from the physical spectrum in this case also. Here it must be emphasized that the maximally reduced form of polarization tensor of $B \wedge F$ theory also has the same form (6.57). This is not surprising since the physical sector of $B \wedge F$ theory is equivalent to massive KR theory whose first class version is the theory (6.43) under consideration now. It is now straight forward to see that the translational group $T(3)$ in the representation $D(p, q, r)$ (3.65) generates the full set of gauge transformations in the theory described by (6.43) also. The action of $D(p, q, r)$ on (6.57) is given by,

$$\{\varepsilon^{ab}\} \to \{\varepsilon'^{ab}\} = D(p, q, r)\{\varepsilon^{ab}\}D^T(p, q, r)$$

$$= \{\varepsilon^{ab}\} + \begin{pmatrix} 0 & -q\varepsilon^{12} - r\varepsilon^{13} & p\varepsilon^{12} - r\varepsilon^{23} & p\varepsilon^{13} + q\varepsilon^{23} \\ q\varepsilon^{12} + r\varepsilon^{13} & 0 & 0 & 0 \\ -p\varepsilon^{12} + r\varepsilon^{23} & 0 & 0 & 0 \\ -p\varepsilon^{13} - q\varepsilon^{23} & 0 & 0 & 0 \end{pmatrix} \tag{6.58}$$

Since in the rest frame $k^a = (M, 0, 0, 0)^T$, (6.58) can be considered to be the gauge transformations of the form (6.49) if

$$f^1 = \frac{1}{im}(q\varepsilon^{12} + r\varepsilon^{13}), \quad f^2 = \frac{1}{im}(-p\varepsilon^{12} + r\varepsilon^{23}), \quad f^3 = \frac{-1}{im}(p\varepsilon^{13} + q\varepsilon^{23}) \tag{6.59}$$

Here one must clearly note that the component f^0 remains completely undetermined and does not depend at all either on the parameters p, q, r of $T(3)$ or on the components of maximally reduced polarization tensor of the theory whereas the other components f^1, f^2, f^3 are determined by these parameters and the elements of the polarization tensor. Interestingly, it is exactly in the same fashion as in the present case (of Stückelberg extended massive KR theory) that gauge transformations of $B \wedge F$ theory are generated by the translational group $D(p, q, r)$ (section 3.5). Hence, analogous to the gauge transformation generated by $W(p, q)$ in massless KR theory, for any given set of (f^1, f^2, f^3) we have a continuum of values for f^0, representing the reducibility of the gauge transformation in the underlying 2-form field both in Stückelberg extended first class version of massive KR theory and in the $B \wedge F$ theory. Therefore, the complete independence of the time-component of f^a on the maximally reduced polarization tensor or on the parameters of the group $D(p, q, r)$ is a consequence of the reducibility of the gauge transformations of these theories.

Dimensional descent relates the gauge transformations in 5+1 dimensional massless KR theory to those in the presently discussed Stückelberg extended massive KR theory just the same way it is related to the gauge transformation of the antisymmetric field of $B \wedge F$ theory.

6.4 summary

We showed that, just like in the case of topologically massive $B \wedge F$ theory, translational group $T(3)$ acts as generators of gauge transformations in gauge theories obtained by Stückelberg embeddings of massive theories having no gauge symmetry. We illustrate these with the examples of Stückelberg extended first class versions of Proca, Einstein-Pauli-Fierz and massive Kalb-Ramond theories in 3+1 dimensions. In each of these cases, we have shown that the representation $D(p, q, r)$ of $T(3)$ generates gauge transformation when acted suitably on the corresponding maximally reduced polarization tensor. The reducibility of the gauge transformations transformation in Stückelberg extended massive Kalb-Ramond theory is manifested clearly in the gauge generation by the translational group $T(3)$. With suitable 4+1 dimensional theories as the respective starting points, dimensional descent can be applied consistently in all the Stückelberg embedded models considered here.

Chapter 7

Conclusions

We have studied several aspects of a variety of planar field theories with particular emphasis on topologically massive gauge theories and the role of translational groups in generating gauge transformations. Attention is also focused on the interrelationship between the different theories. We considered Abelian models with vector, symmetric and antisymmetric second rank tensors as underlying fields. Our methodology relied crucially on the maximally reduced form (i.e., representing just the physical sector) of the polarization vectors and tensors of these theories. The maximally reduced polarization vector/tensor is usually derived by considering a plane wave solution corresponding to a single component in the mode expansion of the basic field of the respective theory. Hence this method of derivation is named the 'plane wave method. In this method, the plane wave solution is substituted in the equation of motion of the theory and confining to a particular reference frame we obtain the explicit form of the maximally reduced polarization vector/tensor corresponding to that frame, using mainly the structure of equation motion itself and finally making a gauge transformation. The plane wave method is used for the derivation of the maximally reduced form of the polarization vector/tensor of every model considered in this work.

To begin with we obtained, using the plane wave method, the polarization vectors of Maxwell-Chern-Simons and Maxwell-Chern-Simons-Proca models in both Lagrangian Hamiltonian frameworks with compatible results. In Lagrangian formalism the polarization vector of MCS theory is calculated in the Lorentz gauge while in the Hamiltonian formalism, the calculation is done in a variety of covariant gauges and it is found that the maximally reduced polarization vector has the same form in all these gauges. The structure of the polarization vectors explicitly display the doublet structure of Maxwell-Chern-Simons-Proca theory consisting of a pair of Maxwell-Chern-Simons models with opposite helicities and different mass parameters. The polarization vector of self-dual model is of exactly the same form as the maximally reduced polarization vector of a Maxwell-Chern-Simons theory with positive helicity whereas there is a similar correspondence between antiself-dual model and a Maxwell-Chern-Simons theory with negative helicity. This is in agreement with the well known equivalence of self and antiself-dual models with a pair of Maxwell-Chern-Simons theories with opposite parities.

We have also made a comparison of the polarization vectors of Maxwell and Maxwell-Chern-Simons theories in different gauge choices and in different reference frames. The time component of the Coulomb gauge polarization vector of Maxwell-Chern-Simons theory is undefined in the rest frame. Like Maxwell theory, basic field in Maxwell-Chern-Simons model can be made to satisfy the spatial transversality condition $\mathbf{k} \cdot \mathbf{A} = 0$ in a boosted frame provided it undergoes a suitable gauge transformation. However, unlike Maxwell theory, the temporal gauge condition $A^0 = 0$ cannot be satisfied simultaneously by Maxwell-Chern-Simons field except in the ultra-relativistic limit, whereas it blows up in the rest frame. In comparison, the Maxwell-Chern-Simons polarization vector satisfying Lorentz gauge condition is well defined in all reference frames including the rest frame. Notice that rest frame is not available in Maxwell theory.

We reviewed, with necessary details, the role of translational subgroup $T(2)$ of Wigner's little group for massless particles in generating gauge transformations in 3+1 dimensional massless gauge theories - Maxwell, massless Kalb-Ramond and linearized gravity theories. In all these cases, it is demonstrated that the transformation of the respective maximally reduced polarization vector/tensor under the representation of $T(2)$ inherited from defining representation of the little group amounts to a gauge transformation in momentum space. Our analysis, however showed that, the gauge transformations generated by the translational group in massless Kalb-Ramond and linearized gravity constitute only certain subsets of the full spectra of gauge transformations available in these two theories. This is attributed to the fact that the gauge freedoms in these second rank tensor theories are represented by arbitrary vector fields with four components while the translational group provides only two parameters. Because of this deficit in the number of parameters the range of gauge transformations generated by $T(2)$ become restricted. We also reviewed the gauge generation by the translational group $T(3)$ in the 3+1 dimensional topologically massive $B \wedge F$ gauge theory. Both massless Kalb-Ramond and $B \wedge F$ theories have reducible gauge transformations. We have shown that the reducibility of gauge transformations is manifested clearly in the gauge generation by translational groups.

Just like $T(2)$ generates gauge transformations in 3+1 dimensional massless theories, translational group $T(1)$ acts as generator of gauge transformations in 2+1 dimensional Maxwell theory. (Massless Kalb-Ramond and linearized gravity possess no physical degrees of freedom in 2+1 dimensions.) We show that $T(1)$ generate gauge transformations in the topologically massive planar gauge theories namely Maxwell-Chern-Simons and Einstein-Chern-Simons theories. A suitable representation of $T(1)$ when acted on the respective maximally reduced polarization vector and tensor generate momentum-space gauge transformations in these theories. The polarization tensor of Einstein-Chern-Simons theory is found to be the direct product of the polarization vector of a Maxwell-Chern-Simons theory with itself.

It has been known that the gauge transformations in a topologically massive theory living in a certain space-time dimensions can be related to those of massless gauge theories in a space-time of one higher dimension through the method of dimensional descent. We presented a short review of dimensional descent from 4+1 dimensional Maxwell and massless Kalb-Ramond theories to 3+1 dimensional $B \wedge F$ theory and derived the gauge generating representation of $T(3)$ with the help of the gauge transformation properties of these higher dimensional theories. Subsequently we have derived, using dimensional descent, the previously mentioned gauge generating representation of $T(1)$ for Maxwell-Chern-Simons and Einstein-Chern-Simons theories in 2+1 dimensions by considering the gauge transformation properties of 3+1 dimensional Maxwell and linearized gravity theories respectively. In this process, we have also considered the massive non-gauge Einstein-Pauli-Fierz theory in 2+1 dimensions whose basic field transforms like a second rank symmetric tensor. A comparison of the polarization tensors of Einstein-Pauli-Fierz theory with those of a pair of Einstein-Chern-Simons theories having opposite helicities suggested a doublet structure for Einstein-Pauli-Fierz theory. This is analogous to the vector case in which Proca theory in 2+1 dimensions is equivalent to a doublet of Maxwell-Chern-Simons theories with opposite helicities. In connection with this analogy, it should be noted that the relative and overall signs of the Einstein term as well as the Pauli-Fierz term in the Einstein-Pauli-Fierz theory should be the same as the conventional ones for the theory to be meaningful. However, the sign of the Einstein-term in Einstein-Chern-Simons theory should be opposite to that the conventional one. Therefore, a theory with all three terms present simultaneously (the tensor analogue of Maxwell-Chern-Simons-Proca theory) is not possible because of the unavoidable conflict of requirements on the signs of the Einstein term. Therefore, the suggestion that Einstein-Pauli-Fierz theory is a doublet of Einstein-Chern-Simons theories needs further investigation.

Next we have considered, the role of translational group in generating gauge transformations in some massive gauge theories obtained by elevating massive non-gauge theories to their first class versions by Stückelberg embedding mechanism. Such theories considered in this work are the Stückelberg extended versions of Proca, Einstein-Pauli-Fierz and massive Kalb-Ramond theories which have vector, symmetric and antisymmetric second rank tensors respectively as the basic fields. The investigation of these theories is done in 3+1 dimensional space-time in which such models are usually studied. However, with suitable modifications the methods and results of our study of these theories are applicable to the corresponding 2+1 dimensional theories as well. We have shown that the representation of $T(3)$ that generates gauge transformation in $B \wedge F$ theory also generates gauge transformations in the above mentioned massive gauge theories in 3+1 dimensions. Analogous to the case of linearized gravity, the gauge transformations generated in the Stückelberg extended Einstein-Pauli-Fierz theory by $T(3)$ constitute only a subgroup of the full set of gauge transformations in the theory. The gauge transformations in the Stückelberg extended massive Kalb-Ramond theory are reducible and this reducibility is clearly exhibited in the gauge generation by the translational group.

Appendix A

Wigner's Little group in 3+1 dimensions

Wigner's little group preserves the energy-momentum vector of a particle and is a subgroup of homogeneous Lorentz group. Here we give a derivation of the explicit expression for the little group for a 3+1 dimensional massless particle [19]. Let

$$W = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \end{pmatrix} \tag{A.1}$$

be a representation of an arbitrary element of the little group that preserves the momentum 4-vector of a massless particle of energy ω propagating along the z-direction:

$$W^a{}_b k^b = k^a, \qquad k^a = \omega(1,0,0,1)^T. \tag{A.2}$$

With $l^a = (1,0,0,1)^T$, we obviously have $W^a{}_b l^b = l^a$ also. Consider a time-like vector $\alpha^a = (1,0,0,0)^T$ which has the following obvious Lorentz covariance properties,

$$(W\alpha)^a(Wl)_a = (W\alpha)^a l_a = \alpha^a l_a = 1, \tag{A.3}$$

$$(W\alpha)^a(W\alpha)_a = \alpha^a\alpha_a = 1. \tag{A.4}$$

Since $(W\alpha)^a = (m_1, p_1, q_1, r_1)^T$ (which is nothing but the first column of W), one can see using (A.3) that $r_1 = m_1 - 1$. Therefore, the form of $(W\alpha)^a$ that satisfies the relation (A.4) is given by

$$(W\alpha)^a = \left(1 + \frac{p_1^2+q_1^2}{2}, p_1, q_1, \frac{p_1^2+q_1^2}{2}\right)^T. \tag{A.5}$$

Now, W acting on the space-like unit vector $\beta^a = (0,0,0,1)^T$ yields $(W\beta)^a = (m_2, p_2, q_2, r_2)^T$. Analogous to (A.3) and (A.3) in this case we have

$$(W\beta)^a(Wl)_a = (W\beta)^a l_a = \beta^a l_a = -1, \tag{A.6}$$

$$(W\beta)^a(W\beta)_a = \beta^a\beta_a = -1. \tag{A.7}$$

It is easy to see that (A.6) and (A.7) restrict the form of $(W\beta)^a$ to $\left(-\frac{p_4^2+q_4^2}{2}, p_4, q_4, 1 - \frac{p_4^2+q_4^2}{2}\right)^T$ which, along with (A.6), when substituted in the relation

$$(W\alpha)^a(W\beta)_a = \alpha^a\beta_a = 0 \tag{A.8}$$

yields $(p_1+p_4)^2 + (q_1+q_4)^2 = 0$. This implies that $p_1 = -p_4$ and $q_1 = -q_4$. Thus the fourth column of W (A.1)is given by

$$(W\beta)^a = \left(-\frac{p_1^2+q_1^2}{2}, -p_1, -q_1, 1 - \frac{p_1^2+q_1^2}{2}\right)^T. \tag{A.9}$$

In a similar fashion, by considering another space-like unit vector $\gamma^a = (0,0,1,0)^T$ we can obtain $(W\gamma)^a = (m_3, p_3, q_3, r_3)^T$, the third column of W. A simple algebra using the properties

$$(W\gamma)^a(Wl)_a = (W\gamma)^a l_a = \gamma^a l_a = 0, \quad (W\gamma)^a(W\gamma)_a = \gamma^a\gamma_a = -1 \tag{A.10}$$

leads to the condition $p_3^2+q_3^2 = 1$ which enables one to make the parametrization $p_3 = \sin\phi,\quad q_3 = \cos\phi$. Hence we have $(W\gamma)^a = (m_3, \sin\phi, \cos\phi, m_3)^T$. Furthermore, since

$$(W\beta)^a(W\gamma)_a = \beta^a\gamma_a = 0 \tag{A.11}$$

we obtain the relation $m_3 = p_1\sin\phi + q_1\cos\phi$ wherein we used (A.9) and the form $(W\gamma)^a$ mentioned above. Therefore

$$(W\gamma)^a = (p_1\sin\phi + q_1\cos\phi, \sin\phi, \cos\phi, p_1\sin\phi + q_1\cos\phi)^T. \tag{A.12}$$

Lastly, we introduce the spacelike vector $\delta^a = (0,1,0,0)^T$ so that $(W\delta)^a = (m_2, p_2, q_2, r_2)^T$ is the second column of W (A.1). Then, just as in the previous case, using the covariance properties

$$(W\delta)^a(Wl)_a = (W\delta)^a l_a = \delta^a l_a = 0,$$

$$(W\delta)^a(W\delta)_a = \delta^a\delta_a = -1,\quad (W\delta)^a(W\alpha)_a = \delta^a\alpha_a = 0$$

we get $(W\delta)^a = (p_1\cos\theta + q_1\sin\theta, \cos\theta, \sin\theta, p_1\cos\theta + q_1\sin\theta)^T$. Contracting with $(W\gamma)_a$ and using the relation

$$(W\delta)^a(W\gamma)_a = \delta^a\gamma_a = 0 \tag{A.13}$$

we deduce that $\theta = -\phi$. Therefore we have

$$(W\delta)^a = (p_1\cos\phi - q_1\sin\phi,\quad \cos\phi,\quad -\sin\phi,\quad p_1\cos\phi - q_1\sin\phi)^T. \tag{A.14}$$

From (A.5), (A.9), (A.12) and (A.14) we then obtain the representation of the little group for a massless particle in 3+1 dimensions as

$$W(p,q;\phi) = \begin{pmatrix} 1+\frac{p^2+q^2}{2} & p\cos\phi - q\sin\phi & p\sin\phi + q\cos\phi & -\frac{p^2+q^2}{2} \\ p & \cos\phi & \sin\phi & -p \\ q & -\sin\phi & \cos\phi & -q \\ \frac{p^2+q^2}{2} & p\cos\phi - q\sin\phi & p\sin\phi + q\cos\phi & 1-\frac{p^2+q^2}{2} \end{pmatrix} \tag{A.15}$$

dropping the subscripts on p, q which are any real numbers. This little group can be written as

$$W(p,q;\phi) = W(p,q)R(\phi) \tag{A.16}$$

where

$$W(p,q) \equiv W(p,q;0) = \begin{pmatrix} 1+\frac{p^2+q^2}{2} & p & q & -\frac{p^2+q^2}{2} \\ p & 1 & 0 & -p \\ q & 0 & 1 & -q \\ \frac{p^2+q^2}{2} & p & q & 1-\frac{p^2+q^2}{2} \end{pmatrix} \tag{A.17}$$

is a particular representation of $T(2)$ - the group of translations in a plane and $R(\phi)$ represents a $SO(2)$ rotation in the plane. It is clear that $W(p,q)$ and are Abelian subgroups of the little group $W(p,q;\phi)$:

$$W(p,q)W(\bar{p},\bar{q}) = W(p+\bar{p}, q+\bar{q}) \tag{A.18}$$

$$R(\phi)R(\bar{\phi}) = R(\phi+\bar{\phi}) \tag{A.19}$$

Moreover, the subgroup $W(p,q)$ is invariant since

$$R(\phi)W(p,q)R^{-1}(\phi) = W(p\cos\phi + q\sin\phi, -p\sin\phi + q\cos\phi). \tag{A.20}$$

Therefore, the little group is not semi-simple. The group generators are given by

$$A = -i\frac{\partial W(p,0;0)}{\partial p}\Big|_{p=0} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{A.21}$$

$$B = -i\frac{\partial W(0,q;0)}{\partial q}\Big|_{q=0} = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{A.22}$$

$$J_3 = -i\frac{\partial W(0,0;\phi)}{\partial \phi}\Big|_{\phi=0} = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.23}$$

It is important to note that the generators A and B can be expressed as a combination of the generators of boosts and rotations[1]:

$$A = J_2 + K_1, \quad B = -J_1 + K_2. \tag{A.24}$$

The Lie algebra of the little group is given by

$$[A, B] = 0, \quad [J_3, A] = iB, \quad [J_3, B] = -iA. \tag{A.25}$$

This is identical to the algebra of $E(2)$ - the Euclidean group in two dimensions, comprising of two mutually commuting translation generators (corresponding to A and B) and a generator of rotation in the plane (J_3). Thus, the algebra of the 3+1 dimensional little group for massless particles is isomorphic to the $E(2)$ algebra [70].

[1]We denote by J_i $(i = 1, 2, 3)$ the generator of rotation about the j-th axis. Similarly, the generator of boost along x_i-direction is denoted by K_i. In this notation, the algebga of the homogeneous Lorentz group is given by $[J_i, J_j] = i\epsilon_{ijk}J_k$, $[J_i, K_j] = i\epsilon_{ijk}K_k$, $[K_i, K_j] = -i\epsilon_{ijk}J_k$.

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