RENORMALIZATION GROUP AS A PROBE FOR DYNAMICAL SYSTEMS

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To
Baba and Mee

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LIST OF PUBLICATIONS

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- **A. Sarkar** and J. K. Bhattacharjee, "*Renormalization Group for nonlinear oscillators in the absence of linear restoring force*" EPL **91** 60004 (2010).
- **A. Sarkar** and J. K. Bhattacharjee, "*Renormalization Group and isochronous oscillators*" Eur. Phys. Jour. D **66** 1-6 (2012).
- **A. Sarkar**, P. Guha, A. Ghose-Choudhury, J. K. Bhattacharjee, A. K. Mallik and P. G. L. Leach, "*On the properties of a variant of the Riccati system of equations*" Jour. Phys. A: Math. Theor. **45**(41), 415101. (2012)
- S. Chakraborty and **A. Sarkar**, "*Oscillators with Parametrically Excited Nonlinearity: Resonance, Anti-resonance and Switch*", arXiv preprint arXiv:1202.6592 (2012).
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- J. K. Bhattacharjee, Sagar Chakraborty, and **Amartya Sarkar**. "*A methodology for classifying periodic orbits*." In Chaos Theory: Modeling, Simulation and Applications, Selected Papers from the 3rd Chaotic Modeling and Simulation International Conference (CHAOS2010) Chania, Crete, p. 120. World Scientific Publishing Company Incorporated, 2011.
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Contents

Chapter 1

Introduction

1.1 Dynamical systems

The study of dynamical systems has fascinated physicists for hundreds of years. It was through Newton's attempts to understand the celestial dynamics of bodies in our solar system that led to the development of calculus [1, 2]. After the advent of calculus the study of dynamical systems modelled through differential equations of motion constituted the major part of dynamical systems study. Despite the elegance and simplicity of such equations of motion, the solution to specific problems has proved to be quite problematic and has engaged the greatest minds in both physics and mathematics over the last few centuries. In particular, solutions to the nonlinear differential equations have proved to be remarkably difficult [3–5]. Apart from obtaining approximate analytical solutions through certain perturbative techniques for weakly nonlinear systems [6, 7], the area of nonlinear dynamical systems remains largely inaccessible to theoretical analyses. In late nineteenth century Poincaré [8] in a seminal work showed that perturbation methods may not always yield correct results, because the series used in such cases diverged. Poincaré then went on to use techniques of analysis coupled with principles of geometry to lay the foundation stones of modern qualitative approach [9, 10] to study differential equations. The modern methods of qualitative analysis of dynamical systems were further enriched by the works of Birkhoff [11], Liapunov [12] and others of the Russian School like Andronov [13], Arnold [14] etc. In 1967 Smale [15] in a landmark paper outlined a number of outstanding problems in the field and in another classic paper, Ruelle and Takens [16] showed the importance of strange attractors in study of complex phenomena like turbulence. Following these works the 1970s and 80s saw a remarkable increase in the volume of research in the field of nonlinear dynamics.

Over the past few years there has been huge volume of research in the field of nonlinear dynamical

systems. A major portion of this research are applications in solid and structural mechanics [17–20] as well as fluid dynamics and the phenomenon of turbulence [21–24]. Apart from this further applications have appeared in the fields of population dynamics [25–27], chemical oscillations [28–30], network theory [31, 32], magneto-hydrodynamics [33] etc and following which the interest in dynamical systems theory has grown in the various disciplines such as quantum physics, chemical physics, biophysics, material sciences, ecology, astrophysics, engineering and applied sciences etc. Consequently a number of researches have concentrated on developing analytical tools for differential equations in general. However, the major trouble with most nonlinear differential equations is that almost always one can't find an exact solution to them. To illustrate this point, after Newton solved the two-body problem (the Earth moving around the Sun), people tried to extend his analytical methods to treat the three-body problem (the motion of the Earth, the Sun and the Moon) but oddly enough it turned out to be extremely difficult to solve. Eventually it was established that the three-body problem was essentially impossible to solve exactly i.e. one can't find explicit expression describing the motion of the bodies [34]. To get around this impasse one has to resort to the modern qualitative approach to dynamical systems, pioneered by Poincaré [8–10]. The trick is instead of asking quantitative questions like the exact position of a body at a certain time one can ask - "Is the system going to be stable forever or will one of the bodies eventually fall out of the system and fly-off into infinity?" It turns out a geometrical approach coupled with tools of analysis more often than not is able to answer such questions satisfactorily.

We now focus our attention specifically to differential equations. The first classification of differential equations that one may achieve is by distinguishing between ordinary and partial differential equations. For example the equation for a damped harmonic oscillator,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega^2 x = 0, \tag{1.1}$$

is an ordinary differential equation (ODE) involving only ordinary derivatives and only one independent variable, the time t. On the other hand, the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

is a typical example of a partial differential equation having both time t, position x as independent

variables. A very general framework for ODEs is provided by the following system,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,...,x_n), \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{x}_n &= f_1(x_1,...,x_n), \end{aligned} \tag{1.3}$$

where the overdots denote differentiation with respect to t. The variables x_i might represent position of an oscillator, concentrations of chemicals in a reaction, population of different species etc. The functions f_i are usually determined by the problem. All differential equations may be cast in the above form, for example Eq. (1.1), may be written as,

$$\dot{x} = y, \tag{1.4}$$

$$\dot{y} = -\frac{b}{m}y - \frac{k}{m}x. \tag{1.5}$$

The above equation is an example of a *linear* differential equation as all the variables on RHS appear to the first power only. If on the other hand terms of higher powers (x^2, x^3,.. etc) or product of terms (xy, x^2y,... etc) or functions ($\sin(x), \cosh(y)$,... etc) of the variables had occurred on the RHS the equation would have been *nonlinear*. A typical example is the ODE describing the motion of a pendulum, given by

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0, \tag{1.6}$$

where θ is the angle of the pendulum from vertical, g is the acceleration due to gravity and L the length of the pendulum. The presence of the *nonlinear* term makes the problem very difficult to solve analytically. What is usually done is to invoke the small angle approximation i.e. for small values of the θ, $\sin\theta \approx \theta$, such that the problem is rendered a linear one. Doing so however one throws out a lot of interesting physics like the motion of the pendulum whirling over the top. It turns out though that the pendulum equation can be solved exactly in terms of elliptic functions although it involves a lot of cumbersome mathematics. In contrast however the problem itself is not too complicated physically. One feels here that there ought to be an easier way to solve the problem and this is where the geometric approach comes into play.

One can imagine the so called *phase space* for any general system of equations (Eq.(1.3)). For a 2-dimensional system, like our pendulum problem which can be rewritten in terms of new variables

($\theta = x_1$ and $\dot{\theta} = x_2$),

$$\dot{x}_1 = x_2, \tag{1.7}$$
$$\dot{x}_2 = \frac{g}{L}\sin x_1; \tag{1.8}$$

$x_1 - x_2$ constitute the phase space. The solution would have to be in terms of two functions $x_1(t)$ (representing the angular displacement) and $x_2(t)$ (representing the angular velocity of the pendulum). We now construct an abstract space with coordinates (x_1, x_2) and in this space the solution would be a point moving along a path, called the *trajectory*. The phase space is filled with trajectories as each point in it may serve as the initial condition. And in principle if we knew the trajectories we would have solved the problem. The geometric approach aims to draw the trajectories without actually solving the system i.e. given a system of equations geometric reasoning alone allows us to draw the trajectories and in the process extracting most of the relevant information about the system.

A phase space for the general system Eq. (1.3) is a space with all the $x_1, ..., x_n$ as coordinates. This is referred to as *n-dimensional system* or a *nth-order system*, n representing the dimension of the phase space. To make things clearer for a body executing motion in 3-D Euclidean space may be represented in a 6-D phase space where the coordinates are the positions in the independent variables x, y and z and the respective velocities i.e. $(\dot{y}, \dot{z})$. The Eq. (1.3), however is not the most general system possible as it does not include explicit time-dependence. Such systems which don't have explicit time-dependence are called *autonomous systems* and on the other hand systems with explicit time-dependence are called *non-autonomous systems*. For example the forced harmonic oscillator,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega^2 x = f\cos\Omega t, \tag{1.9}$$

is an example of a non-autonomous system. To deal with such systems using the phase space technique involves a very simply trick. Every n-th order non-autonomous system may be cast as an $(n+1)$-th order autonomous system of higher dimensionality by introducing new variables. For example let $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = t$, and we can rewrite Eq. (1.9) as,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 + f\cos x_3 \\ \dot{x}_3 &= 1. \end{aligned} \tag{1.10}$$

Now a short summary of the various analytical tools available in literature to find approximate solutions to nonlinear differential equations is in order here. Given any dynamical system the first

thing that one can do is find out its *fixed points*. A *fixed point* of a general system described by Eq. (1.3) is defined as a point (x_1^*,..., x_n^*) such that all $\dot{x}_i = 0$. Physically speaking if a body is sitting at a fixed point of the dynamics it won't move unless perturbed externally. For example, for the pendulum problem described by

$$\dot{x}_1 = x_2, \tag{1.11}$$
$$\dot{x}_2 = \frac{g}{L}\sin x_1, \tag{1.12}$$

($x_1 = x_2 = 0$) is such a point. A fixed point may be *stable* (also called *attractors)* or unstable (also called *sinks*) depending on if the system perturbed from the fixed point comes back to the point or moves away from it, respectively. The simplest thing one can do to check the stability of a fixed point is to assume a small perturbation and check whether the perturbation grows in time or dies down. For a nonlinear differential equation assuming a "small" perturbation allows one to ignore the nonlinear terms and work only with the leading linear terms. This method is the so called *linear stability analysis* and arguably one of simplest yet powerful tool in our hands. Furthermore over the decades a number of methods for obtaining approximate solutions to differential equations have been developed. To mention a few, there is the Lindstedt-Poincaré method, harmonic balance *etc*, multiple scales analysis, Bogoliubov-Krylov method [3–6] etc. More recently a number of new methods have been proposed, *e.g.* non-perturbative method [35, 36], δ-method [37], homotopy perturbation method [38], variational iteration methods [39] *etc*.

1.2 Renormalization group

The theoretical basis of a remarkably large part of physics as we know today can't be fathomed unless one has a minimal understanding of the quantum and statistical field theories. Interestingly the quantum field theories and statistical field theories turn out to be quite related formally (can be related through a continuation to imaginary time) [40–42]. Field theories in its various versions describe the fundamental interactions at the microscopic scale, singular properties of phase transitions at the transition point (e.g. liquid-vapor, superfluid, ferromagnetic, etc), statistical properties of long polymer chains, surface growth and so on. Essentially the field theories provide the most robust framework to describe physical systems characterized by a large number of strongly interacting local *degrees of freedom*. However at its birth, quantum field theory (QFT) was plagued by a rather unexpected problem — the problem of *infinities*. The calculation of most physical results led to infinite results, a problem which was finally solved with the development of the concept of renormalization group.

1.2.1 Renormalization group and quantum field theory

The origin of the concept of Renormalization Group (RG) theory can be traced back right to the early days of development of Quantum Field theory (QFT). For example the quantum mechanics of hydrogen atom posed enormous mathematical difficulties and this lead to the development of QFT. The fine structure of hydrogen atom's spectral lines eventually required the implementation of RG to resolve the problem of infinities . The energy gap between the $2s$ and $2p$ levels of the hydrogen atom, known as the *Lamb shift* [43] was something which came out to be infinite in most of the existing theories. It was Hans Bethe [44] who in his seminal 1947 paper first calculated the Lamb shift and his result was a finite and accurate one [45,46]. This was probably the first instance where renormalization group in its modern perturbative sense was used successfully. While it was true that calculating various physical quantities lead to divergent answers, physicists quickly noticed that it was always the same kind of divergent contributions. One could even in a few cases find a combination which were finite (Weisskopf [47]). In fact Bethe in his calculation of Lamb shift organized the terms so that the infinities cancelled out and the result turned out to be in excellent agreement with experiments. However the physical meaning of this property of cancellation of infinities remained obscure. In 1948 Schwinger solved another troublesome problem [48] — calculating the leading order contribution to the anomalous magnetic moment of the electron. Finally in 1949 Dyson [49], based on works of Feynman, Schwinger and Tomonaga, [48,50,51] put forth a formal proof of cancellation of infinities to all orders of the perturbative expansion. It was clear that the appearance of infinities in QFT was a "cosmetic" problem arising out of improper summation techniques and people quickly realized the correct way "resum" such divergent series and this was the advent of renormalization group methods. Following these developments, using the concept of renormalization in QED people were able to calculate finite results for all physical observables, many of which were experimentally verified and they turned out to be in excellent agreement with theory [52–54].

The general approach is to start with the so called *bare theory* which depends on certain parameters called *bare parameters*. For example the *bare mass* m_0 or *bare fine structure constant* α_0 in case of electron (values in the absence of interactions). Then one introduces in a somewhat unphysical brute force manner a large-momentum-cutoff Λ (which modifies the theory at very short distance $\hbar/\Lambda$). One can now calculate the *renormalized* (measurable) values of the same quantities — *mass* and *fine structure constant* as functions of the bare parameters and the cutoff. The relations so obtained are then inverted to obtain expressions for bare parameters in terms of renormalized ones. The next step is to replace the bare quantities by their renormalized counterparts in the expressions for any other physical observables. The remarkable thing is that in the infinite cut-off Λ limit one obtains finite values for the physical quantities. This prescription has allowed for rather high precision (now experimentally

verified) calculations in QED proving beyond doubt that renormalized QFT is the proper theory to describe electrodynamics in the microscopic scale. Finally it turned out that not all field theories lead to finite results using this process. This led to the important concept of *renormalizable field theories* restricting the structures of physically meaningful field theories.

Although the above described recipe for renormalization met with huge success yet the physical meaning of renormalization or for that matter that of the bare quantities remained obscure. Several attempts were made to understand it. It was thought the divergences occurred due to an unjustified perturbative expansion and a correct summation of expansion in powers of the small parameters would take care of the problem. This was basis for development of so called *axiomatic QFT* which advocated development of non-perturbative approaches from the basic tenets of QFT. Another line of thought was that only renormalized perturbation theory was meaningful and initial bare theories have no physical significance. This led to the BPHZ (Bogoliubov, Parasiuk, Hepp, Zimmerman) formalism [55–57], where the problem of divergences in position space was reduced to mathematically correct definition of singular products of distributions and as a result disguised the problem of divergences as never having existed at all. The more popular approach though was to attribute a physical meaning to the cut-off; a physical meaning that is generated by such physical interactions which can't be described by QFT. Renormalizable theories then can be thought of as theories insensitive to an as yet unknown interaction at shorter length scales. The physical meaning of the renormalization procedure became clearer through the works of Peterman and Stuckelberg(1953) [58], Gell-Mann and Low(1954) [59] and Bogoliubov and Shirkov [60] (1956). These authors noticed that in case of field theories for massless particles like photon, the renormalized perturbation expansion has a peculiar property, something which is a direct consequence of the renormalization process itself. In massive theory, the renormalized charge can be defined through coulomb interaction at rest but this definition can't obviously be applied to massless particles which always travel at speed of light. It then became necessary to introduce an arbitrary mass (or equivalently an energy or a momentum) scale [59] μ to define renormalized charge e. The renormalized charge is related to observed strength of electromagnetic interaction at energy scales of order μ and thus one can in some sense call renormalized charge as an *effective* charge at scale μ. But since the scale μ is arbitrary, one can find other pairs (e', μ') of such scales and corresponding coupling constants (e in this case) which give the same physical results. The set of transformations of the coupling constants (physical parameters) with the change of scale μ, necessary to keep physics constant was called the *Renormalization group* (RG). The variation of the effective couplings (g) with the infinitesimal change of scale was given by a *flow equation*:

$$\frac{dg(\mu)}{d\mu} = \beta(g(\mu)); \qquad \beta(g) = \beta_2 g^2 + \mathcal{O}(g^3) \tag{1.13}$$

As a matter of fact the concept of effective (or renormalized) charge (or coupling) can be introduced for massive theory too. The effective charge in such a scenario might be understood in the following manner: at large distances the intensity of electromagnetic interaction does not vary and the charge has the value measured through Coulomb interaction. However, at distances much shorter than the wavelength $\hbar/mc$ (Compton wavelength), one explores in some sense the interior structure of the particle, leading to screening effects. Due to the Heisenberg uncertainty relations, probing of length scales smaller than the Compton wavelength required energies higher than mc^2 and which allows the possibility of creation of new particles. One of the earliest success of renormalization group (RG) method came when Bogoliubov and Shirkov resolved the so-called "ghost-problem" for renormalizable local QFT models [61]. Following this there were further important results obtained via studies such as the theoretical discovery of the "asymptotic freedom" of non-Abelian vector models [62, 63]. It suffices to say that RG method has allowed for exceedingly precise calculations and predictions be done in QFT. The remarkable agreement between theory and experiment demonstrates that renormalized field theories are the suitable framework to describe electrodynamics in the quantum regime.

1.2.2 Renormalization group and critical phenomena

The next big leap for RG method came from developments in the field of critical phenomena in statistical mechanics. Critical phenomena theory deals with the second order phase transitions or more precisely continuous phase transitions [64, 65]. Physical examples of such phenomena include liquid-vapor transitions, superfluid He-normal fluid transition, magnetic transitions etc. The celebrated Ising model is a simple lattice model which exhibits 2nd order phase transition and critical behavior. These transitions are characterized by collective behavior on large scales near the transition (or critical) temperature (T_c), for example, the correlation length (length scale of the collective behavior) diverges at the critical temperature T_c. Typically such systems depend on two different length-scales near the critical temperature, one being the microscopic length-scale given by the lattice spacing or the range of the forces involved and the other being a dynamic length-scale — the correlation length. The latter scale is the one associated with the large-scale macroscopic phenomena. It is thus fair to assume that the physics near the critical point could be described by a few macroscopic parameters without explicitly worrying about all the initial degrees of freedom. This concept led to the development of the *mean field theories* (MFT).

In the development of theory of critical phenomena the study of two particular physical systems play a central role namely the liquid-vapor system and the ferromagnet. In this context two classical theories (both essential MFTs) — that of liquid-vapor system due to Van der Waals and that for the ferromagnet due to Weiss met with considerable success [66–68]. These theories are *classical* in

the sense that these ignore the "thermal fluctuations", an approximation used in all but a few exactly solvable models. The classical theories due to Van der Waals and Weiss were both successful in many ways as they were able to explain a number of (not all) qualitative features of the phase transitions. As it turned out that these two theories were mathematically identical near the critical point. This particular observation is the basis of Landau's theory of phase transition [64,69,70] which is essentially the most general form of the classical theories (MFTs). Such theories are *mean field* in the sense, a physical variable such as magnetization is replaced by its average value, and the fluctuations about that value are ignored. Such theories lead to some very robust predictions — such as the universality of the singular behavior of thermodynamical quantities at the critical temperature T_c: for example the correlation length ξ always diverges as $(T-T_c)^{-1/2}$ and the spontaneous magnetization M vanishes like $(T_c-T)^{1/2}$. Furthermore these properties were independent of spatial dimension or the details of the microscopic dynamics of the system. Landau's theory stipulates that the physics near a critical point can be described, in some leading approximation, by a few effective macroscopic parameters. Its a quasi-Gaussian theory, in the sense that the remaining correlations between stochastic variables at the microscopic scale can be treated perturbatively. Thus, the macroscopic expectations are given by quasi-Gaussian distributions, in the spirit of central limit theorem.

However, soon after some experiments as well as numerical simulations in simple lattice models were found not to agree with mean field answers. In 1944, Onsager's exact solution to 2D Ising model [71], dealt a further blow to the MFTs and in the years to come it was proved beyond doubt that critical phenomena in two and three dimensions can't be described quantitatively by MFTs. It was found that near the critical point, fluctuations were not negligible [65] and thus MFTs were not self consistent. It turned out that the critical behavior depends on space dimensions, symmetries and some general properties of the models. And yet not all of the universality was lost, there were indications that universality in a more restricted sense still remained. The so called *scaling laws*, first arrived at by B. Widom [72] from phenomenological considerations indicated that there still was some universality in critical phenomena. Widom showed that the equation of state of a physical system near a critical point obeys certain scaling laws (analogous to the law of corresponding states encountered in Van der Waals theory). Widom gave an expression that relates Magnetization M, temperature T and external field H:

$$H = M^{\delta}\Phi\left(\frac{t}{M^{1/\beta}}\right);\ \text{where } t = \left|\frac{T-T_c}{T_c}\right| \tag{1.14}$$

with Φ being a function of a single variable. What was remarkable was Widom's observation that δ, β and T_c may be so chosen to match experimental data from *different materials* (Fe, Ni,...) and that they all satisfy Eq. (1.14). One of the central ideas behind Landau's theory was the assumption that physical phenomena at two different length scales are essentially decoupled. This assumption was not

a drastic one as we actually implicitly assume so even in simplest of physical calculations. For instance one can derive the time period of a pendulum τ from dimensional argument upto a numerical factor: $\tau \propto \sqrt{l/g}$. Implicit to this calculation is that we ignore the internal structure of the pendulum or for that matter the distance of Sun and the Earth, as these length-scales are either too small or too large compared to the size of the pendulum. One simply assumes that at most they lead to extremely small and negligible corrections. Similarly in order to describe the planetary motion through Newtonian mechanics one can, to a very good approximation, ignore the sizes of the celestial bodies replacing them by point objects and also the effect of distant stars. But the failure of the mean field theories indicated that the assumption of decoupling of scales might not always be true in the theory of critical phenomena. As a matter of fact, if one tries to calculate the corrections to MFT, one finds divergences at the critical temperature. In a situation reminiscent of QFT these divergences turn out to depend on the ratio of the correlation length to the microscopic length-scale. Only difference being while in QFT it is the microscopic scale that goes to zero, here its the macroscopic scale that goes to infinity.

It turns out that the infinities met in QFT and statistical field theories have a common origin, i.e. a non-decoupling of widely disparate scales. The trouble arises when one ignores, while calculating any physical variable in one scale, the physical laws of the other scale (as is common practice). Yet on the other hand the fact that some of the universality survive rules out the possibility that the full details of the microscopic dynamics (which would be impossible to know) is absolutely necessary to predict physical phenomena. The first breakthrough to understand this puzzle came through the work of Kadanoff [73] in 1966 who proposed a new strategy to deal with the problem: *coarse graining*. Kadanoff's proposal was based on an intuitive argument that a relation of the form of Eq. (1.14) would follow naturally if one assumes that near a critical point 'the system looks the same at all length scales'. His strategy was to calculate physical quantities by summing recursively over the short-distance-degrees-of-freedom. For example, in case of Ising model one would introduce the concept of "block spins": a new spin variable defined as the sum of spins S_i in a square region of say $2a$(a being the original lattice parameter). Then one goes on to write the Hamiltonian in terms of these new variables and in the process one would obtain a new effective model (or interaction) that could be related to the earlier one through what is now called a RG transformation. This method (called as *coarse graining*) can be repeated so long as the new lattice spacing remains small compared to the macroscopic scale of interest (correlation length). If finally we arrive at an asymptotic form of the effective interaction which is independent of the initial interaction then in principle we have an explanation for the observed universality in critical phenomena. Although Kadanoff's principle provided the basic physical insight into the renormalization procedure but it wasn't quite the correct picture.

The modern understanding of RG is largely due to a series of seminal works by K. G. Wilson [74–77] in 1971 where he developed renormalization group in the context of both quantum and

statistical field theories. Wilson took Kadanoff's vague but powerful insight and developed it into robust operational scheme and unifying Kadanoff's RG with RG as used in QFT. This led to an understanding of universality, as being consequence of existence of a large-distance (IR) fixed points of a general RG. It even became possible to calculate the universal quantities with of help the pre-existing literature (Brezin, Le-Guilloun, Zinn-Justin 1973 [78]). He showed that the RG transformation asymptotically lead to a field theory in continuum space even though to start with one had a lattice model (i.e. the dynamic variable took on discrete values only). For instance in case of the Ising model after a number of iterations, the effective spin variable (which is the local average of large number of spins) takes a dense discrete set of values and clearly can be replaced by a continuum variable. Moreover the initial lattice becomes quite small compared to the iterated lattice and thus one can replace the effective spin variable by a filed $S(x)$ in continuum. The sum over spins now becomes integral over fields analogous to calculation of physical variables in QFT and thus completing the formal analogy with QFT. Further one can verify that the RG fixed point is Gaussian (i.e. it has a Gaussian distribution). At the critical point, in the large-distance limit, the fixed point takes up the form of a free QFT (scalar and non-interacting). Here the correlation length in some sense plays the role of mass and it turns out that in this limit the theory is massless. The weakly perturbed Gaussian model reproduces all universal results predicted by mean field theories. Furthermore in a remarkably striking result, it was discovered that most singular terms are generated by a renormalizable QFT whose large-distance properties can be studied, indicating that renormalizable QFTs can effectively describe large-length-scale properties of critical phenomena. Thus RG in quantum field theories turned out to be an asymptotic form of Wilson-Kadanoff RG.

Following Wilson's work in 1971 physics saw an explosion of research activity in the field which continued for more than a decade or so, see for example [79–83]. Most of the RG calculations utilized perturbation expansion techniques. Today RG has found applications in numerous different fields apart from condensed matter and high energy physics, for example non-equilibrium phenomena, biophysics, turbulence to name just a few. In condensed matter literature there are two versions of RG: the Wilson RG and the Gell-Mann–Low RG. In critical phenomena Gell-Mann–Low RG is based upon perturbation theory while Wilson RG has a more direct geometrical interpretation and is essentially non-perturbative. The connection between the two forms of RG is by no means obvious.

1.2.3 Renormalization group and dynamical systems

Renormalization group theory has been mostly associated with problems in statistical mechanics and quantum field theories as we have discussed. However, equilibrium statistical mechanics or for that matter quantum theories describe a limited subset of a large array of problems. For example,

equilibrium statistical mechanical descriptions fails when there is substantial dynamics present in a problem. Many systems which are not described by traditional quantum theory or equilibrium statistical mechanics can often be expressed as differential equations both ordinary and partial. Differential equations have found applications in a diverse range of fields in natural sciences as convenient descriptions of various dynamics. Dynamical systems theory hence has a very extensive scope, starting from problems like ecological models for competing species to models for market dynamics to modelling chemical reactions and so on and so forth. It describes a plethora of very interesting problems across disciplines such as physics, chemistry, biology, economics etc.

Most differential equations however can't be solved exactly and can only be handled by various perturtabative approaches or asymptotic analyses. This is why perturbation theory and asymptotic analysis constitute such a important topic in mathematical physics and its applications to various natural sciences. Perturbation theory usually refers to collection of iterative methods for the systematic analysis of global behaviour of differential equations. It usually proceeds with the identification of a small parameter, say ε, in the problem such that when $\varepsilon = 0$, the problem is exactly solvable. The global solution to the problem then can be studied via local analysis about ε and the ensuing solution can be expressed by a series of powers of ε:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots . \tag{1.15}$$

Such a series is called a perturbation series. $x_n(t)$ can always be computed in terms of $x_0, x_1, ..., x_{n-1}$ as long as the $\varepsilon = 0$ problem is exactly solvable. Further notice that $x(t)$ is *local* in ε but *global* in t. Usually when ε is small, it's expected that only a few terms of the perturbation series are enough for a well approximated solution. However, the perturbation series can often diverge for all $\varepsilon \neq 0$. Its well known that ordinary perturbation theory can fail in a variety of situations [6]. A well known physical example where ordinary perturbation theory fails is from fluid dynamics when viscosity is treated as a small parameter. As discussed previously Poincaré in his seminal works [8, 9] showed that the perturbation series may not always converge. Hence, in order to glean information from perturbation theory there is a need to develop proper and sophisticated techniques for "summation" of divergent series. This is where one has to resort to various singular perturbation techniques.

One of the best known examples of such techniques is the so called *Lindstedt-Poincaré* technique. Lindstedt-Poincaré method is used for uniformly approximating periodic solutions to differential equations,in cases where ordinary perturbation theory fails. The method simply demands that at every order of perturbation the so called *secular terms* — diverging terms — arising out of straightforward application of perturbation theory be removed. This method has proved to be a particularly successful tool in finding approximate solutions to weakly nonlinear differential equations with finite oscillatory

solutions.

Another rather successful singular perturbation technique is the *multiple-scales* method. Multiple-scales analysis is global perturbation scheme which relies on identifying widely disparate times scales in a given problem. In this method a set of scaled variables, treated as independent (although they are not truly independent) variables in order to remove all secular terms. For example weak dissipation in an oscillator is insignificant on short time scales but become significant on longer scales and classical perturbation theory breaks down in the asymptotic limit. However, there is no prescribed method of choosing the proper time scales and in many cases the choice of the set of scales is rather non-trivial. The justification for the choice usually comes *post hoc*. This is a weakness of this method, nonetheless, multiple-scales analysis is one of the most effective subsumming schemes available.

When the highest order derivative of a given differential equation is multiplied by a small parameter, ε, it usually can lead to narrow regions of rapid vibrations called as *boundary layers*. Such cases constitute yet another class of problems where usual perturbation theory fails. In cases where the small parameter, $\varepsilon \to 0$, *boundary-layer techniques* can be employed. So, when a differential equation has a solution which is slowly varying except in isolated narrow regions, its possible to obtain a fairly good leading order approximation without actually solving the problem. However, the determination of expansion parameters can be subtle and so can be the matching of the solutions inside and outside the boundary layers. There can also be problems where the thickness of the rapidly vibrating region doesn't scale down with reducing ε and in such cases one employs the *WKB method*. There are other singular perturbation techniques as well such as *periodic averaging methods*, *method of matched asymptotic expansions*, *Bogoliubov-Krylov method* etc. In this thesis however, we will be employing a unified approach based on ideas from renormalization group theory to deal with a wide range of differential equations which are not amenable to simple perturbation theory.

In the early 1990s, Chen, Goldenfeld and Oono proposed a new method based on renormalization group (RG) [84, 85] theory to extract asymptotic behavior of solutions to differential equations. This method, which we will refer to as – CGO-RG method henceforth, has proved to be a unified tool for global asymptotic analysis. In fact CGO-RG can be applied to problems involving multiple scales, boundary layers, asymptotic matching, WKB analysis etc and that too without the usual assumptions about the structure of the perturbation series. RG is thus in some sense equivalent to these various singular perturbation techniques used in nonlinear dynamics. However, the chief advantage of RG over traditional perturbative techniques lies in the fact that it does not require any *ad hoc* assumptions, it simply uses a naive perturbation expansion. It is quite well known that RG equations [74] have the peculiar capability to improve the "global" nature of functions obtained in the perturbation theory (which is essentially "local") in quantum field theory (QFT) and statistical mechanics. Let's examine

Landau's theory for continuous phase transitions briefly. The partition function for D-dimensional Ising model can be written approximately as

$$\mathscr{Z} = \int \mathscr{D}\varphi e^{-\int d^D x(1/2(\nabla\varphi)^2 + 1/2 a_0\varphi^2 + b_0\varphi^4)}. \tag{1.16}$$

After a suitable rescaling [see [65] for details] the above can be rewritten with all temperature dependence moved to the fourth order term and the rescaled coefficient, $\overline{b}_0$, now has a temperature-dependence given by,

$$\overline{b}_0 \sim t^{(4-D)/2}. \tag{1.17}$$

It shows that if one attempts a perturbative expansion of the integral in quartic powers, for $D < 4$ the expansion will always diverge near $t = 0$. This happens because of the fact that inclusion of the quartic term in the order parameter introduces fundamentally new physics near the critical temperature. Even if the parameter, b_0 is very small it still persists and quartic model can not be anymore obtained by a smooth deformation of the quadratic one. So, when $b_0 \neq 0$, naive perturbation theory is bound to fail quite similar to what can happen in case of some nonlinear differential equations. In a sense b_0 term is the *singular perturbation* and in this case with use of RG [70] one can remove all divergences and all the critical exponents can be easily calculated.

As already pointed out, a very similar situation can exist in certain differential equations. Much like the quartic term in Landau theory, a nonlinearity, however small, can introduce a fundamentally new dynamics into the system. Similar to the example of Landau's model, the unperturbed solution are not always simply related to the full solution since the perturbation series doesn't always converge. Thus, for reasons quite similar to critical phenomena or QFT, presence of nonlinear perturbations in differential equations give rise to divergences and render the usual perturbation theory useless. Perturbations with such characteristics are usually referred to as *singular perturbations*. Goldenfeld et al demonstrated [cite all there papers] that various singular perturbation methods used for nonlinear differential equations, may be understood as renormalized perturbation theory. Amplitude equations, which are obtainable by various reductive perturbative techniques, are obtained quite naturally with the application of RG to the problems as the renormalization group equations. The analogy between CGO-RG method and the traditional RG used in QFT or statistical mechanics is quite clear. In QFT straightforward perturbative calculations of quantities such as electron charge and mass gives us infinities. In the renormalized theory such quantities are called "bare" quantities and are not directly observable. These bare charge or mass is renormalized via interactions in the theory to give us effective values which are directly observable at certain energy or momentum scale. *Cauchy data*, i.e. simply the initial-value data, is the analogue of bare quantities of QFT. These are then renormalized by application of CGO-RG. In case of periodic solutions its usual to express the solutions in terms of

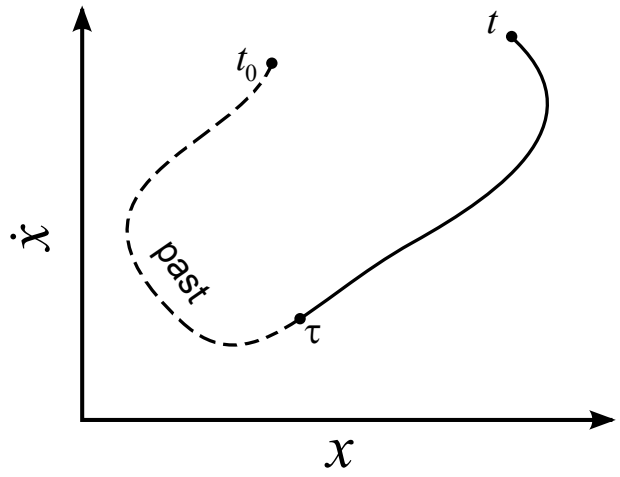

Figure 1.1: On the phase trajectory every point can serve as an initial condition and yet all lead to the same asymptotic $x(t)$. The divergence in the perturbation series comes in some sense from the past and this can be got rid of introducing floating time τ and renormalizing the perturbation series.

amplitudes and phases. In fact any periodic solution, however nonlinear, can in principle be expressed as a Fourier series with amplitude A and phase of the lowest harmonic θ determining the amplitude and phase of the higher order ones. The amplitude and phase are quantities that will *flow*. Amplitude and phase are in some sense analogues of the coupling constants which get renormalized in QFT or critical phenomena. In case of CGO-RG its convenient to use multiplicative renormalization for the amplitude and an additive one for the phase. Typically the RG flow equations may be interpreted as the following: A physical quantity $\mathscr{P}(\alpha,\beta,\mu)$ should have no dependence on the renormalization point μ which is arbitrarily chosen and is absent from the original Lagrangian of the problem, i.e.

$$\frac{d\mathscr{P}}{d\mu} = 0. \tag{1.18}$$

The concept of such a floating renormalization point was first introduced in a classic paper by Gell-Mann and Low [59]. A naive perturbation expansion of the dynamical variable will lead to a divergent answer. If t is the time at which we want to know $x(t)$ and t_0 is the initial time, then $x(t)$ will diverge as $t - t_0 \to \infty$. This is completely similar to divergence in field theories where a physical quantity (e.g. two point correlation function) diverges as the renormalization cutoff $\Lambda \to \infty$. If we are discussing a physical variable, then the answer has to be finite and while this is achieved in field theory by constructing running coupling constants it is done for the differential equations by introducing an arbitrary time scale τ and letting the amplitude and phase depend on τ. The arbitrary time τ is in some sense analogous to the floating renormalization point μ of the Gell-Mann and Low RG. This process leads

to the flow equations of the form:

$$\frac{dA}{d\tau} = f(A,\theta) \tag{1.19}$$

$$\frac{d\theta}{d\tau} = g(A,\theta) \tag{1.20}$$

So, we see that the RG naturally leads to flow equations. Consider an arbitrary phase path (see Fig. 1.1) beginning at time t_0. Typically we are interested in knowing x at some asymptotically later time t. The divergence usually comes from secular terms which have '$(t-t_0)$' (a monotonically increasing quantity) multiplied to them. In some sense the divergence thus comes from the 'past'. RG takes advantage of the fact that along a given trajectory every point can serve as a initial condition without changing the final outcome. By introducing a floating arbitrary time τ, the renormalization process keeps track of the the amplitude, A and phase, θ (which are renormalized to absorb divergences) at different times of the evolution and makes sure that no secular terms are generated.

1.3 Outline of the thesis

In the second chapter of the thesis we describe the RG method for dynamical systems and all its mathematical steps in complete detail. We will be using this method for solving a multitude of other problems throughout this thesis but won't be presenting the intermediate steps in every case. RG can be used to handle wide array of nonlinear differential equations which are normally solved by various singular perturbation techniques. However, we will focus solely on exploring various aspects of periodic motion and application of the renormalization group method to such problems. The study of nonlinear differential equations in a two dimensional dynamical system is of considerable interest to wide range of cross-disciplinary researches. The fact that RG method of Goldenfeld *et al* extracts the asymptotic behavior of solutions to differential equations, makes it particularly handy while analyzing periodic solutions to nonlinear dynamical problems. We will establish in detail the general methodology used to do most of the calculations in this thesis.

The third chapter of this thesis deals with two basic types of periodic solutions encountered for dynamical systems — centers and limit cycles — and how one can use RG to distinguish between them [86]. Centres are a family of initial-condition-dependent orbit surrounding a point and usually encountered in conservative systems whereas limit cycles refer to an isolated closed path in phase space to which all neighbouring trajectories converge. Limit cycles represent self sufficient oscillations i.e. systems which oscillate even in the absence of an external driving force. Oscillators are very important in the field of dynamical systems and yet there isn't any straightforward way of distinguishing between

these two very different kinds of solutions. We propose a way to tell these solutions apart by making use of the RG flow equations. We also illustrate that this very technique can help one ascertain if a fixed point is a focus or a center. Our method of distinguishing focus, center and limit cycle will be shown to be easily extendable to two-dimensional non-autonomous systems and also to the extremely important class of time-delayed dynamical systems.

Chapter four deals with the problem of isochronous centres. As explained, a family of initial condition dependent periodic orbits surrounding a critical point $\mathscr{O}$ is called a center and $\mathscr{O}$ is said to be isochronous if every orbit close enough to $\mathscr{O}$ has the same period. Isochronicity is a widely studied subject not only for its relation with stability theory, bifurcation theory and boundary-value problems [87]. RG can be used very effectively to study isochronous oscillators and to find the so called *period functions* i.e. conditions under which a system becomes isochronous. In fact we demonstrate that RG can be used to construct nonlinear oscillators which are isochronous in nature [88].

In chapter five we tweak our RG method so that it is applicable to problems devoid of any linear terms. Any perturbative method relies on finding a linear problem related to the nonlinear one, around which perturbation analysis can be done. However, in systems lacking any linear terms it is not obvious as to how systems perturbation theory may be applied. We propose in this chapter how RG can be done even in the absence of linear terms [89]. We successfully apply the method to extract correct results for a variant of the second order Riccati equation given by the equation: $\ddot{x}+\beta\lambda x\dot{x}+\lambda^2x^3=0$. In this oscillator the periodic orbit disappears as the parameter, β, is varied. Our RG method captures this disappearance correctly. In course of our research we find a hierarchy (with increasing degree of nonlinearity) of oscillators [90] which are related to second order Riccati equation and share similar physical properties. We study the phenomenon of transition from a periodic state to non-periodic one in some detail.

Until Chapter six we mostly deal with autonomous dynamical systems although we do consider briefly systems with time-dependent excitations in chapter three and show that RG can be successfully implemented in such cases. In chapter six we establish the RG methodology for non-autonomous system and study oscillators near resonances. We show that RG can be an used to analyze resonance phenomenon for various oscillators and not only the excitations which appear as homogeneities in governing differential equation. We consider oscillators where excitations appear as coefficients in the differential equation, i.e. parametric excitation. Applying RG in such cases is a non-trivial problem and can run into trouble unless one perturbs around specific points in the phase space. We study the Hill equation and the Mathieu equation as special case of Hill using RG methodology. We also study the sub-harmonic response of the Duffing oscillator. Further we show that RG may be used to analyze *internal-resonance* and *auto-resonance* as well.

Finally, in Chapter seven we present a very interesting modification to the usual Van der Pol oscillator. We introduce a periodically varying damping coefficient and demonstrate that it can display very rich dynamics. The periodically excited nonlinear Van der Pol oscillator displays exotic resonance phenomenon and can be used to construct switches.

The final chapter is a brief conclusion of this thesis work.

Chapter 2

Renormalization Group Method

2.1 Introduction

Nonlinear dynamics is a truly inter-disciplinary branch of natural science today with applications in not only mathematics and physics but in a diverse range of fields such as chemistry, biology, ecology, engineering, economics etc [91–94]. The history of dynamical systems research can be traced back several hundreds of years with illustrious names such as Galileo, Kepler, Newton, Poincaré etc adorning its pages. But its only in the second half of 20th century that modern science took an enormous interest in the study of nonlinear dynamical systems. In fact research in nonlinear dynamics was very important in various technological advances. For instance, study of nonlinear oscillators played an absolutely pivotal role in the development of technologies such as radio, lasers, radars etc. The recently developed renormalization group (RG) method by Goldenfeld et al [84, 85, 95–98], opened up a new direction of research in nonlinear dynamics. They showed RG can be used as a global and asymptotic analysis tool for ODEs and PDEs. What makes the method so powerful is it starts with a naive perturbation expansion and no other apriori assumptions. In practice, starting with a simple perturbation expansion would mean that due to nonlinearities, "secular terms" are generated in the perturbation series; the secular terms which can be thought as analogous to logarithmically divergent terms in QFT. In a way reminiscent of Gell-Mann-Low type RG, an intermediate floating time τ is introduced and the perturbative series is rewritten in terms of renormalization constants. This method gets rid of the offending secular terms and finally replacing τ by t one obtains globally valid solutions.

It should be noted here that Bricmont and Kupiainen, in an independent work [99], applied a scaling transformation to obtain the asymptotic behaviour of nonlinear diffusion equations which is in some sense equivalent to what Goldenfeld et al have done. Subsequently, Kunihiro showed that RG for

dynamical systems may be understood in terms of classical theory of envelopes [100–103]. He suggested that the RG equation may be interpreted as the basic equation for construction of envelopes for a family of curves (or surfaces in case of partial differential equations). Chen, Goldenfeld, Oono's RG (CGO-RG) has been successfully applied to a wide range of problems. To mention a few, Sasa derived a diffusion-type phase equation describing the evolution of large-scale modulation of spatially periodic patterns in 2D systems [104]; Graham [105] derived a rotationally invariant Newell-Whitehead-Segel amplitude equation appearing in the problem of pattern formation; Maruo et al [106] derived Kuramoto-Sivashinsky (KS) [107] as a phase equation using the RG method. The method has also been applied in cosmology, in order to find asymptotic behaviour of nonlinear equations appearing in the subject [108–111]. De Vega et al also used RG to derive an anomalous transport coefficient [112] relevant in the non-equilibrium states in the early universe and quark-gluon plasma.The RG method has also been demonstrated as a powerful tool for resumming divergent perturbation series appearing in quantum mechanics [113, 114] and also as an effective method for doing time dependent quantum perturbation calculations [115]. The Boltzmann equation has been derived as a RG equation [116]. It has also found applications in accelerator physics [117, 118], astrophysics [119], fluid dynamics [120, 121], material sciences [122, 123] and so on.

Basically, the RG method draws parallels and analogies from traditional RG in field theories to renormalize perturbation series in non-linear dynamical problems otherwise bugged by divergences created by secular terms. The method quite naturally reproduces the results obtained by various singular perturbation techniques. Our interest was in further exploring the potential of this technique in the analysis of nonlinear dynamical systems. In our efforts to explore the potential of the renormalization group technique, the topic of nonlinear oscillators is a natural choice for a couple of reasons: (i) the subject of nonlinear oscillators is a well-researched and fairly mature subject with a rich body of existing works [3, 4, 124]; and (ii) there are numerous very important and interesting problems in the field. Furthermore, a number of abstract mathematical techniques used in the study of dynamical systems appear quite naturally in nonlinear oscillator problems. For most part of this thesis, we will be dealing with nonlinear oscillators with a single-degree-of-freedom i.e. 2-dimensional dynamical systems.

2.2 Failure of perturbation theory

Two dimensional dynamical systems constitute a very important class of dynamical systems with applications across disciplines. To begin with, we will examine the general class of differential equations which can be expressed as

$$\ddot{x} + \omega^2 x = \varepsilon F(x, \dot{x}) \tag{2.1}$$

where $f(x,\dot{x})$ can be any arbitrary nonlinear function of x and $\dot{x}$ and $\varepsilon \ll 1$. Equivalently we can always rewrite the above second order differential equation as two 1st order ones as

$$\dot{x} = y + f_1(x,y) \tag{2.2}$$
$$\dot{x} = -\omega^2 x + f_2(x,y) \tag{2.3}$$

Attempts in solving an ordinary differential equation of the form (2.1) using a naive expansion, $x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, often results in breakdown of the perturbation theory at times t such that $\varepsilon(t - t_0) > 0$ (where t_0 is the initial time) due to the presence of so called *secular terms*. Lets first assume a very simple form for the nonlinear function F and see how secular terms can be generated in perturbation theory. We consider the anharmonic oscillator with a quartic potential, given by

$$\ddot{x} + x = -\varepsilon x^3, \tag{2.4}$$

with the boundary conditions $x(t = t_0) = A_0$ and $\dot{x}(t = t_0) = 0$ where A_0 and θ_0 are constants determined by the initial conditions. We first begin with the naive perturbation expansion in powers of ε, $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$ and inserting it into the Eq. (2.4) and then solving recursively. This leads to, at the zeroth order,

$$\ddot{x}_0 + \omega^2 x_0 = 0 \tag{2.5}$$

to which we can promptly write down the solution, $x_0 = A_0 \cos(\omega t + \theta_0)$. Equating terms of $\mathcal{O}(\varepsilon)$ we obtain the 1st order equation and can use the zeroth order solution to have

$$\ddot{x}_1 + \omega^2 x_1 = -x_0^3 = A_0^3 \cos^3(\omega t + \theta_0) \tag{2.6}$$

We note that $\cos^3(\omega t + \theta_0)$ can be expanded as a Fourier series and the above equation be rewritten as

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{3A_0^3}{4}\cos(\omega t + \theta_0) + \frac{A_0^3}{4}\cos 3(\omega t + \theta_0) \tag{2.7}$$

Note that the '$\cos \omega t$' term on the R.H.S. of equation (2.7) makes it equivalent to a driven harmonic oscillator with driving term having frequency terms same as the natural one. This causes a resonance which is spurious and unphysical since the solution of equation (2.4) is quite well defined. Thus the second term in the perturbation series $\varepsilon x_1(t)$ will grow unbounded with time (such terms are called "secular" terms) and after a while will no longer remain small compared to the zeroth order solution which remains periodic and bounded. So simple perturbation theory breaks down due to generation of secular terms in the perturbation series. At the face of it this seems to be an unsurmountable problem. One way to handle this problem is by using Lindstedt-Poincaré method which proceeds first by

assuming that ω is no longer the frequency of the full dynamics and then demanding that secular terms must vanish at every order of perturbation, which then gives corrections to frequency. However, analogous problems of divergences in QFT and critical phenomenon are easily managed by renormalization group ideas. Similarly the RG method developed by Goldenfeld *et al* proves to be very efficacious in dealing with secular terms and giving convergent solution without having to resort any assumptions like the Lindstedt-Poincaré method.

2.3 Renormalization group

How does one apply the RG principle to the general problem described by Eq. (2.1)? We begin by examining why simple perturbation theory fails. The reason perturbation theory fails in case of the anharmonic oscillator is because we proceed with the assumption that frequency of the dynamics is constant. But in practice, unlike the case of simple harmonic oscillator, the nonlinearity 'x^3' in the problem makes the frequency of the anharmonic oscillator amplitude-dependent. This is never taken into account while doing simple perturbation theory. If one writes down a solution to Eq.(2.7) we have

$$x_1(t) = -\frac{3A_0^3}{8\omega}(t-t_0)\sin(\omega t+\theta) - \frac{A_0^3}{32\omega^2}\left(\cos 3(\omega t+\theta_0) - \cos(\omega t+\theta_0)\right). \tag{2.8}$$

As already pointed out the solution keeps growing with time and breaks down when $\varepsilon(t-t_0) > \mathscr{O}(1)$. Further notice that in Eq.(2.8) the trouble comes from the '$\cos\omega t$' term and not the higher harmonic one. In the asymptotic limit (i.e. $t \to \infty$), the '$(t-t_0)$' in the 1st term of expression of $x_1(t)$ makes it blow-up and hence unphysical. In a sense one can say that the divergence comes from the *past* i.e. $(t-t_0)$. We also note that a periodic solution in 2D can always be expressed as a Fourier series in terms of the two constants of motion — *amplitude* and *phase*. As long as the nonlinearity is small, one can always approximate the periodic orbit (however distorted be the it from a perfect circle) by a Fourier series with the amplitude A_0 and phase θ_0 of the lowest harmonic determining the amplitude and phase of the higher order ones. In order to regularize the perturbation series, RG technique first introduces an arbitrary time τ with a view to split the time interval $(t-t_0)$ into two parts as $(t-\tau)$ and $(\tau-t_0)$. In presence of nonlinearities, the amplitude and phase of the dynamical system are no longer constants and are made dependent on the arbitrary time τ. In practice the terms containing $(\tau-t_0)$ are absorbed into the respective *renormalized* counterparts A and θ of A_0 and θ_0.

This is analogous to divergence in QFT or critical phenomenon where a physical quantity (*e.g.*, two point correlation function) diverges as the cutoff $\Lambda \to \infty$. When we are discussing a physical variable, then the answer has to be finite and while this is achieved in QFT by constructing running

coupling constants, in case of differential equations it is done by introducing an arbitrary time scale τ and letting the amplitude and phase depend on τ. At the end of the RG process one arrives at the so called RG-flow-equations (analogous to the β-function in traditional RG) for A and θ:

$$\frac{dA}{d\tau} = f(A,\theta), \tag{2.9}$$

$$\frac{d\theta}{d\tau} = g(A,\theta). \tag{2.10}$$

RG naturally leads to flow equations solving which one can get the renormalized A and θ in terms of which the perturbation series no longer remains divergent. For autonomous dynamical systems, f and g are generally functions of the amplitude, A, alone. We propose to use flow Eqs. (2.9) and (2.10) to draw certain general conclusions about the asymptotic nature of the solution to various differential equations.

Periodic solutions to various dynamical systems can be differentiated into two distinct varieties: *centers* and *limit cycles*. While the *center* type periodic solution consists of a continuous family of closed orbits in phase space, each orbit being determined by its own initial condition; the *limit cycle* is an isolated orbit in phase space independent of initial conditions with all neighbouring trajectories around it converging onto the periodic orbit — the 'limit'-ing cycle. Limit cycles are very important because the presence of limit cycles in a model facilitates explanation of self-sustained oscillations. Limit cycles appear in wide variety of modern researches in many fields like quantum physics, chemical physics, biophysics, material sciences, ecology *etc.* — see *e.g.,* [125–129] respectively for recent examples. Consequently, a number of research studies are dedicated towards finding out a variety of methods [6, 35–39, 130, 131] (also see [132]) for determining limit cycles in nonlinear problems. Limit cycles are essentially nonlinear in nature. The simplest and arguably most well-known example of a limit cycle occurring in a problem is the Van der Pol oscillator [197].

On the other hand 'centre'-variety solutions (i.e. family of initial-condition-dependent orbits around a point) occur in both linear and nonlinear problems. One often employs *linear stability analysis* as an usual first probe to determine the asymptotic nature of the solution to a second order nonlinear differential equations (equivalently 2D dynamical system). Technically speaking, if a certain fixed point of the system has both eigenvalues imaginary, its called a "centre". Simplest example where such a solution occurs is the simple harmonic oscillator which is linear problem. In case the solution is a centre despite the presence of nonlinearity, its often referred to as a *nonlinear centre*. In this thesis however we will not distinguish between linear and nonlinear centres. However, a word of caution is in order here: linear stability analysis does not always give the correct result as to the existence of a centre. While there are a number of analytical tools available to physicists for probing periodic solutions to

differential equations none of the tools differentiate between these two distinct kind of solutions. Based on extensive studies of various 2D-dynamical systems, here we propose a prescription capable of distinguishing between limit cycles and centres. Although we present no rigorous proof of our statements here or anywhere throughout the thesis, we will demonstrate its validity through numerous examples. *It is our surmise that by merely looking at the structure of the amplitude flow equation it is possible to differentiate between oscillators which are of the center variety and limit cycles.* We shall also illustrate that this very technique can help one ascertain if a fixed point is a center or not. It is worth mentioning that distinguishing center from focus (known as *center problem*) is one of the main and oldest problems in two-dimensional dynamical systems. Our method of distinguishing center and limit-cycle will be easily shown to be extendable to two-dimensional non-autonomous systems and also to the extremely important class of time-delayed dynamical systems.

Centre-like solution is a family of periodic orbits surrounding a point with radius of each orbit being fixed by its own initial condition. This implies that the amplitude A is fixed once the initial condition is set. This suggests that the structure of the amplitude equation for such a scenario must be,

$$\frac{dA}{d\tau} = 0. \tag{2.11}$$

This statement is exact and is not tied to any perturbation theory argument. In order for the periodic solution to be a 'centre' the above condition must hold at every order of perturbation. For the limit cycle on the other hand the structure will be

$$\frac{dA}{d\tau} = f(A) \tag{2.12}$$

and $f(A)$ must be such that the flow has a non-zero fixed point and the value of the fixed point gives the radius of the limit cycle. The limit cycle is stable if the fixed point is stable. Finally, if $f(A)$ is zero only for the trivial fixed point $A = 0$, then we have either a focus or a node in phase space. Yet again the stability of the fixed point in phase space is determined by the stability of the fixed point of the flow equation.
This extremely simple prescription, though not proved rigorously, appeals to one's intuition when one notes that:

1. $A = 0$ means the *assumed* periodic solution has zero amplitude and hence hints at *focus* or a *node*;

2. $f^{(A)} = 0 \forall A \geq 0$ suggests that the amplitude is constant and is fixed at the initial time t_0. Logically the solution has to be family of non-isolated periodic orbits surrounding the fixed point and

therefore existence of center is implied;

3. Vanishing of $dA/d\tau$ at $A = A_i \neq 0$ logically indicates that an isolated periodic orbit of amplitude A_i has to be surrounding the fixed point. If the fixed point A_i of the flow equation is stable then all trajectories irrespective of initial conditions must lead to a periodic solution with the radius A_i, the radius of the limiting cycle.

The functions $f(A)$ and $g(A)$ have to be calculated perturbatively. With any perturbation theory the first step has to be identifying a suitable solution around which the perturbation theory can be carried out. For example in case of the anharmonic oscillator given by Eq. (2.4) we can consider the nonlinearity 'εx^3' as a perturbation around a simple harmonic oscillator given by the equation: $\ddot{x}+x=0$. So one can begin with the solution of a simple harmonic oscillator at the lowest degree and then recursively solve higher order equations to construct the perturbation series. Therefore, application of perturbation theory is possible only if one can locate a linear centre — this is the basic periodic state. Locating such a linear solution which can serve as the unperturbed state can sometimes be straightforward for e.g. in case of the anharmonic oscillator, Lotka-Volterra system etc. But there can be problems where the choice of an unperturbed state can be quite non-trivial. We will discuss more on this issue towards the end of this chapter. First however, we must see how the RG works in practice for a 2nd-order nonlinear differential equation.

2.4 The methodology

In order to establish the RG methodology we take the example of the anharmonic oscillator but now with a damping term and an external forcing term given by

$$\ddot{x}+k\dot{x}+\omega^2 x+\lambda x^3 = F\cos\Omega t. \tag{2.13}$$

where all three parameters k, λ and F are considered to be small. The above equation is referred to as the Duffing oscillator in nonlinear dynamics literature. The Duffing oscillator was first proposed by G. Duffing in 1918 [134]. Later, Holmes et al showed that it arises in the forced vibrations of a cantilever beam in the non-uniform field of two permanent magnets [135, 136]. Duffing oscillator has been used as the one of the most widely used models in nonlinear dynamics literature, see for example [137–140].

To begin with we will drop the external forcing term and consider only the autonomous case ($F=0$) given by

$$\ddot{x}+k\dot{x}+\omega^2 x+\lambda x^3 = 0. \tag{2.14}$$

We have already discussed how due to the presence of the 'x^3' term, straightforward perturbation theory fails in this case. Let's see how RG fares in this scenario. In order to clarify all the steps involved we will presenting the calculation for this example in explicit details. But for most of the other examples in this thesis we won't be presenting calculations in as much detail.

To begin with we notice that a linear centre exists, for $k = \lambda = 0$. Thus the perturbation theory will be built around this limit. We naively expand x in powers of the small parameters λ and k as,

$$x = x_0 + kx_1' + \lambda x_1 + k^2 x_2' + \lambda^2 x_2 + k\lambda x_2'' + \ldots \tag{2.15}$$

Substituting Eq. (2.15) into Eq. (2.14), we obtain for different powers of λ

$$\begin{aligned}
\mathcal{O}(k^0\lambda^0): \qquad & \ddot{x}_0 + \omega^2 x_0 &=& \ 0 && (2.16)\\
\mathcal{O}(k^0\lambda^1): \qquad & \ddot{x}_1 + \omega^2 x_1 &=& -x_0^3 && (2.17)\\
\mathcal{O}(k^1\lambda^0): \qquad & \ddot{x'}_1 + \omega^2 x_1' &=& -\dot{x}_0 && (2.18)
\end{aligned}$$

We work with the initial conditions $x(t=0) = A_0$ and $\dot{x}(t=0) = 0$. Unless otherwise mentioned, we will work with these initial conditions for all of the 2D dynamical systems we handle in this thesis. This is done with no loss of generality and only to make the algebra simpler. We write down the solution to the zeroth order equation (Eq.(2.16)) as

$$x_0 = A_0 \cos \omega t. \tag{2.19}$$

We note that the zeroth order solution x_0 picks up the initial condition and hence for all $i \geq 1$we will have $x_i(t=0) = \dot{x}_i(t=0) = 0$. We can now plug in the zeroth order solution into Eq.(2.17) and Eq.(2.18), which yields:

$$\begin{aligned}
\ddot{x}_1 + \omega^2 x_1 &= -\frac{A_0^3}{4}(\cos 3\omega t + 3\cos \omega t), && (2.20)\\
\ddot{x'}_1 + \omega^2 x_1' &= \omega A_0 \sin \omega t. && (2.21)
\end{aligned}$$

The solutions to the above equations are respectively given by,

$$\begin{aligned}
x_1 &= -\frac{3A_0^3}{8\omega} t \sin \omega t + \frac{A_0^3}{32\omega^2}(\cos 3\omega t - \cos \omega t), && (2.22)\\
x_1' &= -\frac{A_0}{2} t \cos \omega t + \frac{A_0}{2\omega} \sin \omega t. && (2.23)
\end{aligned}$$

We can now write down, the displacement $x(t)$ of the oscillator approximate upto 1st order in k and λ,

$$x(t) = A_0\cos\omega t - \frac{3\lambda A_0^3}{8\omega}t\sin\omega t + \frac{\lambda A_0^3}{32\omega^2}(\cos 3\omega t - \cos\omega t) - \frac{kA_0}{2}t\cos\omega t + \frac{kA_0}{2\omega}\sin\omega t, \quad (2.24)$$

where $t_0 = 0$. As expected we have $(t-0)$ multiplying the secular terms in the above expression. As per the RG prescription we split the time interval $(t-0)$ as 0 to τ and τ to t and rewrite the above expression as,

$$\begin{aligned} x(t) &= A_0\cos\omega t - \frac{3\lambda A_0^3}{8\omega}(t-\tau+\tau)\sin\omega t + \frac{\lambda A_0^3}{32\omega^2}(\cos 3\omega t - \cos\omega t) \\ &\quad -\frac{kA_0}{2}(t-\tau+\tau)\cos\omega t + \frac{kA_0}{2\omega}\sin\omega t. \end{aligned} \quad (2.25)$$

Next, in order to remove the divergences, we introduce two renormalization constants $\mathscr{Z}_1(0,\tau)$ and $\mathscr{Z}_2(0,\tau)$ as

$$A_0 = A(\tau)\mathscr{Z}_1(0,\tau) \quad (2.26)$$

$$0 = \theta_0 = \theta(\tau) + \mathscr{Z}_2(0,\tau) \quad (2.27)$$

Further the renormalization constants have the expansions

$$\mathscr{Z}_1(0,\tau) = 1 + a_1\lambda + a_1'k + \ldots, \quad (2.28)$$

$$\mathscr{Z}_2(0,\tau) = b_1\lambda + b_1'k + \ldots. \quad (2.29)$$

This is done so that the coefficients a_i and b_i, for $(i \geq 1)$, can be chosen order by order to remove divergences appearing at each order. In terms of $A(\tau)$ and $\theta(\tau)$, we can write Eq.(2.25) as

$$\begin{aligned} x(t) &= A(\tau)\left[1 + a_1\lambda + a_1'k\right]\cos(\omega t + \theta(\tau) + b_1\lambda + b_1'k) - \frac{3\lambda A^3}{8\omega}(t-\tau+\tau)\sin(\omega t+\theta) \\ &\quad + \frac{\lambda A^3}{32\omega^2}(\cos 3(\omega t+\theta) - \cos(\omega t+\theta)) - \frac{kA}{2}(t-\tau+\tau)\cos(\omega t+\theta) + \frac{kA}{2\omega}\sin(\omega t+\theta) \\ &= A(\tau)\cos(\omega t+\theta) + (a_1\lambda + a_1'k)A(\tau)\cos(\omega t+\theta) - (b_1\lambda + b_1'k)A(\tau)\sin(\omega t+\theta) \\ &\quad -\frac{3\lambda A^3}{8\omega}(t-\tau+\tau)\sin(\omega t+\theta) + \frac{\lambda A^3}{32\omega^2}(\cos 3\Big(\omega t+\theta\Big) - \cos(\omega t+\theta)) \\ &\quad -\frac{kA}{2}(t-\tau+\tau)\cos(\omega t+\theta) + \frac{kA}{2\omega}\sin(\omega t+\theta) \end{aligned} \quad (2.30)$$

correct to $\mathscr{O}(\lambda)$ and $\mathscr{O}(k)$. At the next step, we chose $a_1' = \frac{kA\tau}{2}$, $a_1 = 0$, $b_1' = 0$ and $b_1 = -\frac{3\lambda A^3}{8\omega}\tau$ in

order to eliminate the '$\tau - 0$' containing terms which in some sense cause the divergence. Now we are left with,

$$\begin{aligned} x(t,\tau) &= A(\tau)\cos(\omega t + \theta(\tau)) - \frac{3\lambda A^3}{8\omega}(t-\tau)\sin(\omega t+\theta) + \frac{\lambda A^3}{32\omega^2}\Big(\cos 3(\omega t+\theta) - \cos(\omega t+\theta)\Big) \\ &\quad -\frac{kA}{2}(t-\tau)\cos(\omega t+\theta) + \frac{kA}{2\omega}\sin(\omega t+\theta). \end{aligned} \tag{2.31}$$

The final step in this process comes from the fact that the arbitrary time, 'τ' does not appear anywhere in the original problem and hence the final solution must be independent of it. Therefore, we now impose the condition that $x(t)$ has to be independent of τ i.e. $\left(\frac{\partial x}{\partial \tau}\right)_t = 0$ and this yields (to the lowest non-trivial order) two independent flow equations (collecting terms multiply '$\cos(\omega t+\theta)$' and '$\sin(\omega t+\theta)$' and equating them to zero separately):

$$\frac{dA}{d\tau} = -\frac{kA}{2}, \tag{2.32}$$

$$\frac{d\theta}{d\tau} = \frac{3\lambda A^2}{8\omega}. \tag{2.33}$$

On can immediately solve the above flow equations by integrating to $A = A_0 e^{-k\tau/2}$ and $\theta = \theta_0 + \frac{3\lambda A^2}{8\omega}\tau$. The final removal of τ requires setting $\tau = t$ and this removes the secular terms. Finally, we have a uniformly valid result

$$\begin{aligned} x(t) &= A_0 e^{-kt/2}\cos\left[\left(\omega + \frac{3\lambda A^2}{8\omega}\right)t + \theta_0\right] \\ &\quad + \frac{\lambda A_0^3}{32\omega^2}\Big(\cos 3(\omega t+\theta_0) - \cos(\omega t+\theta_0)\Big) + \frac{kA_0}{2\omega}\sin(\omega t+\theta_0) \end{aligned} \tag{2.34}$$

which describes a trajectory slowly spiralling into the origin $(x,\dot{x}) = (0,0)$. This is indeed the correct result [3] for the Duffing oscillator with non-zero damping coefficient k.
For $k = 0$, however, we have the conservative anharmonic oscillator

$$\ddot{x} + \omega^2 x + \lambda x^3 = 0 \tag{2.35}$$

for which the fixed point $(0,0)$ in the $x - \dot{x}$ plane (*i.e.* $x - y$ plane) is a centre and as expected the amplitude flow equation for $k = 0$ is,

$$\frac{dA}{d\tau} = 0 \tag{2.36}$$

with the solution given by

$$x(t) = A_0 \cos \Omega t + \frac{\lambda A_0^3}{32\Omega^2} [\cos 3\Omega t - \cos \Omega t] + \mathscr{O}(\lambda^2) \tag{2.37}$$

where Ω given by

$$\Omega = \omega + \frac{3\lambda A_0^2}{8\omega} + \mathscr{O}(\lambda^2) \tag{2.38}$$

which is the frequency of the unperturbed solution plus an amplitude-dependent correction. This is indeed the standard perturbative frequency correction [141] for this oscillator at this order. The solution given by Eq. (2.37) describes a family of periodic orbits with each member being determined by A_0. For the anharmonic oscillator of Eq. (2.35), the phase space trajectory is given by

$$\dot{x}^2 + \omega^2 x^2 + \frac{\lambda}{2} x^4 = \text{constant} = A_0^2 + \frac{\lambda}{2} A_0^4 \tag{2.39}$$

where $(A_0, 0)$ is the initial condition for the trajectory. It is straightforward to check that Eq. (2.36) and Eq. (2.37) are in exact agreement with Eq. (2.39) to $\mathscr{O}(\lambda)$. Thus, perturbatively the correct phase portrait is obtained as expected.

At this point let's direct our attention at a couple of features in the above calculation. The divergence in perturbative solution originates when terms with the lowest harmonic *sine* and *cosine* are generated on the right hand side (inhomogeneous terms) at any order of perturbation. The coefficients of the lowest harmonic *sine* terms contribute to the amplitude flow, $dA/d\tau$ and while that of the lowest harmonic *cosine* terms are responsible for the phase flow, $d\theta/d\tau$. Thus, in order to derive the perturbative contribution at any order to the amplitude flow equation, all one needs to do is to read off the coefficient of the lowest harmonic *sine* term. Similarly for the phase flow equation one needs just the coefficient of lowest harmonic *cosine* term at every order. Finally also note that while the amplitude flow equation gives information about the nature of the oscillator and its amplitude, the phase flow equation gives the corrections, if any, to frequency at every order of perturbation.

Keeping this in mind, if one examines the structure of higher order terms in the above example, one finds that a *sine* term can never be generated on the RHS and hence $dA/d\tau = 0$ at all orders. This question of lowest harmonic *sine* and *cosine* terms is quite general and we can subsequently use this to write down the flow equation by mere inspection. It should be borne in mind however, that although only the lowest harmonic is necessary for writing down the flow equation, it is imperative to have all the relevant harmonics for writing down the actual solution, $x(t)$, at any order. We have demonstrated here that RG method can obtain an uniformly valid result perturbatively starting from a naive perturbation expansions. The problem of secular terms leading to divergent results is taken care of quite naturally

as we express the solution in terms of renormalized amplitude and phase, without having to resort any extra assumptions.

2.5 RG for non-autonomous systems

Now we shift our focus as to how one can apply RG method for a non-autonomous system. There are a few conceptual issues involved when it comes to interpreting the RG calculations for non-autonomous systems. We will try and illustrate these by considering Eq.(2.13) but with non-zero forcing $F \neq 0$ and $\lambda = 0$. We have dropped the nonlinearity just for ease of calculations. The damped driven oscillator is given by

$$\ddot{x} + \omega^2 x + k\dot{x} = F\cos\Omega t \tag{2.40}$$

We are interested in the case where the forcing frequency Ω is very close the natural frequency ω. We know that when there is a non-zero periodic external drive it will tend to force the oscillator to oscillate at the frequency of the forcing. In case the forcing and natural frequencies are very close to each other it can lead to resonance — i.e. a sharp increase in the amplitude of oscillations. Its our intention to find out the response of the amplitude with changes in the forcing frequency (in the region where $(\Omega - \omega) \ll 1$). We are interested in the asymptotic behaviour of the solution. In the presence of forcing the asymptotic solution will oscillate at the drive frequency and thus we look for solutions with frequency Ω. Hence, we rewrite Eq. (2.40) as,

$$\ddot{x} + \Omega^2 x = -k\dot{x} + F\cos\Omega t + (\Omega^2 - \omega^2)x. \tag{2.41}$$

We treat all the parameters k, F and $\Omega^2 - \omega^2$ as small to perturb about the centre $k = F = \Omega^2 - \omega^2 = 0$. we proceed with RG calculations from this point onwards in manner identical as explained in previous section. Doing so we obtain, to the first order in all these small parameters, the flow equations:

$$\frac{dA}{d\tau} = -\frac{kA}{2} - \frac{F\sin\theta}{2\Omega} \tag{2.42}$$

$$\frac{d\theta}{d\tau} = -\frac{F\cos\theta}{2\Omega A} + \Delta\omega \tag{2.43}$$

where $\Delta\omega \equiv \omega - \Omega$. The periodic solution in this case corresponds to the fixed point of the Eqs. (2.42) and (2.43). As discussed earlier the phase flow equation gives the correction to frequency. Since, Ω is maintained externally, it cannot change, implying $d\theta/d\tau = 0$. Also, existence of fixed point requires

$dA/d\tau = 0$. Therefore, the fixed point corresponds to the amplitude

$$A = F/[k^2 + 4(\Delta\omega)^2]^{1/2} \tag{2.44}$$

and the phase

$$\theta = \tan^{-1}[-k/2(\Delta\omega)]. \tag{2.45}$$

This is exactly in accordance with the literature of forced oscillators [3]. The stable non-zero fixed point in the evolution of A corresponds to a limit cycle in accordance with what we have claimed has to happen.

2.6 Advantage over linear stability analysis

The renormalization group method is also quite effective in distinguishing between a centre and a focus. The centre-focus problem is a well-known and unsolved problem in the qualitative theory of 2D dynamical systems. For a fixed of the dynamics with purely imaginary eigenvalues, basically the problem is to distinguish between centre (neighbourhood composed entirely of closed periodic orbits) and focus (neighbourhood composed of spiralling orbits). Linear stability analysis results for the so-called *hyperbolic* fixed points are reliable and robust in the sense that they remain correct as long as the nonlinearity is small. Focus, nodes etc fall under the category of hyperbolic fixed points while centre does not fall into this category and a linear stability result predicting a centre can always be dicey. Even the slightest nonlinearity can change the nature of the fixed point from a centre to a focus. However there are also cases where the linear stability result holds for example, the Duffing oscillator.

Let's consider a case where linear stability analysis predicts a centre which in reality is a focus. Consider the following dynamical system:

$$\dot{x} = -y + \varepsilon a x(x^2 + y^2) \tag{2.46}$$

$$\dot{y} = x + \varepsilon a y(x^2 + y^2) \tag{2.47}$$

Here, ε is a small positive parameter that facilitates a trial perturbative solution of the form: $x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$. The only fixed point of the above system is given by $(x,y) = (0,0)$. Since origin is the fixed point and (x,y) represent the deviation from the origin linearizing the above system is extremely trivial, we simply obtain the linearized system by dropping the nonlinear terms. This can easily be checked by a formal calculation which we don't show here. Thus the linearized system is

simply $x = -y$ and $y = x$ from which we can immediately write down the Jacobian

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.48}$$

One can immediately see that the above Jacobian furnishes purely imaginary eigenvalues suggesting that the fixed point $(0,0)$ is a center for all a. However, at this point let us transform the Eqs. (2.46) and (2.47 into its counterparts in polar coordinates. After some fairly simple algebraic manipulations we obtain the polar form of Eqs. (2.46) and (2.47) as

$$\dot{r} = ar^3 \tag{2.49}$$

$$\dot{\phi} = 1 \tag{2.50}$$

It is very straightforward to analyze this system in this form. It's easy to see that all trajectories rotate about the origin at constant angular velocity ($\dot{\phi} = 1$) and the radial motion depends on the parameter a. In reality, the origin is a stable spiral when $a < 0$ and an unstable spiral for $a > 0$.

Now, lets apply the RG method as prescribed by us in this thesis to this particular problem. The linear centre in this case is for $\varepsilon = 0$ and we can treat the nonlinearity as the perturbation. Following along the same lines as described in the previous sections one obtains the following flow equations in this case:

$$dA/d\tau = aA^3 \tag{2.51}$$

$$d\theta/d\tau = 0 \tag{2.52}$$

One immediately notes that in accordance with our scheme of classifying focus and center, from the above flow equations, one can easily extract the correct information regarding the nature of the fixed point in system Eqs. (2.46) and (2.47): if $a = 0$, $dA/d\tau = 0\ \forall A$ implying that the origin is a center; whereas if $a \neq 0$, $dA/d\tau = 0 iff A = 0$, making the origin a focus.

2.7 Discussions

We have establishes that RG is a simple yet powerful methodology for asymptotic analysis of solutions of dynamical systems. We have focussed on periodic solutions in 2D and have shown that our methodology easily handles the problem of secular terms that can arise in traditional perturbation

theory. However, there are certain caveats to the methodology and this is a good point to discuss few of them. The flow equations are calculated perturbatively and as discussed earlier we need to locate a center to perturb around. The job of locating a basic state around which to perturb can be straightforward for e.g. $\dot{x}_1 = x_2, \dot{x}_2 = -\partial V/\partial x_1$, where V is a general anharmonic potential: $V = x_1^2/2 + \lambda_1 x_1^3/3 + \lambda_1 x_1^4/4$. Here $(x_1, x_2) = (0,0)$ is a linear center around which perturbation theory can be done. Similarly for the Van der Pol oscillator given by $\dot{x}_1 = x_2, \dot{x}_2 = k\dot{x}_1(x_1^2 - 1) + \omega^2 x_1$, the origin is a center for $k = 0$. In case of the Lotka-Volterra model — $\dot{x}_1 = x_1 - x_1x_2, \dot{x}_2 = -x_2 + x_1x_2$ the origin is a saddle and $(1,1)$ is the center. And one can not apply RG straightaway. In order to investigate the periodic solution in this case the first step is to shift the origin of the system to the center at $(1,1)$. This is the first step in the process of determining the function $f(A)$. We will discuss this example in detail in the next chapter.

A somewhat more complicated situation arises in the case of Belushov-Zhabotinsky reaction [142, 143] system. In that case, a mere transfer of origin to the fixed point does not do the trick. This has to be followed by a proper setting of parameters in the problem to make the origin a centre and thus to find a basic state about which to do the perturbative RG. This is the starting point of any kind of perturbative approach. This gives rise to the possibility that we might end up with a situation where the given dynamical system may not have a relevant parameter which can be varied to find a suitable unperturbed state. In many cases this issue can be quite non-trivial. For instance the well known paradigm for a limit cycle is given by

$$\dot{z} = (1+i)z - \beta|z|^2 z \tag{2.53}$$

where $z = x + iy$ is the complex variable and the parameter $\beta > 0$. The only fixed point for this system is the origin and it is an unstable focus for all β which leads us to believe that RG analysis may not be possible here. However, we can overcome this difficulty by considering the more general system given by

$$\dot{z} = (\alpha_1 + i\alpha_2)z - \beta|z|^2 z \tag{2.54}$$

Now, the origin is a stable focus for $\alpha_1 < 0$, unstable focus for $\alpha_1 > 0$ and a center for $\alpha_1 = 0$. It is this center about which one can set up the perturbation theory.

Taking a lesson from the above case we will try and summarize the proposed methodology in the next few lines. The perturbative evaluation of flow equations (i.e. of $f(A)$ and $g(A)$) involves the following initial steps:

1. Find all the fixed points of the system and identify the linear centers.

2. If there are no linear centers, an extension of the parameter space can be attempted to see if a linear center can be located as the parameters are changed.

3. For every linear center, thus located, we need to perturbatively construct $f(A)$ and $g(A)$ and from the form of these function one may draw conclusions about the asymptotic nature of the said solution.

Finally if a linear center is absent even after extension of the parameter space this methodology in its present form can't handle the problem. So one can say that 2D dynamical systems which don't have linear terms at all are not amenable to RG treatment. However, we will show later in this work that there are classes of problems without any linear terms which can also be analyzed by RG using some techniques.

Chapter 3

Centre vs Limit Cycle

3.1 Introduction

Nonlinear systems having periodic solutions represent a very important class of dynamical systems. The fact that the system has a periodic solution makes such system ideal to model a plethora of natural phenomena as examples of periodic phenomena are abound in natural sciences e.g. oscillations in populations of prey and predator in an ecosystem, circadian rhythm, beating of a heart, cycles in weather conditions so on and so forth. One can go on listing such periodic phenomena which we see all around us in the physical world. The most popular method of modelling these varied phenomena is treating them as dynamical systems and using differential equations to represent the growth, decay or oscillations of the variables involved. A class of dynamical systems which show periodic motion can be *conservative systems*. A conservative system is a system where the total energy is always constant independent of the path and its completely reversible. We considered the example of Duffing oscillator in the last chapter and showed that RG can be used to conclude that this oscillator has a 'centre' at origin. The Duffing oscillator without the forcing and damping term: $\ddot{x} + x + \lambda x^3 = 0$ is a good example of a conservative system with periodic solutions. Centres can be very delicate in nature and can be destroyed with slightest of nonlinear perturbations. However, when the system is conservative nonlinear centres tend to be more robust as is the case for the unforced undamped Duffing oscillator. Quite a number of nonlinear oscillators fall under this category.

On the other hand we have the limit cycles which are isolated periodic orbits in the phase space. While centres can exist in both linear and nonlinear systems, limit cycles can occur only in nonlinear systems. The most important kind of limit cycle is the *stable limit cycle*, where all nearby trajectories spiral towards the isolated orbit. Existence of a stable limit cycle in a dynamical system means

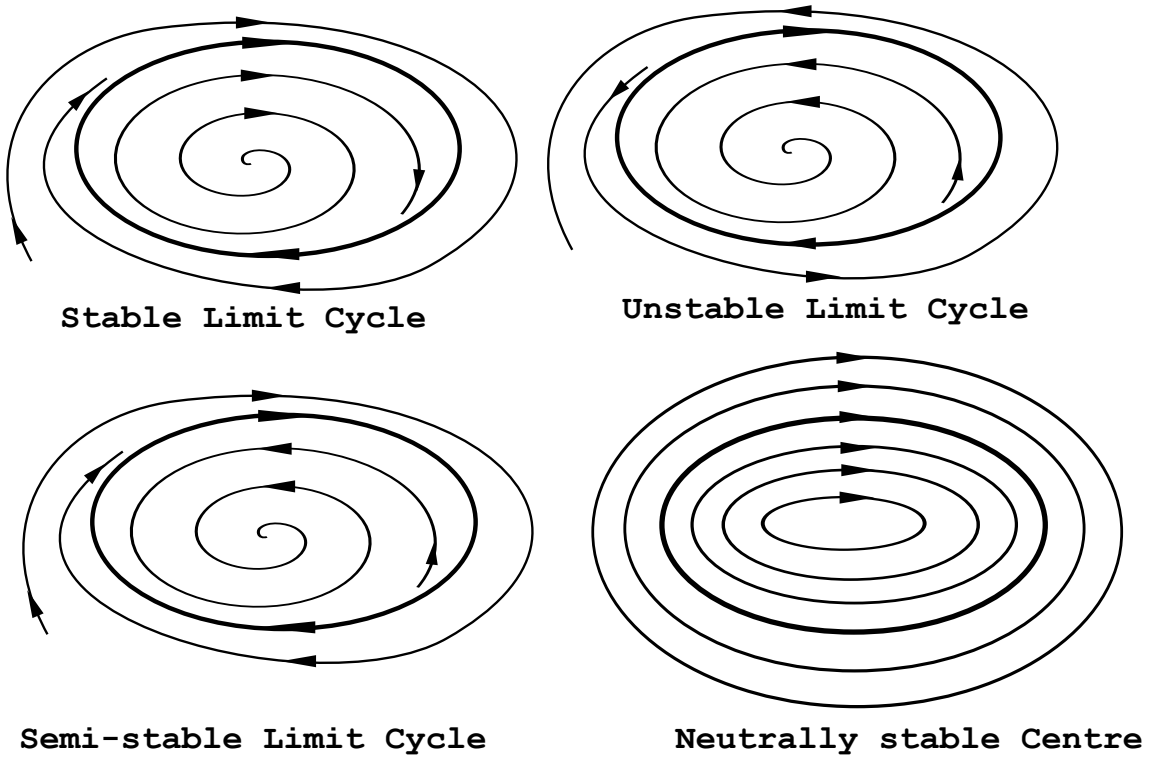

Figure 3.1: Various kinds of periodic trajectories in phase space.

there exists self-sustained oscillations in the system. Dynamical systems capable of having limit cycle oscillations are very important from the point of view modelling real-world systems which exhibit self-sustained oscillations. Some examples of such phenomena from nature include beating of our hearts [144, 145], oscillations in body temperature [146, 147], REM oscillations during sleep [148], hormone secretion [149, 150], chemical reactions that oscillate spontaneously [151–153] etc. There can be other varieties of limit cycles as well: (i) the *unstable limit cycle*, where all neighbouring trajectories tend to move away from the isolated orbit in phase space and (ii) *semi-stable limit cycle*, where trajectories are attracted to the limit cycle on one side and repelled on the other (see Fig 3.1). In a sense *centres* can be thought of as *neutrally stable limit cycles*. Although the lot of research effort has gone into finding ways to determine if a system has a limit cycle, surprisingly little is known about how to do this and this topic still remains a highly active topic of research.

Some open challenges in the field: At the beginning of the 20th century famous mathematician David Hilbert presented what appeared to be 23 daunting mathematical problems [154]. Amongst these ***Hilbert's 16th Problem*** deals with 2-D planar systems and is as yet unresolved [155–157]. The problem consists of two parts, the first part poses a question about relative positions of branches of algebraic curve and traditionally, specialists in the field of real algebraic geometry are interested in this

question. Its the second part of Hilbert's 16th problem which concerns dynamical systems theory. In a 1998 paper Smale [156] restates the problem as follows:

> *Consider the planar differential equation in* $\mathbb{R}^2$,
>
> $$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y) \tag{3.1}$$
>
> *where P and Q are polynomials. Is there a bound K on the number of limit cycles of the form* $K \leq d^q$, *where d is the maximum of the degrees of P and Q, and q is a universal constant.*

This is the modern version of second part of Hilbert's 16th problem and a satisfactory answer to this question has proved to be elusive till date. Traditionally, the above problem is split into further three parts.

The first being: *Is it true that a planar polynomial vector field has a finite number of limit cycles?*

The second part is: *Is it true that the number of limit cycles of a planar polynomial vector field is bounded by a constant depending only on the degree of the polynomials?* It is customary to denote the bound on the number of limit cycles by $H(n)$, which is known as the *Hilbert number*. Note that in case the vector fields are linear they can't have any limit cycles consequently for such systems $H(1) = 0$.

The final part is: *what is the upper bound* $H(n)$*?* If one can answer the third part it implies an answer to the second and which in turn solves the first part. However, apart from the first part of the question none of the other parts are even close to be answered. In 1923, Dulac claimed that the system (3.1) always has a finite number of limit cycles [158]. However, in 1985 Ilyashenko [159] found an error in Dulac's proof. Subsequently, though the first part of the question was successfully resolved [160, 161] and the answer is indeed in the affirmative. But, there's no resolution to the other parts. In fact the simple case of finding $H(2)$ is as yet unresolved. Consequently, people have resorted to answering bits and pieces of this problem, for instance choosing special systems where the bound can be found.

Another particularly daunting problem for planar ODE systems is the so called ***centre-focus*** (CF) problem. This is one of the oldest problems in qualitative theory of differential equations in 2D [14, 162, 163]. The problem is to distinguish between a *centre* and a *focus*. Mathematically speaking, if $\mathscr{P} \in \mathbb{R}^2$ be a fixed point or singular point of the system (3.1), its a *centre* if there is a neighbourhood $\mathscr{U}$ of $\mathscr{P}$ where all the orbits are periodic and on the other hand its a *focus* if their is a neighbourhood $\mathscr{U}$ where all trajectories spiral out or in. Linear stability analysis does distinguish between these two types of fixed point but if linear stability analysis predicts a centre, it's not always reliable [4]. The CF problem has been solved for a limited number of systems only and a general answer is still lacking.

The typical approaches to this problem are to reduce the system to its normal form [164] or find a formal first integral [165].

We present here a method of perturbatively constructing RG flow equations for the "amplitude" and "phase" of periodic solutions to autonomous differential equations in 2D. Its our contention that from the structure of the flow equations, conclusions as to whether a system has a centre or limit cycle can be drawn. Furthermore one can also distinguish between focus and centres using the amplitude flow equation. Our RG method thus touches upon a few aspects of the above-mentioned two rather difficult problems. We have not attempted anywhere to analyze to what extent RG method is capable of answering the above-mentioned problems because first of all our method is perturbative and secondly its motivated by the physical considerations and lacks the requisite mathematical rigour to answer the above questions. Our choice to use oscillators as the topic of our research was motivated by these outstanding problems and some of our results shed some light on these intractable issues. Basically we present a practical way for physicists to find the asymptotic behaviour of a wide range of dynamical systems (in particular oscillator) and to distinguish between centres, focii and limit cycles.

3.2 Centre

In this section, we take up the study of centers using RG. A *centre* refers to a family of initial-condition-dependent periodic orbits surrounding a point. Its a neutrally stable solution and the initial condition fixes the amplitude of the periodic orbit. While dealing with dynamical systems in $\mathbb{R}^2$ only two constants of integration are sufficient to uniquely characterize a solution. Typically a periodic solution can be characterized by two parameters — amplitude (A) and phase (θ). In the perturbative renormalization treatment of such dynamical systems its these quantities - amplitude and phase - that get renormalized. When the approximate solution is expressed in terms of these renormalized amplitude and phase, the perturbative series is uniformly valid and doesn't have any secular terms. Amplitude and phase are in some sense analogous to the coupling constants of a field theory when one does traditional RG. While the RG flow equation (β-function) in traditional RG relates the coupling constants at different energy or momentum scales, in CGO-RG (Chen-Goldenfeld-Oono RG), the amplitude and phase (in effect the *Cauchy* data) get related along a phase trajectory for different 'times'. In case of a centre since the initial condition fixes the amplitude of oscillations, the amplitude can't change and thus $dA/d\tau$ must vanish at every order of perturbation. The condition that "*the amplitude flow equation must vanish at each order of perturbation*" has to be satisfied for a centre. Further, for a given system if the amplitude flow equation vanishes when certain relation between the parameters of the system is satisfied one can obtain necessary conditions for the existence of a centre

in that dynamical system. We will now solve a few examples to show how this works in practice.

3.2.1 Cubic anharmonic oscillator

We have already dealt with the quartic anharmonic oscillator in the previous chapter. We consider the cubic anharmonic oscillator in order to demonstrate that in some cases the corrections to frequency may not come in the first order calculations and the perturbative calculation may have to be carried out to higher orders to obtain the relation. The equation of motion of this anharmonic nonlinear oscillator is

$$\ddot{x} + x + \alpha x^2 = 0. \tag{3.2}$$

It can also be written in the form of two first order differential equations as

$$\dot{x} = y, \tag{3.3}$$
$$\dot{y} = -x - \alpha x^2. \tag{3.4}$$

The first step is to identify all the fixed points (or singular points) in the system and in this case we have two — $(0,0)$ and $(-1/\alpha, 0)$. Linear stability analysis reveals that the point $(0,0)$ is a centre and $(-1/\alpha, 0)$ a saddle. We also notice that a linear centre exists for $\alpha = 0$. Hence the perturbation theory will have to be built around this limit. We are interested in finding out if the linear centre survives the quadratic perturbation or not. We expand x as

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots \tag{3.5}$$

Substituting Eq.(3.5) in Eq.(3.2), we obtain:

$$\ddot{x}_0 + \omega^2 x_0 = 0 \tag{3.6}$$
$$\ddot{x}_1 + \omega^2 x_1 = -x_0^2 \tag{3.7}$$
$$\ddot{x}_2 + \omega^2 x_2 = -x_1 x_0 \tag{3.8}$$

With the initial condition set as $x(t=0) = A_0$ and $\dot{x}(t=0) = 0$, we write the solution of Eq. (3.6) as

$$x_0 = A_0 \cos \omega t \tag{3.9}$$

As per our prescription, x_0 picks up the initial condition and hence $x_i(t=0) = \dot{x}_i(t=0) = 0 \; \forall \; i \geq 1$. Accordingly, Eq.(3.7) can be written as,

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= -\alpha A_0^2 \cos^2(\omega t + \theta) \\ &= -\frac{\alpha A_0^2}{2}\left(1 + \cos 2(\omega t + \theta)\right). \end{aligned} \tag{3.10}$$

to which we can immediately write down the solution,

$$x_1 = -\frac{\alpha A_0^2}{2}\left(1 - \cos(\omega t + \theta)\right) + \frac{\alpha A_0^2}{6}\left(\cos 2(\omega t + \theta) - \cos(\omega t + \theta)\right). \tag{3.11}$$

Notice that there are no secular terms at this order. Both phase and amplitude flow equations at this order are zero. But its easy to see from the structure of x_1, that secular terms will be generated at the next order. Taking the calculation to the next order, we have

$$\begin{aligned} \ddot{x}_2 + \omega^2 x_2 &= -2\alpha A_0 \cos\phi \left(-\frac{\alpha A_0^2}{2} + \frac{\alpha A_0^2}{2}\cos\phi + \frac{\alpha A_0^2}{6}\cos 2\phi - \frac{\alpha A_0^2}{6}\cos\phi\right) \\ &= -\frac{5\alpha^2 A_0^2}{6}\cos\phi + \frac{\alpha A_0^2}{3}(1 + \cos 2\phi) + \frac{\alpha A_0^2}{6}\cos 3\phi \end{aligned} \tag{3.12}$$

where $\phi = (\omega t + \theta)$. At this point we can split the interval 0 to t as 0 to τ and τ to t and we introduce the renormalization constants $\mathscr{Z}_1$ and $\mathscr{Z}_2$, to remove the divergences and obtain the flow equations,

$$\frac{dA}{d\tau} = 0 \tag{3.13}$$

$$\frac{d\theta}{d\tau} = \frac{5\alpha A^2}{12} \tag{3.14}$$

integrating to $A(t) = A_0$ and $\theta = \theta_0 + \frac{5\alpha A^2}{12}\tau$. From the phase flow equation we can immediately write down the frequency-amplitude relation

$$\Omega = 1 + \frac{5\alpha A^2}{12} + \mathscr{O}(\alpha^3) \tag{3.15}$$

The origin, $x = \dot{x} = 0$, is confirmed as a centre by the fact that $dA/d\tau = 0$. The frequency-amplitude relation obtained is also the standard result that can be obtained by other methods [166, 167]. As we have already explained the *sine* term is responsible for the amplitude flow, $dA/d\tau$ and the *cosine* term is responsible for the phase flow, $d\theta/d\tau$. If we examine the structure of higher order terms in the above example, we find that a *sine* term is never generated on the right hand side and hence $dA/d\tau = 0$ for all higher orders as well. So as long as the anharmonicity is weak Eq. (3.2) has a centre at origin. Since,

there is saddle at $(-1/\alpha, 0)$ the amplitude of oscillations has to be small enough so that the trajectory is sufficiently away from the saddle for this result to be valid.

3.2.2 Lotka-Volterra system

As our next example we consider the famous population dynamics model proposed by Lotka and Volterra [168]. The general system is given by the equations,

$$\frac{dx}{dt} = ax - bxy \tag{3.16}$$

$$\frac{dy}{dt} = -cy + dxy \tag{3.17}$$

where x represents the prey population, y represents the predator population while a, b, c and d are positive parameters describing the interactions of the different species. Note that, by definition $x \geq 0$ and $y \geq 0$, because obviously negative population has no meaning. dx/dt repents the rate of change of prey population. It is determined by the fact that by themselves a prey population, say rabbits, will flourish in an ecosystem with plenty of available food (grass) and reproduce proportional to their numbers 'ax'; But when they interact with a predator population, say wolves, their number reduces, hence '$-bxy$'. On the other hand rate of change of predator population, dy/dt, is positively affected by interactions with prey population, so dxy; But left to themselves with no food they tend to die off and hence '$-cy$'. The typical phase portrait is shown in Fig. 3.2 and the cyclic variations in prey and predator populations is shown in Fig. 3.3. This exceedingly simple model has been at the root of

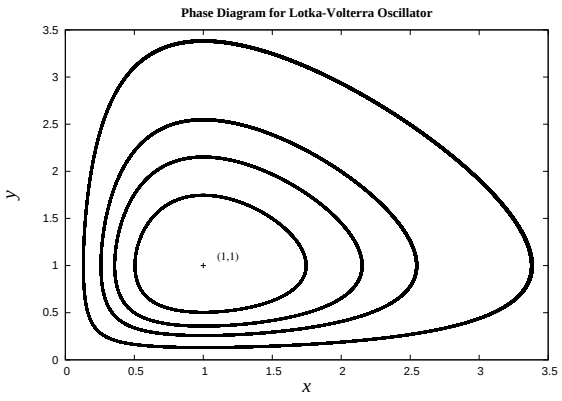

Figure 3.2: A typical phase portrait of the to Lotka-Volterra system obtained by numerically integrating the equation for different initial conditions.

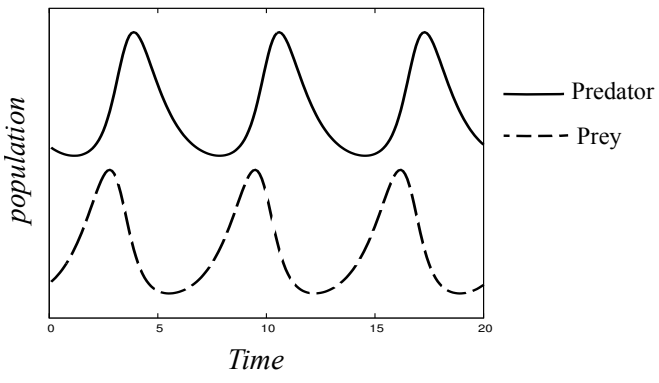

Figure 3.3: The prey and predator populations undergo oscillations (time in arbitrary units). However, notice that the oscillations are out of phase.

numerous investigations into population dynamics over the years. It predicts the existence of periodic

population cycles where the time period plays an important role. It was Frame who initially gave an expression for the time period, T [169]. Grasman et al have also given an asymptotic formula for the period [170]. Apart from having applications in population biology and ecology Lotka-Volterra model and its variations, have in the last few decades extended its applicability to a wide vista of fields. It has found applications in fields like economics [171, 172], network-electronics [173, 174], membrane dynamics of competing neurons [175], neural networks [176] and stochastic dynamics [177] etc.

We will see how RG may be applied in case of this equation and how the amplitude-frequency relation can be derived. Without loss of generality we may take the parameters a, b, c and d as unity and rewrite system (3.16) and (3.17) as,

$$\frac{dx}{dt} = x - xy, \tag{3.18}$$

$$\frac{dy}{dt} = -y + xy. \tag{3.19}$$

This is done in order to simplify the mathematical expressions. Its easy to see that there are two fixed points $(0,0)$ and $(1,1)$. Linear stability analysis reveals the origin to be a *saddle* and that there is a *center* at $(1,1)$. Our interest is in the periodic response of this set of equations. In order to probe that the first step is to shift the origin of the co-ordinate system to the centre, $(1,1)$. We will follow this strategy regularly in our perturbative calculations of the periodic trajectories. Accordingly we define

$$x = X + 1; \qquad y = Y + 1 \tag{3.20}$$

and write the system as:

$$\dot{X} = -Y - XY, \tag{3.21}$$

$$\dot{Y} = X + XY. \tag{3.22}$$

The perturbative theory proceeds by imagining the existence of a coupling constant λ in terms of which we have

$$\dot{X} = -Y - \lambda XY, \tag{3.23}$$

$$\dot{Y} = X + \lambda XY. \tag{3.24}$$

We expand X and Y in powers of λ as $X = X_0 + \lambda X_1 + \lambda^2 X_2 + \cdots$ and $Y = Y_0 + \lambda Y_1 + \lambda^2 Y_2 + \cdots$

respectively to subsequently arrive at the following set of equations at different orders of perturbation:

$$\begin{aligned}
\mathscr{O}(\lambda^0): \qquad \dot{X}_0 &= -Y_0, && (3.25)\\
\dot{Y}_0 &= X_0; && (3.26)\\
\mathscr{O}(\lambda^1): \qquad \dot{X}_1 &= -Y_1 - X_0 Y_0, && (3.27)\\
\dot{Y}_1 &= X_1 + X_0 Y_0; && (3.28)\\
\mathscr{O}(\lambda^2): \qquad \dot{X}_2 &= -Y_2 - (X_0 Y_1 + Y_0 X_1), && (3.29)\\
\dot{Y}_2 &= X_2 + (X_0 Y_1 + Y_0 X_1); && (3.30)
\end{aligned}$$

and so on. Clearly at the lowest order (initial condition being $x = A_0$, $\dot{x} = 0$ at $t = 0$), we have

$$X_0 = A_0 \cos t \quad \text{and} \quad Y_0 = A_0 \sin t. \tag{3.31}$$

At the next order, the equation can be written as

$$\ddot{X}_1 + X_1 = -(L_0 + \dot{L}_0) \tag{3.32}$$

where $L_0 = X_0 Y_0$ and hence plugging in the zeroth order solutions we have

$$\ddot{X}_1 + X_1 = -\left(\frac{A_0^2}{2}\sin 2t + A_0^2 \cos 2t\right). \tag{3.33}$$

There are no resonating terms on the right hand side and thus we have the solution upto $\mathscr{O}(\lambda)$,

$$X_1 = \frac{A_0^2}{6}(\sin 2t - \sin t) + \frac{A_0^2}{3}(\cos 2t - \cos t) \tag{3.34}$$

keeping the initial condition $X_1(t = 0) = \dot{X}_1(t = 0) = 0$ in mind.

At the succeeding order

$$\ddot{X}_2 + X_2 = -(L_{10} + \dot{L}_{10}) \tag{3.35}$$

where $L_{10} = X_0 Y_1 + Y_0 X_1$. Note that Y_1 can be easily obtained from the Eq. (3.28). If now one calculates the RHS of Eq. (3.35) the only *secular term* turns out to be '$A_0^3 \cos t/12$' and which will contribute to the phase flow equation (keeping in mind our previous discussions). If we follow the renormalization

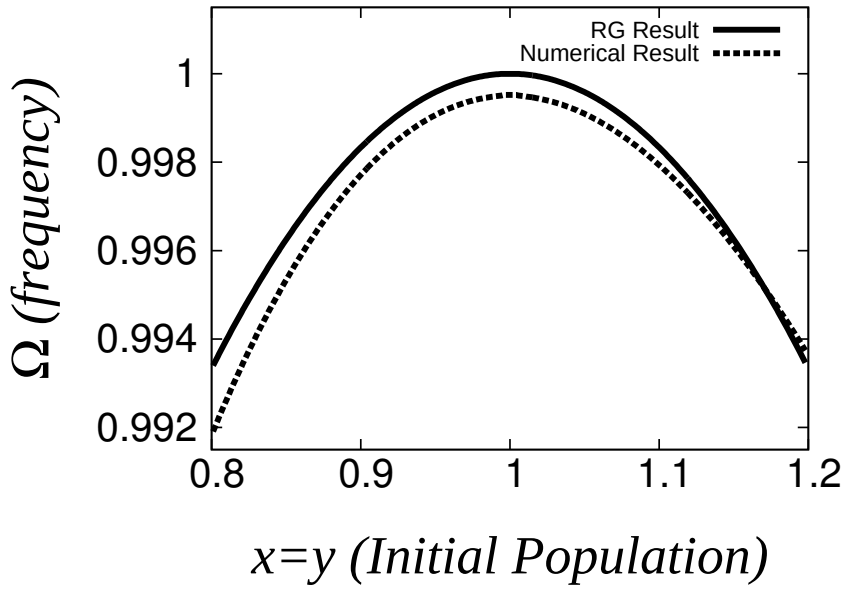

Figure 3.4: The initial population density is plotted against the corresponding frequency obtained from the solutions to Lotka-Volterra equations both by RG and numerically. The figure compares the result from RG calculations with the numerically obtained result.

process, we finally get the flow equations,

$$\frac{dA}{d\tau} = 0 \tag{3.36}$$

$$\frac{d\theta}{d\tau} = -\frac{A^2}{12} \tag{3.37}$$

This ultimately gives an amplitude dependent frequency of

$$\Omega = 1 - \frac{A_0^2}{12} \tag{3.38}$$

which we have verified numerically (see Fig. 3.4). We obtained the amplitude-frequency relation numerically by integrating the Eqs. (3.18) and (3.19) using a Runge-Kutta 4th order algorithm. We have plotted the amplitude-dependent frequency (Ω) given by Eq. (3.38) and numerically obtained result in Fig. 3.4 and as it can be easily seen they are in excellent agreement.

3.2.3 A Liénard system

We next consider a dynamical system which belongs to the family of Liénard systems. The most commonly occurring periodic motions in natural sciences and engineering applications, belong to this second-order autonomous system of differential equations. Examples of Liénard systems include several important oscillators such as Rayleigh, Van der Pol, Duffing etc. Liénard equation of the *first kind* is given by

$$\ddot{x}+\dot{x}\mathscr{F}(x)+\mathscr{G}(x)=0. \tag{3.39}$$

Liénard equation of the *second kind* differs from the above equation in that in the second term, $\dot{x}^2$ instead of $\dot{x}$, multiplies $\mathscr{F}(x)$. We will mainly focus on the Liénard equation of the *first kind* and there exists a rich range of literature on this 2nd order ODE [178–180]. Liénard equation can lead to both limit cycles and family of periodic orbits around a centre depending on its specific form and its often necessary to understand the specific nature of solutions of such systems. Various theorems and methods have been developed over the years to address this question [13, 181–186]. There has also been a lot of interest on issues of integrability of Liénard systems [187–190]. Despite such a rich body of research, an unified way of determining whether a particular Liénard system has a limit cycle or centre is lacking in literature. However, recently RG approach was shown to be effective in providing a unified analysis of the limit cycle and centre in a Liénard system of differential equations [191]. We will take a few examples from this important class of systems to demonstrate the efficacy of our RG method.

The first Liénard system we consider has $\mathscr{F}(x)=-(\alpha+\beta x)$ and $\mathscr{G}(x)=x+x^2$ and thus we have

$$\ddot{x}-\dot{x}(\alpha+\beta x)+x+x^2=0 \tag{3.40}$$

Writing it as a dynamical system,

$$\dot{x} = y \tag{3.41}$$
$$\dot{y} = -x-x^2+y(\alpha+\beta x) \tag{3.42}$$

The only two fixed points of the above system are at $(0,0)$ and $(-1,0)$. Linear stability analysis reveals while the former is a center, the latter a saddle. Our intention is to investigate the oscillatory orbit around $(0,0)$. We redefine α & β as $\alpha=1+\mu$ & $\beta=1$ respectively, and we also introduce small parameters k and λ and rewrite Eq. (3.40)

$$\ddot{x}-k\dot{x}(1+x+\mu)+x+\lambda x^2=0 \tag{3.43}$$

Clearly the perturbation theory has to proceed around the linear center which which is obtained for $k = \lambda = 0$. Accordingly, we expand x in powers of the two small parameters k and λ,

$$x = x_0 + kx_1 + \lambda x_1' + k^2 x_2 + \lambda^2 x_2' + k\lambda x_2'' + \cdots. \tag{3.44}$$

At different orders of perturbation we have,

$$\begin{aligned}
\mathcal{O}(k^0\lambda^0): \quad & \ddot{x}_0 + x_0 = 0 && (3.45)\\
\mathcal{O}(k^1\lambda^0): \quad & \ddot{x}_1 + x_1 = \dot{x}_0(1+\mu) + x_0\dot{x}_0 && (3.46)\\
\mathcal{O}(k^0\lambda^1): \quad & \ddot{x}_1' + x_1' = x_0^2 && (3.47)\\
\mathcal{O}(k^2\lambda^0): \quad & \ddot{x}_2 + x_2 = \dot{x}_1(1+\mu) + x_1\dot{x}_0 + x_0\dot{x}_1 && (3.48)\\
\mathcal{O}(k^0\lambda^2): \quad & \ddot{x}_2' + x_2' = -2x_0x_1' && (3.49)\\
\mathcal{O}(k^1\lambda^1): \quad & \ddot{x}_2'' + x_2'' = x_1'(1+\mu) + x_0\dot{x'}_1 + x_1'\dot{x}_0 - 2x_0x_1 && (3.50)
\end{aligned}$$

and so on. We take the initial condition $x = A_0$, $\dot{x} = 0$ at $t = 0$, and obtain the solutions until $\mathcal{O}(\lambda)$:

$$\begin{aligned}
x_0 &= A_0\cos t && (3.51)\\
x_1 &= (1+\mu)\frac{A_0}{2}(t\cos t - \sin t) + \frac{A_0^2}{6}\big(\sin 2t - 2\sin t\big) && (3.52)\\
x_1' &= -\frac{A_0^2}{2} + \frac{A_0^2}{6}\cos 2t + \frac{A_0^2}{3}\cos t && (3.53)
\end{aligned}$$

Some fairly straightforward algebra yields the corresponding flow equations at this order, given by

$$\begin{aligned}
\frac{dA}{d\tau} &= \frac{kA}{2}(1+\mu) && (3.54)\\
\frac{d\theta}{d\tau} &= 0 && (3.55)
\end{aligned}$$

If we were to have the possibility of a center, the amplitude flow equation must vanish and clearly $\mu = -1$. In this section we focus on the potential non-linear center and work with $\mu = -1$. We need to take the calculation to the next order of perturbation to obtain the amplitude-frequency relation. Without going into the details of the calculations for the next order we simply quote the second-order-perturbation result. At the second order, the flow becomes

$$\begin{aligned}
\frac{dA}{d\tau} &= -k\lambda\frac{A^3}{8} && (3.56)\\
\frac{d\theta}{d\tau} &= -\frac{5}{12}\lambda^2A^2 - \frac{1}{24}k^2A^2 && (3.57)
\end{aligned}$$

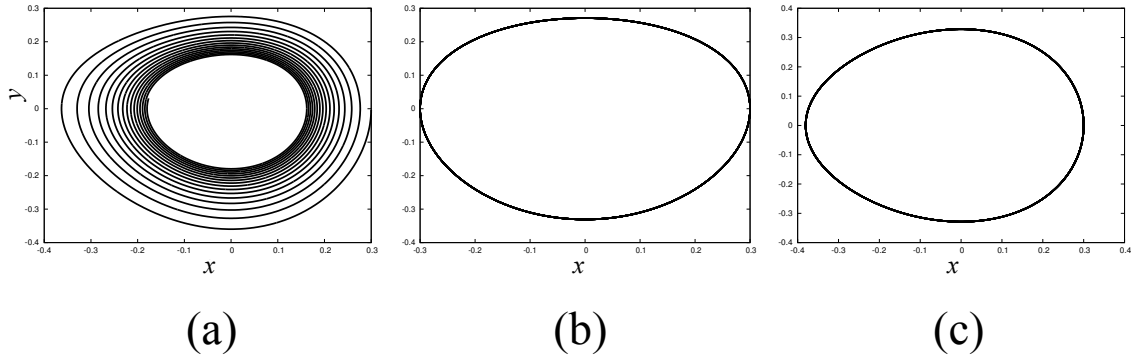

Figure 3.5: Phase portrait for Eq. (3.43) with $\mu = -1.0$, initial conditions $x(0) = 0.3$; $\dot{x}(0) = 0.0$ and varying values of the parameters k and λ (a) $k = 1.0$ and $\lambda = 1.0$ and as expected we get a *focus*;(b) $k = 1.0$ and $\lambda = 0.0$ and as predicted by RG we get a *centre*; (c) $k = 0.0$ and $\lambda = 1.0$ and we get a *centre* as expected.

From the flow equation in accordance with our prescription one can conclude that the origin is an attractor (focus) if both k and λ are non-zero. On the other hand if either or both of k and λ are zero, we have a centre at the origin. The phase flow equation gives the correction to frequency. When $k = 0$ and $\lambda \neq 0$ the corrected frequency $\Omega = 1 - 5\lambda A^2/12$. It must be kept in mind that the periodic solution can exist for only such values of λ for which the frequency, Ω, remains positive. A similar situation occurs for the nonlinear centre when $\lambda = 0$ and $k \neq 0$. The validity of our findings are easily verified numerically. In Fig. 3.5, numerically determined phase portraits are plotted and the results confirm the prediction from RG calculations. The above case is yet another instance where increasing the space of parameters in a dynamical system, makes the system much more interesting. Starting with Eq. (3.40), one would not have access to the different possibilities that we have found here — e.g. the competition between the centre and the focus. This is made possible by considering a more general system by introducing the two parameters k and λ. The general system has been treated in [191].

3.3 Limit cycle

In this section we illustrate our method used on few representative examples of systems where limit cycles occur. In case a system has a stable limit cycle all neighbouring trajectories asymptotically approach the limiting orbit. This suggests that the amplitude phase flow has a non-zero fixed point and all trajectories are such that the amplitude asymptotically approaches this particular value. As discussed earlier, in case limit cycles exist in a system, the RG flow equation for amplitude takes up the form, $dA/d\tau = f(A)$, and the function $f(A)$ is such that it yields non-zero fixed points. The non-zero fixed points, in other words, the non-zero values of A where $f(A)$ vanishes, give the radii of the corresponding limit cycles. Moreover, the stability of the limit cycle is determined from the stability

of the corresponding fixed point of the flow equation.

A word of caution is in order here. Determination of limit cycles using perturbative RG approach can be a tricky business — there is no guarantee that once a limit cycle is determined at a certain order of expansion, taking the perturbative calculations to higher orders would not change its amplitude, stability and basin of attraction. However in the weak nonlinearity limit, our method reproduces the correct perturbative result, for a wide range of problems. When the nonlinear perturbation is weak enough the major contribution to the perturbative series is expected to come from the lowest non-trivial order. Any corrections that come from higher order calculations are minor corrections to the limit cycle radius at best. There is almost no or miniscule chance that the nature of the result itself can change with higher order corrections. Thus, in all our calculations, we inherently assume that the problem is perturbatively renormalizable. We have carried out RG calculations for numerous systems and this assumption seems to hold and no ambiguity arises in any of the cases.

There are a few considerations one needs to keep in mind while doing perturbative RG calculations for determining existence and radius of limit cycle in a given system. For instance, one must be in the proper parameter range for the method to work. We have chosen our examples in such a manner so as to illustrate these subtle points. But before we go into the details of applying RG to such systems we briefly survey the existing methods of determining existence and configuration of limit cycles in a dynamical system.

3.3.1 Existence of limit cycle

The question whether a given system has a limit cycle or not has been a major topic of research over the years. One of main tool which has been applied over the years to determine existence of limit cycles in $\mathbb{R}^2$ is the celebrated *Poincaré-Bendixson Theorem*. Consider the 2D system given by,

$$\dot{x} = f(x,y), \tag{3.58}$$

$$\dot{y} = g(x,y), \tag{3.59}$$

where f and g are in general nonlinear functions of x and y. Whether the above system has a limit cycle can be concluded from the following theorem [3]:

> ***Poincaré-Bendixson Theorem***: *Suppose $\mathscr{R}$ be a closed, bounded region in 2D plane, on which the system $\dot{x} = f(x,y), \dot{y} = g(x,y)$ is regular. If a positive half-path $\mathscr{H}$ lies entirely on R, then either*
>
> i. *$\mathscr{H}$ consists of a closed phase path on $\mathscr{R}$;*

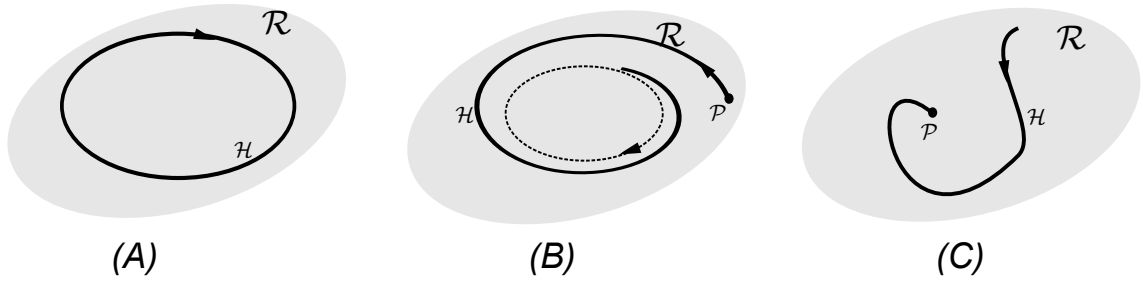

Figure 3.6: (A) $\mathscr{H}$ is a closed orbit, (B) $\mathscr{H}$ approaches a limit cycle; (c) $\mathscr{H}$ approaches an equilibrium point $\mathscr{P}$.

ii. *$\mathscr{H}$ approaches a closed phase path on $\mathscr{R}$*;

iii. *$\mathscr{H}$ approaches an equilibrium point on $\mathscr{R}$.*

A positive *half-path* is simply a path in phase space from a particular point (x_0, y_0) (corresponding to $t = t_0$) onwards, *i.e.*, for all $t \geq t_0$. The theorem confirms what one expects (illustrated in Fig. 3.6): that for any regular system there can be no other limiting cases except either closed paths or various equilibrium points. To apply the theorem practically we present it a little differently as follows:

> *Suppose $\mathscr{R}$ is the finite region of the plane lying between two simple closed curves $D1$ and $D2$, and $\mathbf{F}$ is the velocity vector field for the system (Eqs. (3.58) & (3.59)). If*
>
> 1. *at each point of D_1 and D_2, the field $\mathbf{F}$ points toward the interior of $\mathscr{R}$, and*
> 2. *$\mathscr{R}$ contains no critical points,*
>
> *then the system (Eqs. (3.58) & (3.59)) has a closed trajectory lying inside $\mathscr{R}$.*

The hypothesis of the above statement in illustrated in Fig. 3.7. We will not give a proof of the theorem, which is subtle and rather difficult, requires a background of topology (see [13] and [181] or [192, 193] for details). However, the theorem can be understood intuitively. If one starts on one of the boundary curves, the solution will enter the region $\mathscr{R}$, since the velocity vector points into the interior of $\mathscr{R}$. As time goes on, the solution cannot ever leave $\mathscr{R}$, since every time it approaches a boundary curve, trying to escape from $\mathscr{R}$, the velocity vectors pointing inwards, force it to stay inside $\mathscr{R}$. Since, the phase trajectory can never leave $\mathscr{R}$, the only thing remaining for it to do as $t \to \infty$ is either approach a fixed point — of which there are none within $\mathscr{R}$, by hypothesis — or spiral in towards a closed trajectory. Thus there has to be a a closed trajectory inside $\mathscr{R}$. Apart from the Poincaré-Bendixson theorem there aren't very many strong existence theorems for limit cycles. However, over the years several works have been done on the topic existence of limit cycles and finding out their configuration in phase space. For instance, Delamotte developed a non-perturbative method based on harmonic series expansion for

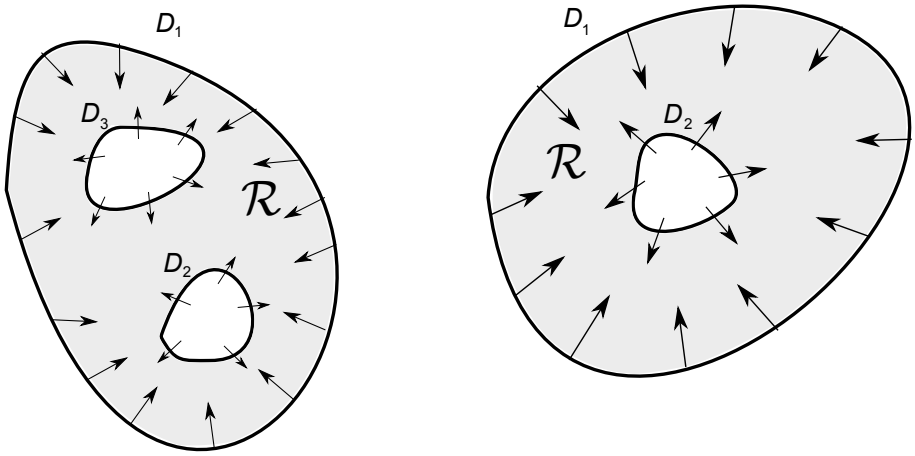

Figure 3.7: To use the Poincaré-Bendixson theorem, one has to search the vector field in $\mathbb{R}^2$ for closed curves D along which all the velocity vectors point towards the same side. Here is a schematic diagram showing such scenarios.

determining configuration of limit cycles in 2D [35,36]; Giacomini et al [130,186] developed a method of determination of limit cycles for a special class of 2D systems. There are several other studies in dynamical systems literature along this direction but most of them have very restrictive applicability.

A discussion on the issue of existence of limit cycles can not be complete without a discussion of the *non-existence theorems* and *criteria*. Without giving any proofs we will briefly mention the negative criteria that are used in the literature. The first such criteria we list is: *Gradient systems can't have closed orbits*. Gradient systems are defined systems which can be expressed as $\dot{\mathbf{x}} = -\nabla V$, $V(\mathbf{x})$ being a continuously differentiable, single-valued scalar function. This statement is very easy to prove [4].

The next criteria is: *If **Liapunov function** exists for a system, closed orbits are forbidden*. Consider a system: $\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x})$, for which $\mathbf{x}^*$ is a fixed point. *Liapunov function* for this system is defined as a continuously differentiable, real-valued function $V(\mathbf{x})$ such that

(a) $V(\mathbf{x}) > 0 \; \forall \; \mathbf{x} \neq \mathbf{x}^*$ i.e. V is positive definite;

(b) $\dot{V} < 0 \; \forall \; \mathbf{x} \neq \mathbf{x}^*$ i.e. all trajectories flow downhill towards $\mathbf{x}^*$.

Then for all possible initial conditions $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$ i.e. $\mathbf{x}^*$ is globally asymptotically stable (see [3] for proof). This means that this system can't have any closed orbits.

Then we have the so called ***Bendixson's negative criterion*** which states:

> *If $\partial f/\partial x$ and $\partial g/\partial y$ are continuous in a simply-connected region $\mathscr{R}$ (i.e. without holes), and*
>
> $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} \neq 0 \quad \text{at any point of } \mathscr{R}$$

then the system (Eqs. (3.58) & (3.59)) has no closed trajectories inside $\mathscr{R}$.

An extension of this theorem is called the ***Dulac's Criterion*** which states: *If $\dot{\boldsymbol{x}} = -\boldsymbol{f}(\boldsymbol{x})$, be a continuously differentiable vector field defined on a simply connected region $\mathscr{R}$ in 2D and there exists a continuously differentiable function $\boldsymbol{g}(\boldsymbol{x})$ such that $\nabla \cdot (\boldsymbol{g}\dot{x})$ doesn't change sign throughout $\mathscr{R}$, then there can be no closed orbits inside $\mathscr{R}$.*

Another often used argument is the ***critical-point criterion*** which simply states: "*A closed trajectory has a critical point in its interior*". If we turn this statement around, we see that it's really a criterion for non-existence. It says that if a region $\mathscr{R}$ is simply-connected and has no critical points, then it cannot contain any limit cycles. For if it did, the Critical-point Criterion says there would be a critical point inside the limit cycle, and this point would also lie in $\mathscr{R}$ since $\mathscr{R}$ has no holes. Note the distinction between this theorem, which says that limit cycles enclose regions which do contain critical points, and the Poincare-Bendixson theorem, which implies that limit cycles tend to lie in regions which dont contain critical points. The difference is that these latter regions always contain a hole and the critical points are in the hole(See Fig. 3.7).

The chief trouble with all these methods listed above is that none of them are systematic in nature. One has to apply these theorems and criteria on a case by case basis. There is no well-defined manner in which one can obtain the suitable curves (with all velocity vectors directed inside), enclosing a region without critical points, so that Poincaré-Bendixson theorem can be applied; Neither is there anyway of systematically determining the *Liapunov function* or for that matter determining a suitable ***g(x)*** satisfying the *Dulac's criterion*. Liénard himself established conditions on $f(x)$ of Liénard System (3.39) which guarantee the existence of a single limit cycle. Subsequently there have been some progress in dealing with the existence problems in certain Liénard systems (see for e.g. [194], [195]). A satisfactory resolution to the question of existence or non-existence of limit cycles is far from being achieved. It can safely said that there is no general algorithm to prove or disprove the existence of limit cycles in a given 2D system.

3.3.2 Van der Pol oscillator

We will now demonstrate with help of a representative set of examples how RG can be used to answer the issue of existence of limit cycle in planar systems, in the weak nonlinearity limit. Although perturbative RG is far from being a fool-proof approach to this question it has the advantage of being systematic. This makes it a very bright prospect in dealing with question of existence, stability, configuration and number of limit cycles in a given weakly nonlinear system. As our first example we will consider the celebrated Van der Pol oscillator [197–200] (which is a Liénard system with

$f(x) = \varepsilon(x^2 - 1)$ and $g(x) = \omega^2 x$), first proposed by Dutch physicist B. Van der Pol to describe triode oscillations in electrical circuits, in 1927. This system is represented by the equation

$$\ddot{x} + \varepsilon\dot{x}\left(x^2 - 1\right) + \omega^2 x = 0 \tag{3.60}$$

Van der Pol equation is a well known paradigm for limit cycles and in the strong nonlinear limit it exhibits relaxation oscillations [4]. Typical phase portraits for the oscillator with weak and strong nonlinearity are shown in Fig. 3.8. We will investigate this equation in the weak nonlinear limit. When

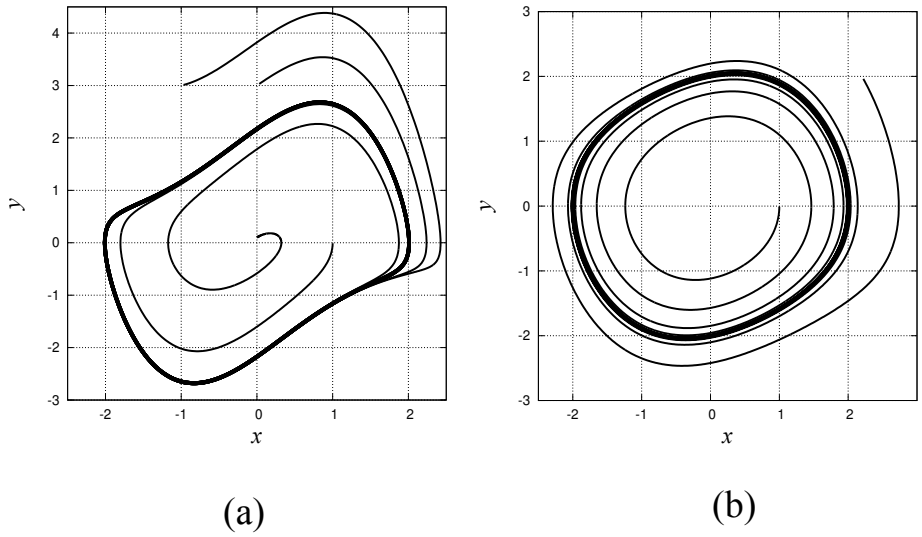

Figure 3.8: (a)Phase portrait for $\varepsilon = 1.0$. (b) Phase portrait for $\varepsilon = 0.20$

written as a 2D dynamical system $\dot{x} = y$, $\dot{y} = -\varepsilon y(x^2 - 1) - \omega^2 x$, the only fixed point is at the origin which is a focus and its stability depends on the sign of ε; The focus is stable for $\varepsilon < 0$ and its unstable for $\varepsilon > 0$. Notice that, the fixed point is a center for $\varepsilon = 0$ and we base our perturbation calculation around this limit.

We begin with a naive expansion for x as

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots \tag{3.61}$$

At different orders of ε,

$$\ddot{x}_0 + \omega^2 x_0 = 0 \tag{3.62}$$

$$\ddot{x}_1 + \omega^2 x_1 = -\dot{x}_0(x_0^2 - 1) \tag{3.63}$$

As usual we work with initial condition $x = A_0$ at $t = 0$ and $\dot{x} = 0$ at $t = 0$. Keeping this in mind, we can write down the solutions at the lowest order

$$x_0 = A_0 \cos(\omega t + \theta_0) \tag{3.64}$$

We can plug in this above answer to the $\mathscr{O}(\varepsilon)$ Eq. (3.63) to obtain

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= -\dot{x}_0(x_0^2 - 1) \\ &= A_0\left(\frac{A_0^2}{2} - 1\right)\sin(\omega t + \theta_0) + \frac{A_0^3}{4}\sin 3(\omega t + \theta_0) \end{aligned} \tag{3.65}$$

to which we can immediately write down the solution,

$$x_1 = \frac{1}{2}\left(A_0 - \frac{A_0^3}{4}\right) t\cos(\omega t + \theta_0) - \frac{A_0^3}{32\omega^2}\Big(\sin 3(\omega t + \theta_0) - 3\sin(\omega t + \theta_0)\Big) \tag{3.66}$$

leading to the answer at order $\mathscr{O}(\lambda)$

$$x = A_0\cos\phi + \varepsilon\Big[\frac{1}{2}\left(A_0 - \frac{A_0^3}{4}\right) t\cos\phi - \frac{A_0^3}{32\omega^2}\left(\sin 3\phi - 3\sin\phi\right)\Big], \tag{3.67}$$

where $\phi = (\omega t + \theta_0)$. As before we split the interval 0 to t as 0 to τ & τ to t and define the renormalization constants $\mathscr{Z}_1$ and $\mathscr{Z}_2$ by the relation

$$A_0 = A(\tau)\mathscr{Z}_1(0,\tau) \tag{3.68}$$

$$\theta_0 = \theta(\tau) + \mathscr{Z}_2(0,\tau) \tag{3.69}$$

The renormalization constants $\mathscr{Z}_1$ and $\mathscr{Z}_2$ can be expanded as

$$\mathscr{Z}_1(0,\tau) = 1 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \ldots \tag{3.70}$$

$$\mathscr{Z}_2(0,\tau) = \beta_1\varepsilon + \beta_2\varepsilon^2 + \ldots \tag{3.71}$$

To $\mathscr{O}(\varepsilon)$, we now have

$$\begin{aligned} x(t) &= A(1 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \ldots)\big[\cos\phi - (\beta_1\varepsilon + \beta_2\varepsilon^2 + \cdots)\sin\phi\big] \\ &\quad + \frac{\varepsilon}{2}\left(A - \frac{A^3}{4}\right)(t - \tau + \tau)\cos\phi - \frac{\varepsilon A^3}{32\omega^2}\big(\sin 3\phi - 3\sin\phi\big) \end{aligned} \tag{3.72}$$

The choice

$$\alpha_1 = -\frac{\varepsilon}{2}\left(A-\frac{A^3}{4}\right)\tau \tag{3.73}$$

$$\beta_1 = 0 \tag{3.74}$$

removes the secular terms upto this order i.e. in some sense divergence from the *past*. We are now left with a renormalized perturbation series

$$\begin{aligned} x(t) &= A\cos(\omega t+\theta)+\frac{\varepsilon}{2}\left(A-\frac{A^3}{4}\right)(t-\tau)\cos(\omega t+\theta) \\ &\quad -\frac{\varepsilon A^3}{32\omega^2}\Big(\sin 3(\omega t+\theta)-3\sin(\omega t+\theta)\Big) \end{aligned} \tag{3.75}$$

Amplitude (A) and phase (θ) are now functions of the arbitrary time τ. Since, τ doesn't appear anywhere in the original problem. This can also be thought of along these lines — τ can be chosen arbitrarily along a trajectory and $x(t)$ cannot depend on where one puts the initial condition. So, we impose the condition that $(\partial x/\partial\tau)_t = 0$ since This leads to the flow equations

$$\frac{dA}{d\tau} = \frac{\varepsilon A}{2}\left(1-\frac{A^2}{4}\right)+\mathscr{O}(\varepsilon^2), \tag{3.76}$$

$$\frac{d\theta}{d\tau} = 0+\mathscr{O}(\varepsilon^2). \tag{3.77}$$

The remaining τ-dependence in $x(t)$ and any divergence thereof is removed by setting $\tau = t$. The argument behind this goes somewhat like this — the floating point τ is chosen between the initial time $t_0(=0$ in this case) and the $\tau - t_0$ part is absorbed into the renormalized A and θ; But since τ is chosen arbitrarily it can be taken arbitrarily close to t (which is our point of interest). Thus replacing τ by t we obtain the result upto this order,

$$x(t) = A\cos(\omega t+\theta)-\frac{\varepsilon A^3}{96\omega^2}\Big(\sin 3(\omega t+\theta)-3\sin(\omega t+\theta)\Big) \tag{3.78}$$

From the structure of the flow equation, in accordance with our prescription, we have a limit cycle. The amplitude flow equation has a fixed point at $A^* = 2$ and hence this suggests that we have a limit cycle of radius 2. This indeed is the right result for weakly nonlinear Van der Pol oscillator [3,4]. Integrating

the amplitude and phase we have

$$A(t) = \frac{A_0}{\left[e^{-\varepsilon t} + \frac{A_0^2}{4}(1-e^{-\varepsilon t})\right]^{-1/2}} + \mathscr{O}(\varepsilon^2 t) \tag{3.79}$$

$$\theta(t) = \theta_0 + \mathscr{O}(\varepsilon^2 t). \tag{3.80}$$

The final uniformly valid result is given by

$$x(t) = A\sin\omega t + \frac{\varepsilon}{96\omega^2}A^3\left[\cos 3\omega t - \cos\omega t\right] + \mathscr{O}(\varepsilon^2). \tag{3.81}$$

Its trivial to see that this solution approaches a limit cycle of radius 2 in the asymptotic limit. The power of our prescription lies in the fact that we need not do the above integrations and find a solution always. From our criteria the non-zero fixed point of the amplitude flow is the radius of the limit cycle.

In order to show that the prediction of limit cycle remains valid and no ambiguities arise in the perturbation result we perform the calculation for the next order of perturbation. The second order perturbative calculation proceeds along the same lines and after repeating the renormalization process for this order we have the flow equations:

$$\frac{dA}{d\tau} = \frac{\varepsilon A}{2}\left(1-\frac{A^2}{4}\right) + \mathscr{O}(\varepsilon^3) \tag{3.82}$$

$$\frac{d\theta}{d\tau} = -\frac{\varepsilon^2}{8}\left(1-\frac{A^4}{32\omega^4}\right) + \mathscr{O}(\varepsilon^3). \tag{3.83}$$

The second order calculation is important in two respects: First of all it shows that our initial assumption of perturbative renormalizability isn't violated and the result remains unambiguous; Secondly, the multiple time scales involved in the problem are clearly demonstrated. From the two flow equations its easy to see that the usual time scales used in multiple-scale-analysis appear naturally $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t, \cdots$. Since the radius is $A = 2$, the frequency correction reads

$$\Omega = \omega - \frac{\varepsilon^2}{16\omega^4}. \tag{3.84}$$

This simple example illustrates almost all the salient features involved in a typical RG calculation for 2D oscillators having limit cycles.

3.3.3 Hopf Bifurcation and the RG method

Hopf bifurcation (also called Poincaré-Andronov-Hopf bifurcation) refers to the local birth or death of a limit cycle (self-excited oscillation) from an equilibrium as a parameter, say μ, crosses a critical value. Basically, it is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues (from linearization around the fixed point) crosses the imaginary axis in complex plane. It is the simplest example of a dynamic bifurcation as opposed to static ones which involve only equilibrium states. Typically, a Hopf bifurcation occurs, when as μ is varied, the complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary (where we have a centre) and then the flow becomes unstable. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher. As the fixed point becomes unstable, one can expect to see a small-amplitude limit cycle branching out from the fixed point. The Hopf bifurcation theorem states the above in mathematically precise manner [201–203]. It lays down the generic assumptions under which the above picture remains true. Hopf bifurcations come in two varieties — *super-critical* and *sub-critical*. We have already seen an example of a super-critical Hopf bifurcation when we considered the Van der Pol oscillator. Consider the Eq. (3.60), the fixed point depends on the parameter ε and as one varies it from negative values (where the focus at origin is stable) through $\varepsilon = 0$ (where we have a centre) to positive values (where the focus becomes unstable), and we have a limit cycle. What is important to note here is that Hopf-bifurcation always occurs via centre, which means we always have an appropriate basic unperturbed solution ($\mu = 0$), about which perturbative RG can proceed. Lets now consider one example each for super-critical and sub-critical Hopf bifurcations and see how RG fares in dealing with them.

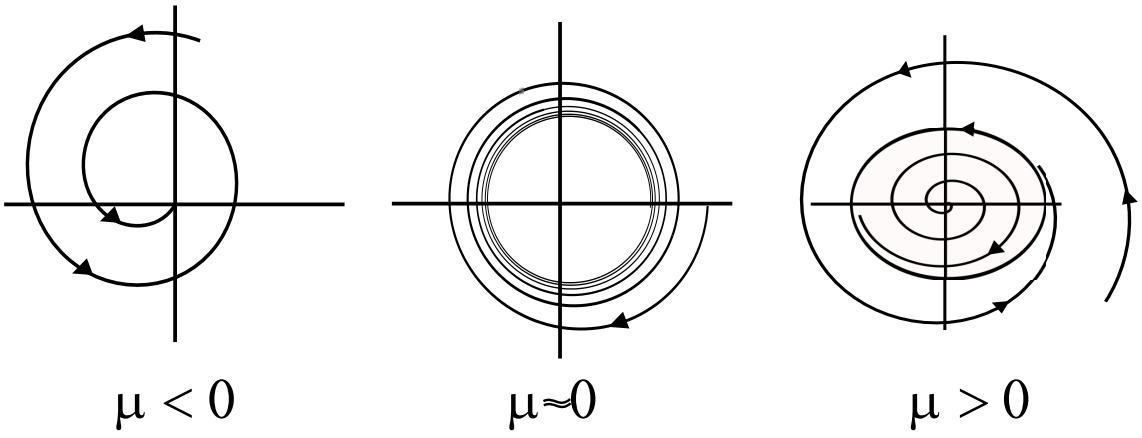

Figure 3.9: Super-critical Hopf bifurcation in 2D plane.

Consider the system, which is a popular paradigm for ***super-critical Hopf*** bifurcation,

$$\dot{x} = \mu x - \omega y - bx(x^2+y^2) - cy(x^2+y^2)^2, \tag{3.85}$$
$$\dot{x} = \omega x + \mu y - by(x^2+y^2) + cx(x^2+y^2)^2. \tag{3.86}$$

where ω, b and c are positive parameters. The only fixed point in this system is the origin, which is a focus and whose stability depends upon the value of the parameter μ. When $\mu < 0$, the origin is a stable spiral while for $\mu > 0$ it is an unstable one which makes it the bifurcation parameter. Further it's easy to see that for $\mu = b = c = 0$, we have a centre. We can do the RG calculation around this centre, by treating the parameters μ, b and c as small. Further for simplicity of calculations (without any loss of generality), we are taking $\omega = 1$. Following the usual method of renormalization, one obtains to the 1st order in all the small parameters, the following flow equations:

$$\frac{dA}{d\tau} = A(bA^2 - \mu) + \mathscr{O}(b^2)\,, \tag{3.87}$$
$$\frac{d\theta}{d\tau} = -cA^2 + \mathscr{O}(c^2). \tag{3.88}$$

So from the flow equation it may be concluded that when $\mu < 0$ we have a stable focus at origin with trajectories spiralling in. When $\mu = 0$, we have only the 'A^3' term in amplitude flow, which suggests locally the fixed point has been destabilized. However, when $\mu > 0$, we have two possible fixed points in the flow equation: $A = 0$, which corresponds to an unstable focus and $A = \sqrt{\mu/b}$, which corresponds to a stable limit cycle. From the phase flow, we have a c-dependent correction to the frequency of infinitesimal oscillations, which affects the large-amplitude regime. This is obviously an example of supercritical bifurcation. This can be seen intuitively in the following manner: When the real parts of the eigenvalues are negative, the fixed point is a stable focus (Fig. 3.9); however, when they cross zero and become positive, the fixed point becomes an unstable focus (Fig. 3.9) with trajectories spiralling out. But this change of stability is a local one and the phase portrait sufficiently far from the fixed point remains qualitatively unaffected. If the nonlinearity is such that it makes the flow from faraway come in, then one can expect that, at some distance from the fixed point where the near-flow and far-flow balance out (Fig. 3.9), we will have an isolated periodic orbit.

Next, we consider an example where a ***sub-critical Hopf*** bifurcation occurs. Sub-critical bifurcations are in some sense more dramatic in nature. Unlike the simple scenario of super-critical Hopf, in sub-critical Hopf bifurcation, the trajectories must jump to a distant attractor (either another fixed point or limit cycle) or to infinity. In higher dimensions, the trajectory may jump to a chaotic attractor (e.g. Lorenz system). It suffices to say while analyzing problems involving sub-critical Hopf bifurcations

one needs to be more careful. Consider for example, the system given by,

$$\dot{x} = \mu x - y + x\left[\left(x^2+y^2\right) - \left(x^2+y^2\right)^2\right] - by\left(x^2+y^2\right)^2, \tag{3.89}$$

$$\dot{y} = x - \mu y + y\left[\left(x^2+y^2\right) - \left(x^2+y^2\right)^2\right] + bx\left(x^2+y^2\right)^2. \tag{3.90}$$

The only fixed point of the above system is the origin. Writing down the linearized system about the origin one immediately obtains the eigenvalues, $\lambda = \mu \pm i$. So, as μ changes from negative values to positive one the fixed point changes from a stable spiral to an unstable one. This, obviously suggests some kind of Hopf-bifurcation occurs here. Treating the parameters — μ, b and c as small and perturbing around $\mu = b = c = 0$, one can do perturbative RG calculations to obtain the RG flow equations for this system. The flow equations, at the lowest no-trivial order, in this case turn out to be:

$$\frac{dA}{d\tau} = A(A^4 - A^2 - \mu), \tag{3.91}$$

$$\frac{d\theta}{d\tau} = -bA^2. \tag{3.92}$$

The amplitude flow equation can now be analyzed for its fixed points and their stability which will in turn give us the phase space configuration of the solutions. For $\mu < 0$, the fixed point at the origin is stable and there two more fixed points: $A^{*2} = 0.5 \pm \sqrt{0.25+\mu}$. As long as $\mu > -0.25$, we have two positive A^{*2} suggesting we have two limit cycles. Since, the focus at origin is stable, obviously smaller of the limit cycles has to be unstable and the larger one stable. This is indeed confirmed by the amplitude equation. At $\mu = -0.25$, the stable and unstable limit cycles collide and annihilate each other. This is an example of a global bifurcation called the *fold* or ***saddle-node bifurcation***. However, we are interested here only in the Hopf-bifurcation that occurs at $\mu = 0$. As the value of μ is increased from negative values the radius of the smaller unstable limit cycle keeps decreasing. Until at $\mu = 0$, the unstable limit cycle ceases to exist and the origin becomes an unstable focus. Hence, the trajectories that used to remain in the vicinity of the stable focus at origin when $\mu < 0$, have to jump to large-amplitude limit cycle all of a sudden at $\mu = 0$, which remains the only stable attractor for $\mu \geq 0$. This is an example of a sub-critical Hopf bifurcation. Its worth noting that this system exhibits hysteresis: Once the large-amplitude oscillations are switched on, they can't be switched off by bringing μ back to zero. Rather, the oscillations persist until $\mu = -0.25$, where a *saddle-node bifurcation* occurs. Whenever a limit cycle is formed via a Hopf bifurcation (the most common mechanism of by which a limit cycle can some to exist), we have a centre for $\mu = 0$ (the bifurcation parameter). And thus be it super-critical or Subcritical RG can be applied without any trouble. In fact the structure of the amplitude equation gives us a clear idea as to whether a certain Hopf-bifurcation is sub-critical or super-critical

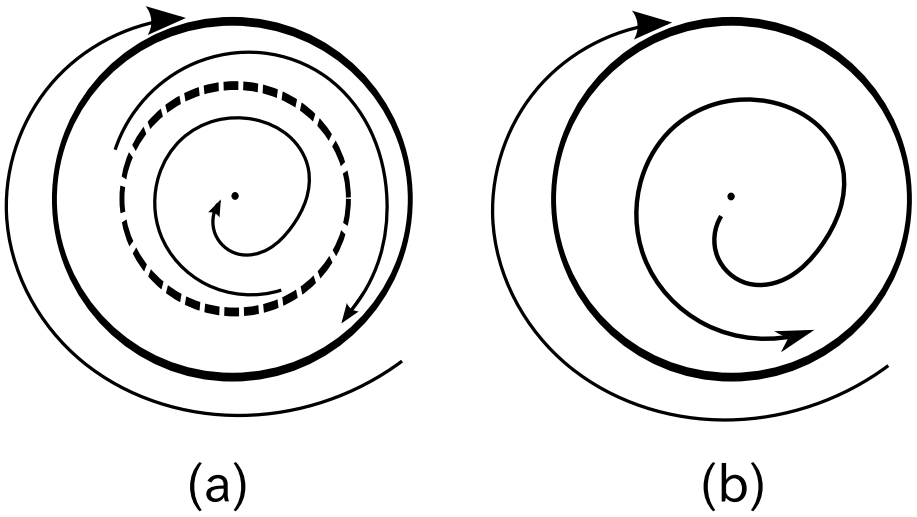

Figure 3.10: Subcritical-Hopf bifurcation. (a) $\mu < 0$: There is unstable cycle between the stable focus and stable limit cycle; (b) $\mu > 0$: The unstable limit cycle has shrinked and merged into the focus making it unstable and there is only a large-amplitude stable limit cycle left.

3.3.4 Liénard equation

In 2D systems a limit cycle may be created or destroyed in four ways - *Hopf bifurcation*, *Saddle-node bifurcation*, *infinite-period bifurcation* and *homoclinic bifurcation*. While Hopf bifurcation is *local* bifurcation in the sense that it can be analyzed in the vicinity of a fixed point the other three are *global* in nature, as they involve a large region of phase space. Having dealt with Hopf-bifurcation we now turn our attention to a limit cycle generated by one of the other bifurcations, namely ***homoclinic bifurcation***. Fortunately, we have already dealt with a system which shows this kind of global bifurcation. We now return to the Lienard equation of Eq. (3.40)

$$\ddot{x} - k\dot{x}(1 + x + \mu) + x + \lambda x^2 = 0$$

and recall that for $\mu = -1$ and of the two parameters k and λ either one being zero and the other non-zero this system has a centre around the origin. We now ask the question what happens when $\mu + 1 \neq 0$. Its easy to see that apart from the fixed point at origin, the above system has a saddle at $(-\lambda, 0)$. We find the flow equations, for $\mu + 1 \neq 0$,

$$\frac{dA}{d\tau} = (\mu+1)\frac{kA}{2} - k\lambda\frac{A^3}{8} \tag{3.93}$$

$$\frac{d\theta}{d\tau} = (\mu+1)\frac{A}{6} - \frac{5}{12}\lambda^2 A^2 - \frac{1}{24}k^2 A^2 \tag{3.94}$$

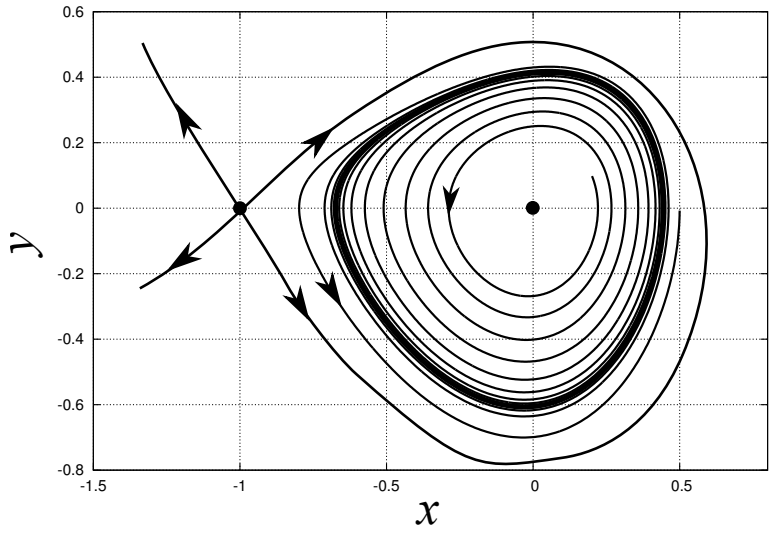

Figure 3.11: Homoclinic Bifurcation: Limit cycle for the Liénard system, Eq. (3.43), with $\lambda = k = 1$ and $\mu = -0.9$.

From the flow equations we can say that there is a limit cycle in the system if $k \neq 0$ and $\lambda \neq 0$; and $\mu + 1$ and λ have the same sign. Further, the oscillatory solution is stable if k also has the same sign as $\mu + 1$ and unstable otherwise. Opposite signs of k and λ do not allow for the existence of a stable limit cycle — a fact easily ascertained by numerically integrating the ODE. Further, numerically the critical value of the parameter μ where the limit cycle is destroyed by the trajectory crashing into the saddle point is determined to be $mu_c = 0.87$. For values of μ, smaller than μ_c one obtains limit cycle oscillations, where the limit-cycle solution passes close to the saddle and is distorted but remains stable. At $\mu = mu_c$ the trajectory goes through the saddle giving rise to a homoclinic path and hence the name of the bifurcation. The typical phase path for the oscillator is shown in Fig. 3.11. At this order of perturbation the critical value of μ from RG is $\mu_c = -1$. In order to improve the result one needs to do the next order perturbative RG calculations.

Liénard's theorem states, the conditions under which the Liénard system of first kind, Eq. (3.39) has a limit cycle. In effect Liénard theorem says that with $\mathscr{G}(x)$ odd, if $\mathscr{F}(x)$ is odd we have a (*centre*) or if its even then a (*limit cycle*). In our system, the limit cycle for $k \neq 0$, $\lambda \neq 0$ is obtained by simultaneously relaxing the conditions on $\mathscr{F}(x)$ and $\mathscr{G}(x)$ in the Liénard system. In our case both $\mathscr{F}(x)$ and $\mathscr{G}(x)$ are of mixed parity.

We now consider briefly one more example from Liénard systems, Eq. 3.39, to demonstrate how RG handles situations where a dynamics has multiple limit cycles. Consider for example the Liénard system with $\mathscr{F}(x) = \varepsilon(x^2-1)(x^2-6)$ and $\mathscr{G}(x) = x$. Thus the system reads,

$$\ddot{x} + \varepsilon(x^2-1)(x^2-6) + x = 0, \tag{3.95}$$

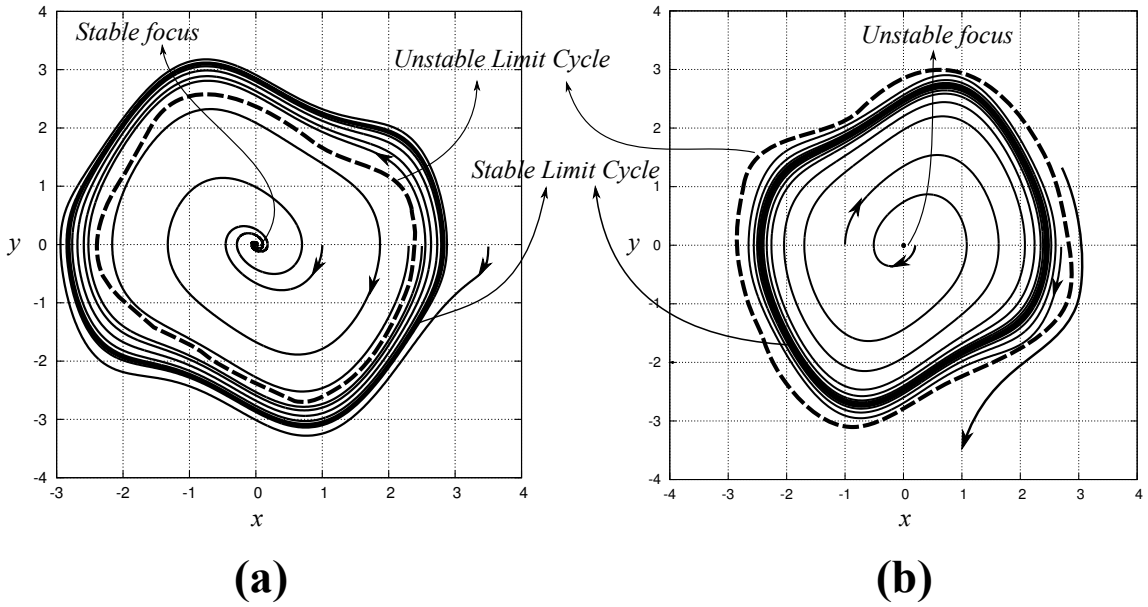

Figure 3.12: Phase portrait for Eq. (3.95)— (a) $\varepsilon = 0.1$: There is unstable limit cycle ($r = \sqrt{6}$) between a stable focus at origin and stable limit cycle ($r = \sqrt{8}$); (b) $\varepsilon = -0.1$: The small limit cycle now is stable while the focus is unstable and so is the large-amplitude limit cycle.

α being much smaller than unity. The RG calculation for this oscillator proceeds similar to the one for Van der Pol oscillator. The flow equation at $\mathscr{O}(\varepsilon)$ is determined to be

$$\frac{dA}{d\tau} = \frac{\varepsilon A}{16}\left(A^2-6\right)\left(A^2-8\right) + \mathscr{O}(\varepsilon), \frac{d\theta}{d\tau} = 0 + \mathscr{O}(\varepsilon). \tag{3.96}$$

The amplitude flow equation has three fixed points: $A = 0$, $A = \sqrt{6}$ and $A = \sqrt{8}$. The non-zero fixed points correspond to two limit cycles with radii $\sqrt{6}$ and $\sqrt{8}$ while $A = 0$ corresponds to a focus at the origin. When the parameter ε is positive, the focus at origin and the larger limit cycle are stable while the smaller limit cycle is unstable. If however, the parameter ε is negative, the system has only a single stable limit cycle with radius $\sqrt{6}$; The focus at origin and the larger limit cycle are unstable. Numerical integration of the above system promptly confirms the RG predictions (see Fig. 3.12). So, RG quite naturally handles scenarios where multiple limit cycles are present surrounding an equilibrium point. Typically for systems with multiple limit cycles around a fixed point, the amplitude flow equation will have the form $dA/dt = f(A)$ and number of non-zero roots of the equation: $f(A) = 0$, gives the number of limit cycles in this system while the values of A, where $f(A)$ vanish correspond to the radii of the limit cycles. Not only does RG predict the number of limit cycles present, but also their configuration in phase space and their stability.

3.3.5 Glycolytic oscillator

A very important aspect that one needs to keep in mind while doing perturbative RG calculation, is that shifting the coordinates to the fixed point of interest greatly reduces the algebra and also provides useful insight into the dynamics. We now turn to another example which clearly illustrates the usefulness of shifting of origin. We draw this example from the field of biology and the oscillator is so-called Glycolytic oscillator [204, 205]. **Glycolysis** is a fundamental biochemical process by which living cells obtain energy by chemically breaking down sugar. It was observed in yeast cells that the process of glycolysis can take place in an self-sustained oscillatory fashion. The simplest mathematical model describing the oscillator is that of Selkov [206,207] and is a 2-dimensional system. The variable x is the concentration of ADP (adenosine diphosphate) and y that of F6P (fructose-6-phosphate). The dynamics is given by

$$\dot{x} = -x+(a+x^2)y \tag{3.97}$$

$$\dot{y} = b-(a+x^2)y \tag{3.98}$$

where 'b' is the rate of fructose production by the substrate and 'a' is the rate at which fructose decomposes (converts to ADP). It should be noted that the presence of ADP catalyzes this conversion and hence 'a' is augmented to $a+x^2$. In order to set up the perturbation theory, to locate the limit cycle and determine its configuration we need to determine the locus of Hopf bifurcation points. The fixed point of the system is at

$$x^* = b, \qquad y^* = b/(a+b^2) \tag{3.99}$$

The fixed point is a stable focus for a certain parameter range and an unstable focus for certain others. The general Jacobian in this case is given by,

$$J = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{pmatrix}. \tag{3.100}$$

After some algebra one can determine the conditions under which the fixed point (x^*, y^*) is stable. In fact, the crossover from stable to unstable focus occurs on the boundary curve which is a locus of points in the a-b plane where a Hopf bifurcation occurs. The curve is given by $2a = \sqrt{1+8b^2} - (1+2b^2)$ and is shown in Fig. 3.13. For points in the shaded region the fixed point is an unstable focus and for these values of (a,b) a limit cycle can be shown to exist by invoking Poincaré-Bendixson theorem [4]. However, we are interested in confirming the existence and finding the configuration of limit cycle in the system via RG method. For purposes of doing perturbative RG calculations, we shift the origin to

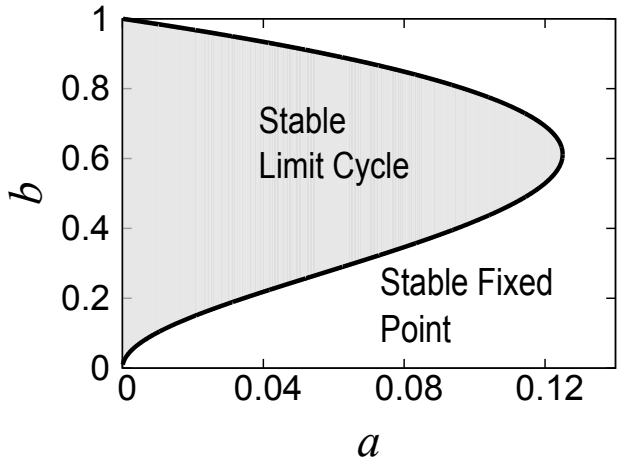

Figure 3.13: The figure shows the phase diagram of the glycolytic oscillator given by Eqs. (3.97) and (3.98). The curve (solid line) — $2a = \sqrt{1+8b^2} - (1+2b^2)$ separates the figure into two regions. For parameters in the shaded region, one gets limit cycle while unshaded region corresponds to parameters giving rise to a stable focus. Linear stability analysis predicts center for parameters on the curve.

the fixed point and use the new coordinates X, Y given by

$$x = b + X \tag{3.101}$$

$$y = \frac{b}{a+b^2} + Y \tag{3.102}$$

To use perturbation theory, we chose (a,b) close to the boundary. Setting $b = \sqrt{3/8}$ (the turning point of the curve), we take $a = 1/8 - \delta$ to consider a point slightly inside the boundary. Clearly, δ is small and positive. To $\mathscr{O}(\delta)$, the equation of motion now reads,

$$\dot{X} = \frac{1}{2}(X+Y) + \mathscr{L}(X,Y) \tag{3.103}$$

$$\dot{Y} = -\frac{3}{2}X - \frac{Y}{2} - \mathscr{L}(X,Y) \tag{3.104}$$

where the operator $\mathscr{L}$ is given by

$$\mathscr{L}(X,Y) = \delta(3X - Y) + \sqrt{\frac{3}{8}}X(X+Y) + X^2Y \tag{3.105}$$

We note that we can write Eqs. (3.103) and (3.104) as 2nd order ODEs as,

$$\ddot{X} + \frac{X}{2} = \dot{\mathscr{L}} \tag{3.106}$$

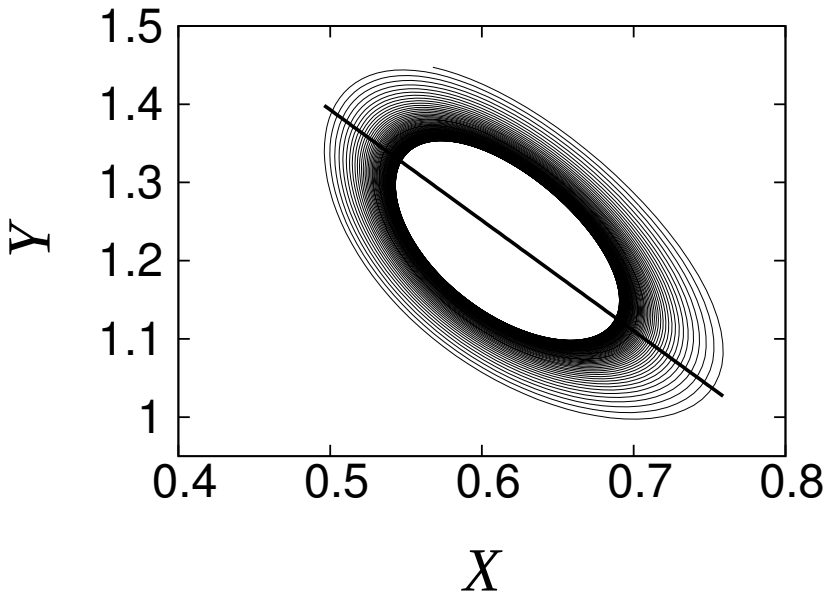

Figure 3.14: Limit cycle in glycolytic oscillator for $a = 0.124$, $b = \sqrt{0.375}$ and $\delta = 0.001$.

Its necessary to keep in mind that $\mathscr{L}$ has to be expanded in amplitude and the parameter δ. As we will see the amplitude will actually emerge to go as $\delta^{1/2}$ for small δ. At the zeroth order we have the solution,

$$X_0 = A_0 \cos\left(\frac{t}{\sqrt{2}} + \theta_0\right) \tag{3.107}$$

$$Y_0 = \sqrt{3}A_0 \cos\left(\frac{t}{\sqrt{2}} + \theta_0 + \pi - \tan^{-1}\sqrt{2}\right) \tag{3.108}$$

The frequency of the basic oscillatory trajectory is $\frac{1}{\sqrt{2}}$ and the axis of the orbit is tilted at an angle $\pi - \tan^{-1}\sqrt{2}$ to the X-axis. The amplitude of the limit cycle oscillation is found from the amplitude flow equation which at the lowest non-trivial order is determined as,

$$\frac{dA}{d\tau} = 2\delta A - \frac{3A^3}{8} \tag{3.109}$$

The frequency correction, as usual, comes from phase flow

$$\frac{d\theta}{d\tau} = -\frac{\delta}{\sqrt{2}} + \frac{A^2}{4\sqrt{2}} \tag{3.110}$$

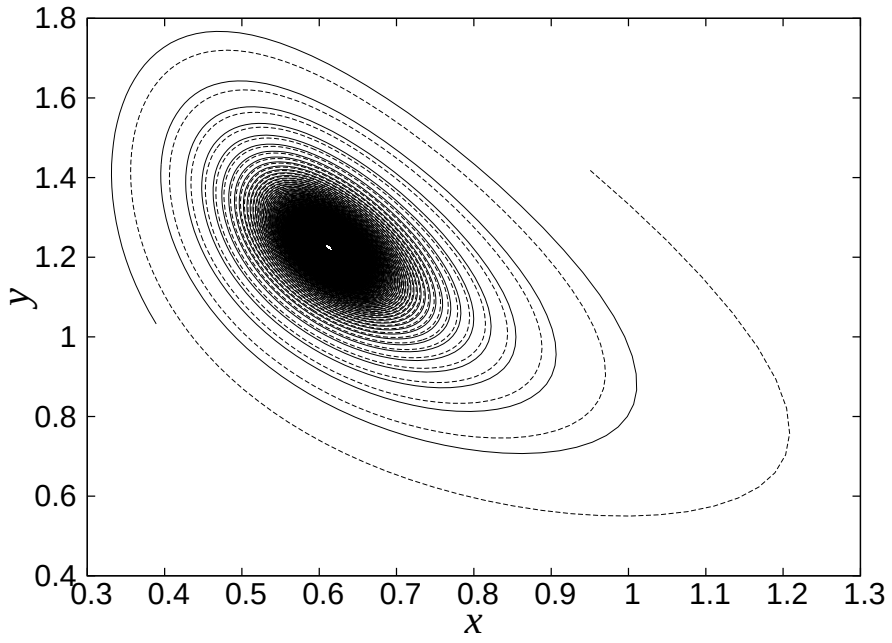

Figure 3.15: Stable focus in glycolytic oscillator for $a = 0.124$, $b = \sqrt{0.375}$ and $\delta = 0.0$. Linear stability analysis wrongly predicts a centre for these values.

and thus the correction to the zeroth order value of $\frac{1}{\sqrt{2}}$ according to perturbative RG is $\Omega = \frac{1}{\sqrt{2}} - \frac{\delta}{\sqrt{2}} + \frac{A^2}{4\sqrt{2}}$. The stable fixed point of the amplitude flow, $A^{*2} = 16\delta/3$, gives us the radius of the limit cycle for $\delta \ll 1$. A typical small-δ orbit obtained numerically, is shown in Fig. 3.14 and bears out the correctness of the above flow. This technique can also be used to probe limit cycles in the more complicated model of Cera *et al* [208] or [205].

One may now recall from the fixed point analysis we did for the glycolytic oscillator [defined by equations (3.97) and (3.98)], for parameter values (a,b) lying on the curve given by: $2a = \sqrt{1+8b^2} - (1+2b^2)$ (Fig. 3.13) the system is supposed to have a *centre*. Or at least that is what linear stability analysis result suggests. For instance, consider a specific value of the parameters, say $(a,b) = (1/8, \sqrt{3/8})$, linear stability predicts a centre for these values. However, numerical integration of the system for the said values reveal that in fact the fixed point turns out to be a stable focus for these parameter values (see Fig. 3.15). Now, from the amplitude flow equation for this oscillator with the value of $(a,b) = (1/8, \sqrt{3/8})$, one obtains the flow equation, [setting $\delta = 0$ in Eq. (3.109)]: $\partial A/\partial\tau = -3A^3/8$ (to the lowest order). And it's not surprising that in accordance with our prescription, for parameter values, $(a,b) = (1/8, \sqrt{3/8})$, the origin is actually *not* a center rather a stable attractor. In linear stability analysis, by dint of the very nature of the basic assumptions of the analysis, the fixed point is shielded from full bombardment of non-linear terms. However, RG method doesn't suffer from any such drawbacks. In this sense, our RG methodology can serve as *non-linear fixed point analysis* in

scenarios where distinguishing between a focus and centre is imperative.

3.3.6 Chemical oscillators

Our next example belongs to a class of chemical oscillators, the famous Belousov-Zhabotinsky system. This system is extremely important in the field [209, 210] and has a rather colourful history behind it. In the 1950s, a Russian biochemist by the name Boris Belousov was attempting to mimic Kreb's cycle in laboratory, by mixing citric acid and bromate ions in sulphuric acid solution in presence of a catalyst. To his utter surprise, he observed that the mixture turned bright yellow to colourless and then back to yellow again in cycles little over a minute. Thus, he discovered this reaction system quite by accident and in those days the concept that chemical reactions can undergo self-sustained oscillations was so alien that nobody accepted his findings. It wasn't until 1961 when a young chemist Zhabotinsky confirmed Belousov's findings and got his name attached to this oscillator. Stories apart, this particular oscillator has been modelled variously by various theorists. A recent version [142, 143] of the reaction system uses a two variable system (chlorine dioxide-iodine-malonic acid reaction)

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \tag{3.111}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right) \tag{3.112}$$

where the variable x and y are the concentrations of the intermediaries I^- and ClO_2^- which vary on a much faster time scale than ClO_2, I_2 and $Malonic\ acid$. The constants 'a' and 'b' are parameters which depend on the rate constants and the *approximately* constant concentrations of the other reactants. We note that this system has one fixed point $x = a/5$ and $y = 1+x^2 = 1+a^2/25$. Our first step is thus to shift the origin to $(a/5, 1+a^2/25)$ and use the variables

$$x = X + \frac{a}{5} \tag{3.113}$$

$$y = Y + 1 + \frac{a^2}{25} \tag{3.114}$$

We perform linear stability analysis of the resulting system about the fixed point $X = Y = 0$ and it turns out to be a centre for $b = b_c$, where b_c given by

$$b_c = \frac{3a}{5} - \frac{25}{a} \tag{3.115}$$

For parameter values $b < b_c$, the origin is an unstable focus and its stable for $b > b_c$. In order to do perturbative RG, we pick a value of 'a' and choose $b = b_c - \delta$, where $\delta \ll b_c$. We carry out the perturbation analysis for the variables X and Y by assuming that the amplitude is small. After some tedious but simple algebra the flow equation is determined as

$$\frac{dA}{d\tau} = -\frac{a}{5}\delta\Omega A + \frac{\Omega A^3}{\left(1+\frac{a^2}{25}\right)^2}\left[\frac{3a^4}{125} - 3a^2 - 315 + \frac{1875}{a^2}\right] \tag{3.116}$$

where $\Omega^2 = a\left(1+\frac{a^2}{25}\right)\left(\frac{3a}{5} - \frac{25}{a}\right)$. From the amplitude flow it can be concluded that a limit cycle exists for positive values of δ. It turns out that as we measure the value of 'a' for which limit cycles can exist, there is a cyclic-fold bifurcation at $a = a_c \simeq \sqrt{191.43}$ — obtained by setting the expression inside square bracket to zero.

Another equally famous chemical oscillator is the so-called ***Brusselator*** reaction-diffusion system. It was proposed by Prigogine and Lefever in 1968 as an model for *autocatalytic reaction*. The name is a portmanteau of "Brussels" (Prigogine is from the city) and "oscillator" and this has set a trend of naming oscillators such as Oregantor, Sao Paolator etc. The Brusselator is a minimal mathematical model which can model oscillating behaviour. It models the reaction system,

$$\begin{aligned} A &\rightarrow X \\ B+X &\rightarrow Y+D \\ 2X+Y &\rightarrow 3X \\ X &\rightarrow E \end{aligned}$$

The ODE describing the above simplified model is given by

$$\dot{X} = A - (B+1)X + X^2 y + D_x \frac{\partial^2 X}{\partial x^2}, \tag{3.117}$$

$$\dot{Y} = BX - X^2 Y + D_y \frac{\partial^2 Y}{\partial x^2}. \tag{3.118}$$

where $A > 0$, $B > 0$, D_x and D_y are constants. We treat here the homogeneous systems and thus the spatial derivatives vanish automatically. The above system has a fixed point at $(X^*, Y^*) = (A, B/A)$. Shifting the variables as $\mathscr{X} = X - X^*$ and $\mathscr{Y} = Y - Y^*$, we have the system

$$\dot{\mathscr{X}} = (B-1)\mathscr{X} + A^2\mathscr{Y} + (B/A)\mathscr{X}^2 + 2A\mathscr{X}\mathscr{Y} + \mathscr{X}^2\mathscr{Y} \tag{3.119}$$

$$\dot{\mathscr{Y}} = -B\mathscr{X} - A^2\mathscr{Y} - (B/A)\mathscr{X}^2 - 2A\mathscr{X}\mathscr{Y} - \mathscr{X}^2\mathscr{Y} \tag{3.120}$$

Linear stability analysis of the above system gives the eigen values $\lambda = -(A^2+1-B) \pm \sqrt{(A^2+1-B)^2 - 4A^2}$. Its clear that at the critical point $B_c = 1+A^2$, there is a Hopf bifurcation, as long as the quantity within the square-root — $(A^2-B+1)^2-4A^2$ is less than zero; When $(A^2-B+1) > 0$, there is a stable focus at the origin and we can see that as A^2+1-B is changed from positive values to negative ones, the focus at origin changes from being stable to unstable. One can do the RG calculation in a similar manner as our previous example — the BZ system. We can choose parameters in such a manner that we have a unstable focus at origin which allows a limit cycle to exist. For instance we can take $B = A^2+1+\delta$ which ensures we are in the regime where we have a unstable focus. We can do the standard perturbative RG calculations from this point onwards. We quote the final result here without giving details of the RG calculations. The amplitude flow equation for this system is given by:

$$\frac{dR}{d\tau} = \delta(1+A^2)R - \frac{2+A^2}{A^2}R^3. \tag{3.121}$$

From the above result its clear that the system has a stable limit cycle.

3.3.7 Koch-Meinhardt reaction diffusion system

The Koch-Meinhardt reaction diffusion system [211], is popular model for the generation of Turing patterns. For our present purpose only the reaction part is relevant and hence we drop the diffusion part. In order to do RG for this system one has to be bit creative. The system is given by the equations,

$$\dot{x} = -x + \frac{x^2}{y} + \sigma \tag{3.122}$$

$$\dot{y} = -y + x^2 \tag{3.123}$$

The variables x and y are the number densities of two species which are responsible for the pigments in the pattern. The pigment (x) is slowly diffusing and auto-catalytic and it also promotes the growth of the antagonistic fast diffusing component (y). The rate of growth of x from the environment is given by the constant σ. The above system has a fixed point at $x = 1+\sigma$, $y = (1+\sigma)^2$. As we have made abundantly clear, the first step in doing RG for these kind of problems is shifting the origin to the fixed point of interest, $(1+\sigma,(1+\sigma)^2)$ in this case. The new coordinates are defined as $x = X+(1+\sigma)$ and $y = Y+(1+\sigma)^2$. In terms of the new variables X and Y, the dynamics now becomes,

$$\dot{X} = -X + \frac{2(1+\sigma)X - Y + X^2}{(1+\sigma)^2 + Y} \tag{3.124}$$

$$\dot{Y} = 2(1+\sigma)X - Y + X^2 \tag{3.125}$$

In perturbation theory, we are interested in the infinitesimal oscillation around the origin and hence we can expand the denominator in Eq. (3.124) to re-write it as

$$\begin{aligned}\dot{X} &= -X+\frac{2}{1+\sigma}\left[X-\frac{Y}{2(1+\sigma)}+\frac{X^2}{2(1+\sigma)}\right] \\ &\quad\times\left[1-\frac{Y}{(1+\sigma)^2}+\frac{Y^2}{(1+\sigma)^4}+\ldots\right] \\ &= X\frac{1-\sigma}{1+\sigma}-\frac{Y}{(1+\sigma)^2}+\frac{X^2}{2(1+\sigma)^2}-\frac{2XY}{(1+\sigma)^3} \\ &\quad+\frac{Y^2}{(1+\sigma)^4}+\frac{2Y^2X}{(1+\sigma)^5}-\frac{Y^3}{(1+\sigma)^6}+\ldots \end{aligned} \tag{3.126}$$

we write down the Jacobian for the system and the eigenvalues λ of a linear stability analysis around the center are found from the expression:

$$\left(\lambda-\frac{1-\sigma}{1+\sigma}\right)(\lambda+1)+\frac{2}{1+\sigma}=0, \tag{3.127}$$

which gives,

$$\lambda=-\frac{\sigma}{1+\sigma}\pm\sqrt{\frac{\sigma^2}{(1+\sigma)^2}-1}. \tag{3.128}$$

As one can see that we are now faced by a problem — for no positive σ, can the origin be an unstable focus and hence it would seem there can be no limit cycle in the system. However at this point, we introduce a new parameter into the system. We consider are more general system given by

$$\dot{x} = -x+\frac{x^2}{y}+\sigma, \tag{3.129}$$

$$\dot{y} = -ay+x^2, \tag{3.130}$$

where, the new parameter a is a decay constant. Now, the fixed point is at $a+\sigma$, $y=\dfrac{(a+\sigma)^2}{a}$ and using a coordinate system $(\mathscr{X},\mathscr{Y})$ centred around this fixed point, we have

$$\dot{\mathscr{X}} = -\mathscr{X}+\frac{2(a+\sigma)\mathscr{X}-a\mathscr{Y}+\mathscr{X}^2}{\dfrac{(a+\sigma)^2}{a}\left[1+\dfrac{\mathscr{Y}}{(a+\sigma)^2}\right]} \tag{3.131}$$

$$\dot{\mathscr{Y}} = 2(a+\sigma)\mathscr{X}-a\mathscr{Y}+\mathscr{X}^2 \tag{3.132}$$

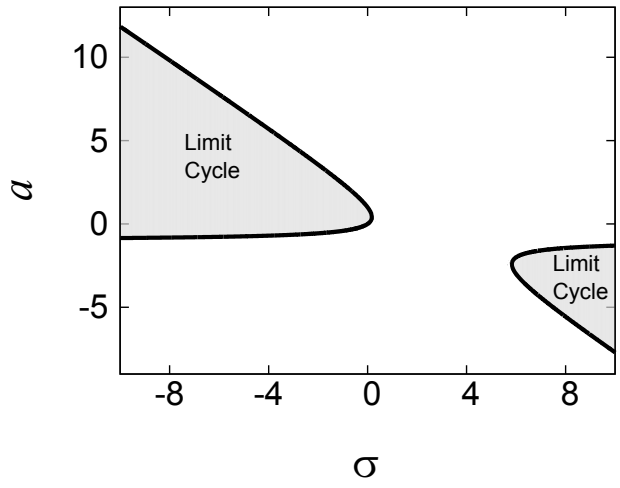

Figure 3.16: The figure shows the phase diagram of Eqs. (3.122) and (3.123). The shaded regions represent the parameter values for which limit cycle solutions can occur.

We again do linear stability analysis about the fixed point $(0,0)$ and it turns out that its a centre when the condition:

$$a = \frac{1-\sigma}{2} \pm \sqrt{\frac{(1-\sigma)^2}{4} - \sigma} \tag{3.133}$$

is satisfied. This curve is shown in Fig. 3.16. The interior region has the unstable focus and hence in this part of the parameter space there is the possibility of a limit cycle. The range of σ is limited to $0 < \sigma \leq 3 - 2\sqrt{2}$ and 'a' lies between 0 and 1. And as expected, the limit cycle at $a = 1$ and $\sigma \neq 0$ (the old system) is ruled out. But , now we have the possibility of a limit cycle in a different parameter range. This example reiterates our initial claim that in situations where the system appears not amenable to perturbation theory expanding the parameter space sometimes allows the system to be analyzed via perturbative RG and one has access to wider range of dynamics. We are not giving details of RG calculations for the above system which turn out be rather cumbersome but its clear that perturbative RG may be done for the above system by choosing suitable values for the parameters (such that they fall in the shaded region of Fig. 3.16).

3.4 Summary

We have considered several examples of 2D dynamical systems displaying both limit cycle or centre type oscillations. In case of oscillators, the methodology is clear — the amplitude flow in case of centres must vanish at every order of perturbation whereas the phase flow equations gives the amplitude-dependent correction to the frequency. When the phase flow also vanishes identically at

each order of perturbation, it means we have oscillation with amplitude-independent frequency; Such systems are called isochronous oscillators, which will be our major point of interest in the next chapter.

On the other hand limit cycle oscillators are an altogether different proposition. Confining ourselves to 2-D examples having a single equilibrium point enclosed by a single limit cycle, we saw that for each such system, the proposed methodology has been successful in finding out the stable/unstable limit cycle and the unstable/stable focus that may be present inside. The instances, where there exist multiple limit cycles surrounding a focus or node, are naturally taken care of in our methodology. The RG flow equation for amplitude — $dA/d\tau = f(A)$ — yields several non-zero fixed points (corresponding to the radii of the limit cycles) confirming that we have multiple limit cycles. The stability of the limit cycle is determined by the corresponding fixed point in the flow equations. For systems where a basic unperturbed oscillatory solution is easily found the application of th RG method is quite clear. In cases where the limit cycle is generated by Hopf bifurcation we saw that the center, changes to an unstable focus once the parameter is shifted from the critical value and we have a stable limit cycle. For the opposite shift, it would be a stable focus and an unstable limit cycle.

At this point, we must also point out that all the examples we have discussed in this chapter happen to have only *focus* and *no node* at the centre of the limit cycle. In principle, a limit cycle can also surround a unstable node. This is a rare occurrence but examples exist [4]. A well-known example having a limit cycle surrounding a node is the following system

$$\dot{x} = -y + ax(x^2+y^2-1); \qquad \dot{y} = x + by(x^2+y^2-1).$$

where a and b are both positive. Linear stability analysis easily shows that the above system has a stable node at the origin. The presence of a node at the centre doesn't affect the RG methodology as long as one can find a centre for some parameter values of a given system. The centre in the above case is obtained for $a = -b$ and following perturbative analysis around $a+b=0$, our methodology detects the limit cycle — $x^2+y^2=1$. It turns out to be stable for $a+b>0$ and under this condition the equilibrium point inside at origin is an unstable node for $ab > -1$.

We have also given examples where an unperturbed solution is missing for any combination of parameter values. Many such systems can be handled by simply introducing a new parameter and considering a more general system, for instance, linear terms with fixed parameters in the system may be used to introduce variable parameters such that they may be tuned to yield a centre. RG calculations can then be carried about this centre. However, there exists a class of systems where lack of linear terms makes it difficult for this methodology to be implemented, e.g., $\ddot{x} + \lambda x^3 = 0$; $\dot{x} = (x^2+y^2)[x-y-px(x^2+y^2)]$, $\dot{y} = (x^2+y^2)[x-y-py(x^2+y^2)]$; $\ddot{x} + \beta\lambda x\dot{x} + \lambda^2 x^3 = 0$, etc. In such situation a different line of attack is in order. A subset of this class of systems can, fortunately,

be tackled by making use of method of equivalent linearization [89] in tandem with perturbative RG. Suitable nonlinear terms can be manipulated by using self-consistent linearization techniques to render the system in a form that allows for a linear centre to be found for certain parameter-values and thus a perturbative RG treatment (see Chapter 5).

In concluding, we may say that perturbative RG is a highly efficient tool for probing oscillatory solutions in 2D systems. In fact, the method described here is easily generalizable to 2N-dimensional oscillators. In principle one can apply this method to perturbatively analyze systems with several coupled oscillators. Its possible to apply the system to arbitrary N-dimensional systems by making use of a vectorial formulation of this method [101]. Finally, it can be said that this method has truly wide applicability and great practical utility for researchers across disciplines.

Chapter 4

Renormalization group and isochronous oscillators

4.1 Introduction

As we have already asserted plane differential systems, particularly oscillators, are widely studied for their relevance in applications in several physical, chemical, biological, engineering systems which can be modelled by means of equations of the form

$$\ddot{x} + h(x, \dot{x}) + x = 0. \tag{4.1}$$

A main problem in the study of 2nd order ODEs is figuring out the existence and properties of oscillating solutions. We have amply demonstrated in previous chapters that RG method is exceptionally useful in analyzing periodic solutions both isolated periodic solutions (*limit-cycles*) or non-isolated ones (*centres*). The latter case, *centre*, usually occurs when the system is integrable, for instance, dynamical systems generated by conservative forces e.g. quartic anharmonic oscillator, the LotkaVolterra equations, etc. If in addition to having a family of periodic orbits around a equilibrium point, all the orbits have the same time period, such an oscillator is called an *isochronous oscillator*. Isochronous oscillators arise in a variety of physical systems, for example, the *simple harmonic oscillator* (SHO) is isochronous in nature.

Isochronous systems have been quite a fascinating subject ever since Galileo first identified such a system - the simple harmonic oscillator. Eminent names such as Newton, Huygens, Jacobi, Poincare etc. have delved into the investigation of isochronous systems. In the last century the focus has mainly been on two-dimensional cases, as they are related to Hilbert's 16th problem. For an exhaustive study of

various isochronous centres of vector fields in the plane see [212] and the references therein. Nonlinear isochronous systems are interesting for obvious physical reasons. Isochronous systems belong to the important class of conservative systems and they are also studied for their integrability properties. Besides, several examples of isochronous systems exist in real life and applications. One such major area is in the field of signal communication and transmission. For instance, in a dynamical system that arises in the telecommunications industry [213] — atmospheric distortion can affect the transmission of high-speed digital signals and this distortion is removed by introducing tapped delay devices, which act as adaptive equalizers. The ODE modelling this has the property that small changes in the initial conditions can alter the amplitude, but not the frequency, of oscillations.

In the language of mathematics, the critical point, $\mathscr{P}$ of a system is said to be a *centre*, if it has a punctured neighbourhood, $\mathscr{V}$, around it where all the trajectories are closed orbits. The stability of a centre can't be confirmed by linear stability analysis. In fact, a non-isolated cycle is *Lyapunov stable* [4] if and only if all neighbouring orbits have the same period. This motivates the definition of isochronicity: a centre $\mathscr{P}$ is said to be isochronous if all the orbits in its immediate neighbourhood, $\mathscr{V}$ have the same minimal period. Further, one can define a function $\mathscr{T} : \mathscr{V}(\mathscr{P}) \to \mathbb{R}^+$ called the *period function* of the centre whose value at each point is given by the time period of the orbit passing through it. Obviously, in case of isochronous centres the period function is a constant.

An isochronous system is characterized by the presence of an *open domain* of initial data (*Cauchy set*) such that all motions evolving from it are completely *periodic* with a fixed frequency. We also assume that all the evolving trajectories remain inside the domain at all times and that the domain has the same *dimensionality* as the phase space. Typically, presence of nonlinearity makes the frequency of oscillators dependent on the amplitude i.e. the *Cauchy set*. But for isochronous oscillators the frequency remains independent of the initial data. Thus, in a two-dimensional dynamical system, isochronicity implies that the time period of all cycles surrounding a fixed point, $\mathscr{P}$, is independent of the amplitude of motion (*i.e.* independent of the initial condition).

We have already seen that perturbative renormalization group method is able to predict asymptotic behaviour of various nonlinear oscillators by constructing amplitude and phase flow equations. For, *centres*, the amplitude is fixed by the initial data and the amplitude flow vanishes at every order of perturbation. On the other hand the phase flow typically gives the amplitude-dependent correction to the frequency. In case of, *isochronous centres*, quite obviously there can't be any amplitude-dependent corrections to the frequency. So it stands to reason that for isochronous oscillators the *phase flow* must also be identically zero at each order of perturbation. This condition gives us the opportunity to analyze isochronous oscillators using the renormalization group technique.

4.2 A brief historical survey

Before we show how RG can be used in analysis of isochronous systems we present here a brief historical survey of the research in isochronous dynamical systems. The history of study in isochronous system pre-dates even the advent of differential calculus. It was Galileo in 16th century who first considered the question of isochronicity when he studied the classical pendulum. The basic problem was — *identifying a curve in the vertical plane with the property that a body under the gravity, starting from rest from any point of the curve and moving along it takes the same time to get the inferior point*. The question of isochronicity was next discussed in great detail by Huygens (who was considered as a very skillful watchmaker) in 1673 [214]. Huygens was able to resolve the above problem when he demonstrated that the cycloidal pendulum has a motion whose period is independent of the amplitude. Hence, the *cycloidal pendulum* has isochronous oscillations in contrast to the simple pendulum in which the period of oscillation increases with amplitude. The simple pendulum has the equation of motion:

$$\ddot{\theta} + \sin\theta = 0 \tag{4.2}$$

and is usually treated as isochronous only in the small amplitude regime where the nonlinear term $\sin\theta$ can be replaced by θ and thus pendulum can be approximated by SHO. On the other hand, the simplest expression for the cycloidal pendulum is given by the equation of motion

$$\ddot{s} + \omega^2 s = 0 \tag{4.3}$$

where s is the arc length and ω a real constant. Huygens deduced that, since the involute of a cycloid is a cycloid then if the chops (side metal plates which act as a constraint) were cycloidal, the bob of a pendulum would swing along a cycloidal path, instead of a circular arc of the simple pendulum, and the pendulum would then be isochronous (See Fig. 4.1). Although Eq. (4.3) expresses the equation of motion of a cycloid in terms of the arc length s, in the parametric form, the equation of motion reads

$$x = r\left[\theta(t) - \sin\theta(t)\right], \tag{4.4}$$

$$y = r\left[\cos\theta(t) - 1\right]. \tag{4.5}$$

Note that the above equation is a nonlinear one. Huygen's discovery of this cycloid pendulum is thus even more significant because its actually a nonlinear oscillator which executes isochronous oscillations.

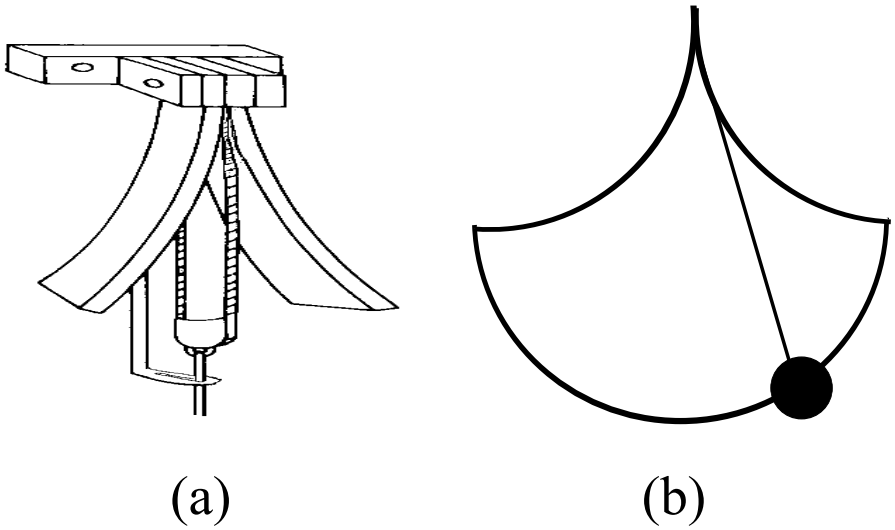

Figure 4.1: (a) Huygen's cycloidal pendulum. (b) Schematic diagram of the pendulum. The cycloid and the involute.

We consider the general class of oscillators which can be expressed as

$$\dot{x} = -y + \sum_{n=2}^{N} \mathscr{X}_n(x,y) \tag{4.6}$$

$$\dot{y} = x + \sum_{n=2}^{N} \mathscr{Y}_n(x,y) \tag{4.7}$$

where $\mathscr{X}_n$ and $\mathscr{Y}_n$ are polynomials of degree n. A centre is called a *non-degenerate* one if the linearized vector field has two non-zero eigenvalues. Based on Poincaré's work [8] on the conditions under which the origin of system (Eqs. (4.6) & (4.7)) is a centre, its been established that a centre is isochronous if, and only if, it is locally linearizable, i.e. , there is a local analytic change of variables which transforms system (Eqs. (4.6) & (4.7)) to its linearization — $\dot{x} = -y,\ \dot{y} = x$; — around the singular point [215, 216]; In other words the isochronous centre is always non-degenerate [217]. This property provides an algorithm to find the conditions under which a given centre is isochronous. Its a common practice to use this property to compute the *period constants* of dynamical systems.

Over the years , the existence of isochronous centres for several classes of differential systems has been studied extensively. One strategy has been to study the isochronous systems by analyzing the behaviour of period functions around a centre. For example, Urabe studied planar systems coming from differential equation of the form: $\ddot{x} + g(x) = 0$, and gave necessary and sufficient conditions for isochrony of oscillations for these systems [218]. Chicone et al have studied the period function properties in a certain class of systems, where they showed that the period functions become important

in the study of bifurcations of critical points leading to limit cycles [219, 220]. Isochronous systems have been studied via the period function by many authors, see for example [221–224].

The problem of characterizing and identifying isochronous centre however, has been the major focus of researchers in the field. It was Loud [225] who first classified isochronous systems with homogeneous quadratic nonlinearities. Later, Pleshkan [226] considered systems with homogeneous cubic nonlinearities and classified them. Several other authors have subsequently considered systems with homogeneous cubic nonlinearities [227,228]. The Kukles system and its isochronous centres have been studied in [229]. Systems of higher degree have received considerably less attention; however, there are results for systems with quartic and quintic homogeneous nonlinearities (see for e.g. [230, 231]). It was shown in [232] that some systems can have coexisting isochronous and non-isochronous centres. Even a study of systems with a fractional vector field have been undertaken [233]. In a recent work [234] it has been argued that although the space of the isochronous potentials is fairly large, up to a shift $x \to x+a$ and the addition of a constant, all rational isochronous potentials can be reduced to either the linear harmonic oscillator potential: $V(x) = \frac{1}{2}\omega^2 x^2$ or the isotonic potential: $V(x) = \frac{1}{8}\omega^2 x^2 + \frac{c}{x^2}$ [235–237]. However, there exist other classes of isochronous potentials described by irrational potentials. For a really comprehensive literature survey of the field, see [212].

However, until recently it was generally believed that occurrence of isochronous nonlinear oscillators is quite rare. In recent times this myth has been busted by the seminal work of Calogero *et al* [238–243] and has been responsible for renewing the interest in the topic of isochronous oscillators. Calogero *et al* have developed a ingenious method of systematically constructing isochronous Hamiltonian systems. Calogeros formalism is quite remarkable for its inherent simplicity — as it employs the Hamiltonian of the linear harmonic oscillator, in a novel manner to generate isochronous Hamiltonians. In this method, a real autonomous Hamiltonian $H(p,q)$ can be transformed into an Ω-modified form, *i.e.*, $\tilde{H} = \frac{1}{2}\left[H(p,q)^2 + \Omega^2 Q(p,q)^2\right]$, which is isochronous. Here H is the new momentum and Q the collective canonically conjugate coordinate, satisfying the Poisson bracket $\{Q,H\} = 1$. Ω is an arbitrary constant which is related to the isochronous time period as, $T = 2\pi/\Omega$. Finally, by inverting H and Q one can obtain expressions for p and q. This has led to the conclusion that "isochronous systems are not rare" [242]. This area of research focuses mainly on the construction of isochronous systems and has been widely exploited in recent literature, for example by Ghose Choudhury and Guha [244, 245], Francoise [246], Chandrasekhar *et al* etc.

Calogero *et al* concentrate on constructing new isochronous systems. But what about methods to establish whether a given system is isochronous or not. It is in this direction, that RG method proves to be very useful. There are methods in the literature which suggest ways to establish whether a given system is isochronous or not. For instance, in [248] an algorithmic method to find isochronous

system has been provided based on an idea first proposed by Pleshkan [226]. Chouikha *et al* also have developed a computer implementation of a method of finding isochronicity based on Urabe's criterion [218]. In an interesting work [250,251] the authors have obtained by an analysis of an Urabe-function, necessary and sufficient conditions for isochronicity of cubic Liénard-type systems. A simple proof of the Urabe criterion based on a formula from Landau and Lifshitz's volume on mechanics [178] is obtained by Sabatini [249]. Based on this algorithm several research studies on isochronous systems have come out in the recent years, see [252–254]. In this work, we propose to find using perturbative RG techniques whether a given system is isochronous or not. Also, for any given system RG can be used to find the conditions under which the system can become isochronous. Our method is based on purely physical considerations and doesn't utilize any of the several techniques we have listed above.

4.3 The method

The renormalization group approach, as we have amply demonstrated by now, is effective in analyzing periodic orbits in dynamical systems. In Chapter 3 we have shown its useful in differentiating between centre like oscillations and limit cycles for 2D dynamical systems [86, 89]. In addition, we show here that the RG method can also be used for identifying isochronous oscillations and perturbatively determining conditions for isochronicity.

Let's recall our RG method briefly. The first step is to find all the fixed points in the system and identify either a linear centre or a point that can be made a linear centre by a suitable choice of parameters (the parameters may, in some cases, have to be introduced). The perturbation theory is set up around the centre. The orbit will be characterized by two constants — the amplitude A and the phase θ, fixed by the two initial conditions associated with a second order ODE. The secular terms appearing in the perturbative answer are then removed by the renormalization process — by introducing a two renormalization parameters which depend on a running time-like constant τ. Finally one obtains RG flow equations for the amplitude and phase. The flow equations for autonomous dynamical systems typically have the form,

$$\frac{dA}{d\tau} = f(A), \tag{4.8}$$

$$\frac{d\theta}{d\tau} = g(A). \tag{4.9}$$

If the solution is of the centre variety, then the initial condition sets the amplitude of motion and hence the amplitude is fixed, leading to: $dA/d\tau = 0$. The phase flow equation, on the other hand, normally furnishes the nonlinear amplitude-dependent corrections to the frequency. But, for an isochronous cen-

tre, naturally there are no corrections to the frequency and hence, the phase flow must also vanish. So, in case of isochronous oscillators the following condition must hold true at every order of perturbation:

$$f(A) = 0; \qquad g(A) = 0 \quad \text{identically.} \tag{4.10}$$

If for a dynamical system the function $g(A)$ doesn't vanish, we can immediately conclude that the system is not isochronous. Typically $g(A)$ depends on the parameters of the system and it may vanish under certain non-trivial combination of values of the parameters. Say the system, has n parameters $\alpha_1, ..., \alpha_n$, then at every order of perturbation we get a relation between these parameters for which $g(A)$ vanishes. These will be however, only the necessary conditions for isochrony. But suppose we have a 2-parameter system; In order to uniquely specify the isochronicity condition, we need to find just two independent conditions which can be found by doing perturbation calculations for two different orders where secular terms are generated. One may argue that there is no guarantee that two independent conditions will be generated at two different orders and hence no guarantee that one can perturbatively find out the isochronous conditions. While this is true and poses a potential problem, however, in the numerous examples we have handled this is never the case. Of course when there are several parameters one needs to do perturbation calculations till higher orders which can be rather cumbersome but with help of algebraic manipulation softwares one can systematically use RG to pin down the necessary and sufficient conditions for isochronicity.

One of the systems we study explicitly is:

$$\dot{x} = y(1+x), \tag{4.11}$$

$$\dot{y} = f_0(x) + y f_1(x) + y^2 f_2(x), \tag{4.12}$$

where $f_0(x) = x + \alpha_1 x^2 + \alpha_2 x^3 + \alpha_3 x^4 \cdots$, $f_1(x) = \beta_1 x + \beta_2 x^2 + \cdots$ and $f_2(x) = \gamma_1 + \gamma_2 x^2$. For $\alpha_3 = \gamma_2 = 0$, a hierarchy of relations between the coefficients is found for isochronous centre at the origin. We find that if in addition $\alpha_1 = -\gamma_1$, the only isochronous cases possible are $\beta_1 = \beta_2 = 0$ and $\alpha_2 = 0, \gamma = 1/2$ or $\alpha_2 = -1, \gamma = 2$. This is where the power of the RG method lies. Without actually trying to solve the dynamics, we can get definite answers about the parameter relations required for isochronous oscillations. The condition of isochronicity we find for this system hasn't been reported before in literature, to the best of our knowledge.

4.4 Examples

4.4.1 A simple case

Apart from simple harmonic oscillators, the other paradigm [235–237] of an isochronous system that has been quite well known since a long time, is the rather unexpected case of the so called isotonic potential $V(x) = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$. As a first check that our statement regarding isochronous oscillators holds true, we considered this particular oscillator. The equation of motion is given by,

$$\ddot{x} = -x + \frac{1}{x^3}. \tag{4.13}$$

The above equation is known as the Ermakov-Pinney equation in literature [255, 256]. The Ermakov-Pinney equation has been the focus of many research studies of nonlinear dynamical systems, for a survey see [257]. The equation also arises in variety of physical situation for instance in scalar field cosmology [258], Feshbach-resonance of BECs [259], elasticity [260, 261], quantum mechanics [262]. A linear stability analysis of the system reveals that the linear centre is at $x = \pm 1$ and its enough to study the orbit around only one of them (there is an infinite barrier at $x = 0$ and hence the two centres are completely decoupled). We shift the origin to the centre according to our strategy and write Eq. (4.13), in terms of the new variable $y = x - 1$,

$$\begin{aligned} \ddot{y} &= -(1+y) + (1+y)^{-3}, \\ &= -4y + 6y^2 - 10y^3 + 15y^4 - 21y^5 + \cdots. \end{aligned} \tag{4.14}$$

One can question the validity of Taylor expanding $(1+y)^{-3}$, but this remains valid in the regime of small oscillations around the centre. For book-keeping purposes we introduce a parameter λ and rewrite Eq. (4.14) as

$$\ddot{y} + 4y = \lambda 6y^2 - \lambda^2 10y^3 + \lambda^3 15y^4 - \lambda^4 21y^5 + \cdots. \tag{4.15}$$

Now we can expand y as a perturbation series in powers of λ

$$y = y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \cdots, \tag{4.16}$$

and proceed as earlier. At different orders of λ we find the following differential equations:

$$\lambda^0: \quad \ddot{y}_0 + \Omega^2 y_0 = 0 \quad ;(\Omega = 2), \tag{4.17}$$

$$\lambda^1: \quad \ddot{y}_1 + \Omega^2 y_1 = 6y_0^2, \tag{4.18}$$

$$\lambda^2: \quad \ddot{y}_2 + \Omega^2 y_2 = 12y_0y_1 - 10y_0^3, \tag{4.19}$$

$$\lambda^3: \quad \ddot{y}_3 + \Omega^2 y_3 = 12y_0y_2 + 6y_1^2 - 30y_0^2y_1 + 15y_0^4, \tag{4.20}$$

and so on. As usual, we work with the initial conditions $y = A_0$, $\dot{y} = 0$ at $t = 0$. We let y_0 absorb the initial condition as: $y_0 = A_0$, $\dot{y}_0 = 0$, so that for all the subsequent orders — $y_i(t=0) = \dot{y}_i(t=0) = 0$. We are not quoting the calculations in detail here but explicitly solving the above set of equations one can confirm that no secular terms are generated. We have done the calculations up to 6th order in λ and the flow equations up to $\mathscr{O}(\lambda^6)$ are:

$$\frac{dA}{d\tau} = 0 + \mathscr{O}(\lambda^7); \qquad \frac{d\theta}{d\tau} = 0 + \mathscr{O}(\lambda^7). \tag{4.21}$$

Our calculations show that secular terms ($\cos\Omega t$ or $\sin\Omega t$) are never generated in the inhomogeneous part of the equations at different orders of perturbation and that our condition (4.10) for isochrony is satisfied by this well-known isochronous oscillator.

4.4.2 Loud system

We consider here systems with homogeneous quadratic nonlinearities. Loud first studied quadratic systems for possible isochronous behaviour. He proved that a nonlinear quadratic system has an isochronous centre at origin if and only if there exists a linear transformation that can take the system to a system of the following form [225]:

$$\dot{x} = -y + Bxy \tag{4.22}$$

$$\dot{y} = x + Dx^2 + Fy^2. \tag{4.23}$$

with $B \neq 0$ and the ratios D/B and F/B assuming either one of the following pair of values : $(0,1)$, $(-1/2,2)$, $(0,1/4)$ and $(-1/2,1/2)$. Note that for the parameter $B \neq 0$, with a suitable rescaling of

time one can express the above system as

$$\dot{x} = -y + xy, \tag{4.24}$$
$$\dot{y} = x + Dx^2 + Fy^2. \tag{4.25}$$

In order to pin down all the four possible combinations of the parameters (D,F) one needs to carry our RG calculations to at least two non-trivial orders of perturbation (which for this system turns out to be 3rd order in amplitude). The calculation is messy and just reproduces a known result. Our purpose, in considering this system is simply to illustrate how one may use RG to find the conditions for isochronous centre. So, instead of trying to pin down all four of the possible (D,F) combinations, we will use RG for the $D = 0$ case where we should get two possible values of F, i.e. $F = 1$ or $1/4$. The system now reads,

$$\dot{x} = -y + xy, \tag{4.26}$$
$$\dot{y} = x + Fy^2. \tag{4.27}$$

At the lowest order we have simply

$$\dot{x}_0 = -y_0; \qquad \dot{y}_0 = x_0, \tag{4.28}$$

to which we can immediately write the solutions $x_0 = A_0\cos(t+\theta_0)$ and $y_0 = A_0\sin(t+\theta_0)$. We can use the zeroth order solutions to write down the equations at second order as

$$\dot{x}_1 = -y_1 + x_0y_0 = -y_1 + \frac{A_0^2}{2}\sin 2(t+\theta_0), \tag{4.29}$$
$$\dot{y}_1 = x_1 + Fy_0^2 = x_1 + \frac{FA_0^2}{2}\left[1-\cos 2(t+\theta_0)\right]. \tag{4.30}$$

We can combine the above two equations to write down a 2nd order ODE for x_1 as,

$$\begin{aligned} \ddot{x}_1 &= -\dot{y}_1 + A_0^2\cos 2(t+\theta_0) \\ &= -x_1 - \frac{FA_0^2}{2}\left[1-\cos 2(t+\theta_0)\right] + A_0^2\cos 2(t+\theta_0) \\ \Rightarrow \ddot{x}_1 + x_1 &= -\frac{FA_0^2}{2} + \frac{A_0^2}{2}(F+2)\cos 2(t+\theta_0), \end{aligned} \tag{4.31}$$

which leads to

$$x_1 = -\frac{FA_0^2}{2}(1-\cos\phi) - \frac{A_0^2}{6}(F+2)(\cos 2\phi - \cos\phi), \tag{4.32}$$

where $\phi = (t + \theta_0)$. We can use the above expression and the relation (4.30) to write down y_1 as

$$\begin{aligned} y_1 &= -\dot{x}_1 + A_0^2 \sin 2\phi \\ \Rightarrow y_1 &= \frac{A_0^3}{4}(2F+1)\sin\phi - \frac{A_0^3}{6}(2F+1)\sin 2\phi. \end{aligned} \tag{4.33}$$

At the next order of perturbation we have the equations,

$$\dot{x}_2 = -y_2 + x_0 y_1 + x_1 y_0 \tag{4.34}$$

$$\dot{y}_2 = x_2 + 2F y_0 y_1 \tag{4.35}$$

We can use the expressions for x_0, y_0, x_1 and y_1 in the above two equations to obtain a 2nd order ODE for x_2. To derive the RG flow equations at this point, we need only to calculate the secular terms which contribute to the flow. After some simple algebra we obtain,

$$\ddot{x}_2 + x_2 = \frac{A_0^3}{12}(4F-1)(F-1)\cos\phi + \text{higher harmonics}. \tag{4.36}$$

This leads to the RG flow equations to this order,

$$\frac{dA}{d\tau} = 0; \tag{4.37}$$

$$\frac{d\theta}{d\tau} = -\frac{A^3}{24}(4F-1)(F-1). \tag{4.38}$$

According, to our conditions for isochrony both the flow equations must vanish identically at each order. For this to happen F can take up the values $F = 1$ and $F = 1/4$. This result is in agreement with Loud's result. Calculating the flow equations for the next order one can confirm that no discrepancies arise and for either of the above values of F the flow equations vanish. The above calculation serves as an example as to how one might systematically pin down the parameter values for which a given system can have isochronous oscillations.

4.4.3 Cherkas System

The next problem is a rather well studied problem — a dynamical system introduced by Cherkas [263]. He first studied this system and gave the conditions under which the system has centre-type solutions. However, the conditions found therein were later found to be incomplete. The complete set of necessary and sufficient conditions can be found in references [265] and [266]. Hill *et al* [267] gave certain conditions for which the origin becomes an isochronous centre. We deal with this multi-

parameter system in detail, so that our methodology becomes clear for dealing with these systems. The Cherkas system has the form

$$\dot{x} = y(1+x), \tag{4.39}$$
$$\dot{y} = -x - a_1x^2 - a_2x^3 - a_3x^4 - a_4(x+a_5x^2)y - a_6y^2. \tag{4.40}$$

Sabatini *et al* [249] and Chouikha [264] et al have also studied this system and have given some conditions for isochronicity. We study this system by perturbative RG approach and obtain a condition for isochronicity not reported before in literature. One can write the Cherkas system as a second order differential equation. Differentiating Eq. (4.39) w.r.t t, we have

$$\ddot{x} = (1+x)\dot{y} + y\dot{x}, \tag{4.41}$$

We use Eq. (4.40) to cast the above equation as,

$$\begin{aligned} \ddot{x}+x &= -(1+a_1)x^2 - (a_2+a_1)x^3 - (a_2+a_3)x^4 - a_3x^5 - a_4\dot{x}(x+a_5x^2) \\ &\quad +(1-a_6)\frac{\dot{x}^2}{1+x}, \qquad (4.42) \\ &= -\beta x^2 - (a_1+a_2)x^3 - (a_2+a_3)x^4 - a_3x^5 - a_4\dot{x}(x+a_5x^2) \\ &\quad +\alpha\dot{x}^2\left[1-x+x^2-x^3+\ldots\right]. \qquad (4.43) \end{aligned}$$

where $\beta = 1+a_1$ and $\alpha = 1-a_6$. Notice that the Cherkas system is actually a Liénard system : $\ddot{x} + \dot{x}f_1(x) + \dot{x}^2 f_2(x) + g(x) = 0$, having both '$\dot{x}$' (first kind) and '$\dot{x}^2$' (second kind) terms. We next introduce a book keeping parameter λ, to keep track of the order of nonlinearity and thus it is distributed as follows:

$$\begin{aligned} \ddot{x}+x &= -\lambda\beta x^2 - \lambda^2(a_1+a_2)x^3 - \lambda^3(a_2+a_3)x^4 - \lambda^4 a_3x^5 - \lambda a_4x\dot{x} - \lambda^2 a_4a_5\dot{x}x^2 \\ &\quad +\lambda\alpha\dot{x}^2(1-\lambda x+\lambda^2x^2-\lambda^3x^3+\ldots). \end{aligned} \tag{4.44}$$

As usual we begin by naively expanding x as,

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots$$

Using the above expansion in Eq. (4.44), at different orders, we get

$$\begin{aligned}
\mathscr{O}(\lambda^0): \qquad \ddot{x}_0 + x_0 &= 0, && (4.45)\\
\mathscr{O}(\lambda^1): \qquad \ddot{x}_1 + x_1 &= -\beta x_0^2 + \alpha \dot{x}_0^2 - a_4 x_0 \dot{x}_0, && (4.46)\\
\mathscr{O}(\lambda^2): \qquad \ddot{x}_2 + x_2 &= -2\beta x_0 x_1 - (a_1+a_2)x_0^3 + 2\alpha \dot{x}_1 \dot{x}_0 - \alpha \dot{x}_0^2 x_0 \\
&\quad -a_4(x_1\dot{x}_0 + x_0\dot{x}_1) - a_4 a_5 \dot{x}_0 x_0^2. && (4.47)
\end{aligned}$$

The initial conditions we use are: $x(t=0) = A$ and $\dot{x}(t=0) = 0$, which we impose wholly on x_0, so that for all subsequent orders we have the initial conditions: $x_n(0) = \dot{x}_n(0) = 0$. At the lowest order $\mathscr{O}(\lambda^0)$, the solution is $x_0 = A_0 \cos(t+\theta_0)$. Using this in Eq. (4.46) we have the solution,

$$x_1 = A_0^2 \Big[\frac{\alpha-\beta}{2} + \frac{\alpha+\beta}{6}\cos 2\phi - \frac{a_4}{6}\sin 2\phi - \frac{2\alpha-\beta}{3}\cos\phi + \frac{a_4}{3}\sin\phi \Big], \tag{4.48}$$

where $\phi = t + \theta_0$. Inserting these solutions from the first two orders into the Eq. (4.47), we calculate the coefficients of the secular term and thus find that the condition for the existence of a centre *i.e.* $dA/d\tau = 0$ (vanishing of the coefficient of the $\sin\phi$ term), which turns out to be

$$\begin{aligned}
& a_4(\beta - \alpha - a_5) = 0 \\
\textit{i.e.} \qquad & a_4 = 0 \quad \text{or} \quad \beta - \alpha = a_5. && (4.49)
\end{aligned}$$

The condition for isochronicity comes from the vanishing of the phase flow $d\theta/d\tau = 0$ which gives the following expression:

$$\frac{\alpha^2}{3} - \frac{\alpha}{4} + \frac{a_4^2}{12} = \frac{3}{4}(a_1+a_2) + \frac{5}{6}\alpha\beta - \frac{5}{6}\beta^2. \tag{4.50}$$

Notice that the above two conditions (4.49) and (4.50) don't contain the parameter a_3. So it doesn't take into account the effect of the terms containing a_3. Hence, these conditions hold only for $a_3 = 0$ in the system (4.39) and (4.40). In order to obtain conditions involving a_3, one needs to take the calculation to higher orders. Further, notice that the conditions obtained above are only the *necessary conditions*. So, they need to be satisfied for the existence of an isochronous centre for $a_3 = 0$, but obviously do not guarantee it's existence. However, let's consider the case where even the parameter a_4 is absent (the case considered by Ghose-Choudhury and Guha [244]), then with both $a_3 = a_4 = 0$, the isochronicity condition for the system, (4.39) and (4.40), turns out to be

$$\frac{5}{6}\alpha(1+a_1) - \frac{5}{6}(1+a_1)^2 + \frac{3}{4}(a_1+a_2) = \frac{\alpha^2}{3} - \frac{\alpha}{4}, \tag{4.51}$$

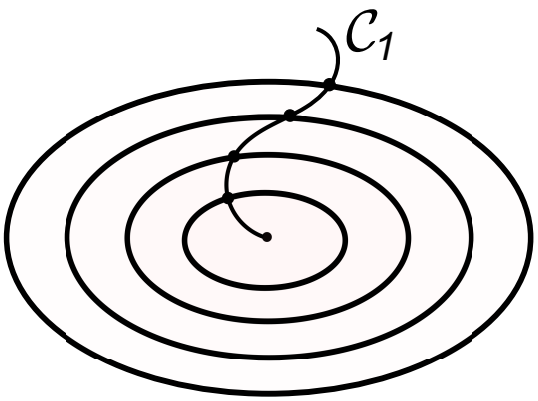

Figure 4.2: A family of closed orbits around a fixed point. For isochronous centres, starting the evolution of the dynamics from the points where $\mathscr{C}_1$ cuts the individual orbits, all the orbits cut the curve again at the same time.

where $1+a_1=\beta$. In order to check the validity of the condition above we have carried out numerical integration of the Cherkas system with $a_3=a_4=0$. We found that parameter values satisfying the condition (4.51) are all isochronous in nature. The way one checks whether a system is isochronous or not is by numerically determining the phase trajectory and performing a simple test. Suppose we have an isochronous centre around a fixed point $\mathscr{P}$, then the phase space looks like a family of closed orbits around the fixed point. Starting from the fixed point one can always draw a curve $\mathscr{C}_1$, intersecting every member of the family of closed orbits (See Fig 4.2). In case, the centre is an isochronous one, if we start evolution of the dynamics at the points (on every member of the family of closed orbits) where $\mathscr{C}_1$ cuts each member of the family of closed orbits then the next time when each trajectory cuts the curve $\mathscr{C}_1$ must all be the same. This simple test can be performed by numerically integrating the ODEs and determining the phase space trajectory.

It is worth noting that all of the isochronous cases of the Cherkas system found in the literature with $a_3=a_4=0$ satisfy the condition given in Eq. (4.51). Further, in the interesting special case where $\alpha=\beta$, this condition reduces to

$$\left(\alpha-\frac{3}{2}\right)^2=\frac{9}{4}a_2. \tag{4.52}$$

In the two isochronous cases [244] we have encountered so far in the literature, with $\alpha=\beta$, the results are in accordance with Eq. (4.52). The usefulness of such constraints lies in the fact that in one's search for isochronous systems, one knows in which subspace of the parameter space the search must be limited. This is a major advantage of perturbative RG method of determining isochronicity. One can systematically determine the parameters for which isochrony is possible.

For $a_3\neq 0$, the calculation needs to be carried through for two more orders and at the end of a long

but straightforward calculation one obtains the condition:

$$\frac{5}{4}\alpha^4+\frac{65}{48}\beta^4+\frac{355}{48}\alpha^2\beta^2-\frac{65}{12}\alpha\beta^3-\frac{55}{12}\alpha^3\beta-\frac{31}{24}\alpha^3-\frac{5}{12}\alpha\beta^2+\frac{83}{48}\alpha^2\beta-\frac{39}{16}\beta^2(a_1+a_2)$$
$$+\frac{69}{12}\alpha\beta(a_1+a_2)-\frac{27}{8}\alpha^2(a_1+a_2)-\frac{1}{3}\alpha\beta+\frac{83}{192}\alpha^2+\frac{7}{4}\beta(a_2+a_3)-\frac{3}{2}\alpha(a_2+a_3)$$
$$-\frac{1}{16}\alpha(a_1+a_2)+\frac{3}{64}(a_1+a_2)^2-\frac{5}{8}a_3-\frac{\alpha}{8}=0. \tag{4.53}$$

Once again, all the $a_3 \neq 0$ cases noted by Ghose-Choudhury and Guha satisfy the above constraint. When $a_3 = a_4 = 0$ and $\alpha = \beta$ the above constraint takes on a much simpler form, given by

$$\frac{\alpha^3}{48}-\frac{1}{16}\alpha^2(a_1+a_2)+\frac{19}{192}\alpha^2+\frac{1}{4}\alpha a_2-\frac{1}{16}\alpha(a_1+a_2)+\frac{3}{64}(a_1+a_2)^2-\frac{\alpha}{8}=0. \tag{4.54}$$

For the Cherkas system, when $a_3 = a_4 = 0$ and $\alpha = \beta$, all the isochronous cases must satisfy the above condition. Keeping in mind the fact that $\alpha = \beta = 1 + a_1$, the above constraint gives a relation between a_1 and a_2 much like the constraint (Eq. (4.52)) obtained at the previous order. So between the two equations, Eqs. (4.52) and (4.54), for the two parameters which must be satisfied for isochronicity, we can uniquely specify these parameters. The two conditions (Eqs. (4.52) and (4.54)) constitute the *necessary* and *sufficient* conditions for an isochronous Cherkas system, with $a_3 = a_4 = 0$ and $\alpha = \beta$. We can now eliminate a_2 between Eqs. (4.52) and (4.54) and this leads to a cubic equation in α, given by

$$4\alpha^3-24\alpha^2+45\alpha-27=0. \tag{4.55}$$

Solving the above cubic equation in α we find that there are only two roots of the above cubic equation, viz. $\alpha = 3$ and $\alpha = 3/2$ (which is a repeated root). These two cases correspond to the parameter values of: $a_2 = 1$ and $a_2 = 0$ respectively and are the only possible values for which the Cherkas system (when $a_3 = a_4 = 0$ and $\alpha = \beta$) is isochronous. In order to find isochronous cases of Cherkas with non-zero a_3 and a_4 one needs to take the perturbative calculations to even higher orders to obtain further relations that can then used to completely specify those parameters. The calculations for higher orders of perturbation in Cherkas system with all parameters non-zero is long and tedious, but straightforward.

In order to establish the efficiency of our method we consider another cubic systems as defined by Bardet *et al* [268] given by,

$$\dot{x} = -y(1+a_1x)+a_2x^2+a_3x^3,, \tag{4.56}$$
$$\dot{y} = x+b_3x^2+b_5x^3-y(b_2x-b_4x^2)-b_1y^2. \tag{4.57}$$

We have done RG calculations for the above set of equations and found conditions for isochronicity of

the centre at origin. The necessary conditions obtained are

$$(2a_2-b_2)\left[\frac{a_1+b_1}{3}-\frac{b_3+a_1}{6}+\frac{3}{4}b_3-\frac{b_1}{4}+\frac{a_1}{2}\right]+a_2(2b_1+a_1)-b_4-3a_3=0, \tag{4.58}$$

and

$$\frac{(a_1+b_1)}{3}(b_3+b_1+2a_1)(b_3+b_1+2a_1)-\frac{b_3+a_1}{6}(7b_1-5b_3+2a_1)-\frac{3}{4}(b_5+a_1b_3-a_2b_2)$$
$$-\frac{a_1(a_1+b_1)}{4}+\frac{1}{12}(2a_2-b_2)^2=0. \tag{4.59}$$

We have then checked that all the isochronous cases mentioned in Theorem 4.1 of reference [268] satisfy the criteria given above.

4.4.4 Ricatti equation

We finally turn to an oscillator which belongs to the of Liénard type of system, given by

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\Omega x=0. \tag{4.60}$$

For our example, $f(x)=kx$ and $g(x)=(k^2/9)x^3+\Omega x$. This oscillator, Eq. (4.61), has been widely treated in the literature and has been referred to as by various names. The oscillator may be interpreted as a cubic anharmonic oscillator with a nonlinear damping force, $kx\dot{x}$. The equation has also been described as generalized Emden-type equation [280]. This equation arises in a variety of physical problems — such as modeling thermodynamic fusion of pellets [271], one-dimensional analogue of Yang-Mill's boson gauge theory [272], equilibrium configurations of a spherical cloud subject to the laws of thermodynamics [273] etc. This system is related to the Ricatti hierarchy of equation and is a variant of the 2nd order Ricatti equation [272]. We refer to this system simply as the Riccati equation. We have dealt with this oscillator and its various unique properties in much greater detail in the next chapter. In this chapter however we will concentrate on the isochronous properties of this oscillator. Recently, Chandrasekhar *et al* [269] have shown that for a positive Ω, this nonlinear oscillator admits non-isolated periodic orbits with the unusual property that the frequency of oscillations remains independent of amplitude and the same as that of the linear harmonic oscillator: $\ddot{x}+\Omega x=0$. For our purposes we will show that the isochronous periodic solution that Eq. (4.60) admits, also follows the criterion we have put forward in this paper, *i.e.* $dA/d\tau=d\theta/d\tau=0$. Assuming that k is small, we

expand x in powers of k as follows:

$$x = x_0 + kx_1 + k^2x_2 + k^3x_3 \ldots \tag{4.61}$$

We work with the initial conditions $x(t_0) = A$ and $\dot{x}(t_0) = 0$ and absorb the initial conditions wholly at zeroth order, as was done previously. Proceeding henceforth according to our prescription, we calculate the flow equations perturbatively. We find by perturbative calculations upto the third order that no secular terms are generated for this oscillator. Thus, the flow equations as per our expectations turn out to be $dA/d\tau = d\theta/d\tau = 0$, right up to order $\mathscr{O}(k^3)$ and we therefore have an indication of an isochronous type oscillator. Numerical simulations of the oscillator represented by Eq. (4.60) reveal that for positive values of Ω, one obtains isochronous non-isolated orbits with time periods $T = 2\pi/\sqrt{\Omega}$. Using our RG approach we have shown here that for a oscillator to execute isochronous oscillations, it must obey the criterion (4.10) we have proposed.

4.5 A new family of isochronous systems

Our intention in this section is to see whether or not one can make use of the RG condition (4.10) for isochronicity to construct a new isochronous oscillator. Nonlinear terms in an oscillator equation generate secular term and if we can find a combination of nonlinear terms which cancel out each other's contributions (to secular terms) at every order it would mean that we have an isochronous oscillator in accordance with our condition. To this end we begin by looking at systems of the general form,

$$\dot{x} = -y + \alpha\mathscr{N}_x(x,y), \tag{4.62}$$

$$\dot{y} = x + \beta\mathscr{N}_y(x,y), \tag{4.63}$$

where $\mathscr{N}_x$ and $\mathscr{N}_y$ are homogeneous nonlinearities. The strategy we adopt is to see how nonlinearities generate nonlinearities at various orders of perturbation and then to choose suitable $\mathscr{N}_x$ and $\mathscr{N}_y$ such that they can lead to cancellation in the flow when parameters are right and in effect making the oscillations isochronous. We were able to identify a number of quadratic and cubic nonlinearities using our method. However, we could find no cases outside the systems already known in the literature to be isochronous. An extensive literature survey, indicated that all of the isochronous systems having homogeneous quadratic nonlinearities, and in all likelihood, all those with cubic nonlinearities have already been identified. We therefore moved on to homogeneous quartic nonlinearities in hopes of finding potentially new isochronous systems. Although there exist several previous works to identify

isochronous quartic systems [212, 230]. To our utter surprise we were immediately rewarded with a fairly simple system which, to the best of our knowledge, hasn't been identified as an isochronous oscillator before. The general system is given by,

$$\dot{x} = -y + \alpha x^4, \tag{4.64}$$

$$\dot{y} = x + \beta x^3 y, \tag{4.65}$$

and we will show how one can find the specific α and β values for which the system is isochronous. Notice that the choice of nonlinearity is crucial and they must be chosen in such a way that they contribute to the same (either 'cos' or 'sin') kind of term; otherwise, it can't lead to cancellation of flow. For instance, the choice $mcN_y = y^4$ would have generated cos terms and can't combine with secular contributions of 'x^4' and thus lead to trivial isochronicity conditions $\alpha = \beta = 0$. The choice 'x^4' and 'x^3y', however contribute to the same secular term and serve our purposes here. The terms 'x^4' and 'x^3y', being of even power, will not generate any secular term at lowest nontrivial order (4th order in amplitude) of perturbation. But, at the next nontrivial order (7th order in amplitude) we get an expression, relating α and β which then must vanish for isochronous oscillations. We now proceed to see if this is at all possible. At the lowest order of perturbation the equations are: $\ddot{x}_0 = -y_0$; $\ddot{y}_0 = x_0$, to which the solutions are $x_0 = A_0 \cos\phi$ and $y_0 = A_0 \cos\phi$ where $\phi = t + \theta_0$. At the next order we have the equations,

$$\begin{aligned} \dot{x}_1 &= -y_1 + \alpha x_0^4 \\ &= -y_1 + \alpha \cos^3\phi \\ \Rightarrow \dot{x}_1 &= -y_1 + \frac{\alpha A_0^3}{8}(3 + 4\cos 2\phi + \cos 4\phi) \end{aligned} \tag{4.66}$$

and

$$\begin{aligned} \dot{y}_1 &= x_1 + \beta x_0^3 y_0 \\ &= x_1 + \beta A_0^4 \cos^3\phi \sin\phi \\ \Rightarrow \dot{y}_1 &= x_1 + \frac{\beta A_0^3}{8}(2\sin 2\phi + \sin 4\phi). \end{aligned} \tag{4.67}$$

One can combine the Eqs. (4.66) and (4.67) to write down a 2nd order ODE in x_1,

$$\begin{aligned} \ddot{x}_1 &= -\dot{y}_1 - \frac{\alpha A_0^4}{2}(\sin 4\phi + 2\sin 2\phi) \\ &= -\frac{A_0^3}{4}(4\alpha + \beta)\sin 2\phi - \frac{A_0^3}{8}(4\alpha + \beta)\sin 4\phi \end{aligned} \tag{4.68}$$

which leads to the answer

$$x_1 = -\frac{A_0^4}{4}(4\alpha+\beta)\sin\phi + \frac{A_0^4}{12}(4\alpha+\beta)\sin 2\phi + \frac{A_0^4}{120}(4\alpha+\beta)\sin 4\phi \tag{4.69}$$

The expression for y_1 is obtained from Eq. (4.66) and is given by

$$y_1 = \frac{3\alpha A_0^4}{8} + \frac{A_0^4}{5}(4\alpha+\beta)\cos\phi - \frac{A_0^4}{6}(\alpha+\beta)\cos 2\phi - \frac{A_0^4}{120}(\alpha+4\beta)\cos 4\phi \tag{4.70}$$

At the next order $\mathcal{O}(A_0^7)$, we have the equations,

$$\dot{x}_2 = -y_2 + 4\alpha x_1 x_0^3 \tag{4.71}$$

$$\dot{y}_2 = -x_2 + 3\beta x_1 x_0^2 y_0 + \beta y_1 x_0^3. \tag{4.72}$$

Its obvious that secular terms will be generated at this order. In order to find the conditions for isochrony we need only to calculate the coefficients for the secular terms generated at this order and the conditions under which the secular terms vanish is nothing but the condition of the isochronicity. Calculating the coefficients of the secular terms from the above two equations using expressions for x_0, y_0, x_1 & y_1 we get the following condition for vanishing of the coefficient

$$\begin{aligned} 336\alpha^2 - 357\alpha\beta + 21\beta^2 &= 0 \\ \Rightarrow 21(16\alpha-\beta)(\alpha-\beta) &= 0. \end{aligned} \tag{4.73}$$

The values of α and β which satisfy the above condition thus can be isochronous. It was quite possible that at the next order we might get another relation between the parameters but it turns out that at the next order we get back the above condition again. This led us to check whether this condition itself is enough for isochronicity. We numerically checked that whenever either of the relations

$$\alpha = \beta \qquad \text{or} \qquad 16\alpha = \beta. \tag{4.74}$$

is satisfied the system given by Eqs. (4.64) and (4.65) is isochronous. The case $\alpha = \beta$ has been identified as an isochronous one in the review paper by Sabatini *et al* [212] as part of a more general case (system $H4_1$ in their paper) that they report. The $\alpha = 16\beta$ case, however, hasn't been identified as an isochronous case in any of the literature we have encountered.

Having identified this new isochronous oscillator, we noticed a curious fact that this equation is actually related to the Riccati equation, Eq. (4.60), we considered in the previous section. Actually, we can

write down the system given by Eqs. (4.64) and (4.65), as a 2nd order ODE in the following manner

$$\begin{aligned}
\ddot{x} &= -\dot{y} + 4\alpha x^3 \dot{x} \\
&= -x - \beta x^3 y + 4\alpha x^3 \dot{x} \\
&= -x - \beta x^3(-\dot{x} + \alpha x^4) + 4\alpha x^3 \dot{x} \\
\ddot{x} &= (4\alpha + \beta) x^3 \dot{x} + \alpha\beta x^7 - x. \qquad (4.75)
\end{aligned}$$

For the parameter values $\alpha = \beta = -1$ and we have the equation

$$\ddot{x} + 5x^3\dot{x} + x^7 + x = 0. \qquad (4.76)$$

Eq (4.60) (with $k = 3$ and $\Omega = 1$) is given by

$$\ddot{x} + 3x\dot{x} + x^3 + x = 0 \qquad (4.77)$$

The above two oscillators, Eqs. (4.76) and (4.77), are both isochronous and actually form a pattern. They belong to the general class of equations given by the generalized form,

$$\ddot{x} + (2n+3)x^{2n+1}\dot{x} + x^{4n+3} + x = 0; \qquad n = 0, 1, 2, 3, \cdots, \qquad (4.78)$$

identified first in [90]. We have carried out RG calculations for the above system suspecting that all the instances of this class of systems might be isochronous. The calculations confirmed our suspicion and the above system is indeed isochronous for all non-negative integers n. We have confirmed this fact numerically as well. We next cast the Eq. (4.78) in terms of two first order equations and obtain

$$\begin{aligned}
\dot{x} &= -y + \alpha x^{2n+2}, \qquad (4.79) \\
\dot{y} &= x + \beta x^{2n+1} y. \qquad (4.80)
\end{aligned}$$

We ask the question: what is the condition for isochronicity for the above generalized form? Would it give a generalized set of conditions for which the system becomes isochronous? In fact, RG calculations for the above system furnish the following two conditions,

$$\begin{aligned}
&\text{(I)} \qquad \alpha = \beta, \\
&\text{(II)} \qquad \alpha = \frac{1}{(2n+2)^2}\beta.
\end{aligned}$$

Notice, that the condition (I), when $\alpha = \beta = 1$, is actually equivalent to system given by Eq. (4.78). However, condition (II) gives us another set of parameter values for which isochronicity can occur. We have numerically verified that the system, Eqs. (4.79) and (4.80), becomes isochronous whenever either of the conditions (I) or (II) is satisfied. One must note here the for $n = 0$, the system (Eqs. (4.79) and (4.80)) reduces to

$$\dot{x} = -y + \alpha x^2, \tag{4.81}$$

$$\dot{y} = x + \beta xy. \tag{4.82}$$

which is obviously the subset of Loud system (Eqs. (4.26) and (4.27)) which we have already considered. The conditions of isochrony for the above system, simply are

$$\alpha = \beta; \qquad \alpha = \frac{1}{4}\beta \tag{4.83}$$

which is equivalent to the result we obtained for this particular system previously.

4.6 Conclusion

We have presented here a criterion, based on, perturbative RG method, which must be satisfied for isochronous oscillations to be possible in 2-D dynamical systems. Without giving a formal proof we have argued based on physical considerations that for a given centre to be isochronous both RG flow equations must identically vanish at each order. We have tested our hypothesis on several well known isochronous systems and it has proved to be a correct one. We have further checked if the condition (4.10) holds for the numerous isochronous cases listed in the review paper by Chavarriga *et al* [212] and it turns out that it holds for all the cases. Although a formal proof of our assertion would go a long way in establishing our claim, the condition itself is easy to understand intuitively once one understands the role played by the flow equations for a periodic solution. The amplitude flow equation provides information whether one has self-sustained oscillations (limit cycle) or neutrally stable (centre) solutions. On the other hand the phase flow equation gives the nonlinear amplitude-dependent correction to the assumed frequency (frequency of the basic unperturbed solution). For neutrally stable solutions, a centre, the amplitude doesn't flow and hence the amplitude flow must vanish at every order. If in addition the centre is isochronous the frequency can't have any amplitude dependence and it must remain fixed; which simply means that the phase flow must also vanish for isochronicity. For a given system, using the prescription presented here one can systematically identify all of the parameter combinations for which isochronous oscillations can occur.

The limitation of our methodology is being perturbative, it provides only the *necessary* and not necessarily the *sufficient* conditions for isochronicity. However, as we show using examples this is not a problem that can't be remedied. For the restricted Cherkas system ($a_3 = 0,\ a_4 = 0\ \&\ \alpha = \beta$), we find both the necessary and sufficient conditions. In order to find necessary and sufficient conditions for isochrony its only necessary that one finds enough independent conditions relating the parameters. In order to find necessary number of conditions one may need to do higher order perturbation theory. Being recursive in nature doing the higher order calculations poses no technical difficulties except perhaps the tediousness of it. However, the tedious-higher order calculations necessary to obtain enough relations for all parameters can be performed easily with help of computer algebra packages like *MATHEMATICA* or *MAPLE*.

The Ricatti oscillator constitutes a rather interesting case, where even after addition of nonlinearity the oscillator frequency remains independent of amplitude and the same as that of the linear harmonic oscillator. We also found that it is related to a family of nonlinear oscillators with increasingly higher order nonlinearities which possess the same property. We consider this family of periodic orbits in further detail in the next chapter. using our RG method. This opens up the possibility of identifying more such nonlinear oscillators which admit oscillations with amplitude-independent frequency. Furthermore this method can serve as a robust way to perturbatively construct the so called *period functions* which must vanish for isochrony.

We discuss next how does our method compare to the other methods for finding new isochronous systems. In this regard the method developed by Calogero et al is very efficient and systematic. Basically, the trick introduced by Calogero and Leyvraz — amounts to a simple change of the variables — which deforms the original system and its time evolution in such a manner that the deformed system turns out be isochronous. In this process though it modifies the system itself. Using this trick any autonomous dynamical system can be modified or extended to obtain a new autonomous dynamical system, involving a constant T (whose value can be arbitrarily assigned), such that all its solutions are isochronous. An interesting aspect of the new system is that although the dynamics is isochronous, it mimics closely the dynamics of the original system over time scales much smaller than T. These findings suggest two things - (i) 'isochronous systems are not rare' [243] and (ii) the time evolution of the new isochronous systems obtained by the above techniques can be quite complex. Calogero *et al* have thus given a method to derive new isochronous systems rather than determining whether a given planar dynamical system is isochronous or not. The RG technique, on the other hand, can be employed to determine perturbatively whether a given oscillator is isochronous or not and, if not, whether there are conditions under which it can become isochronous. In addition, RG used intelligently can also be employed to find new isochronous systems albeit along an entirely different philosophy than that of Calogero *et al*. Our method is much more closely related in philosophy to the that of Chouikha et al

(based on Urabe's criterion) which can be employed to find conditions of isochronicity for a given system. However, the method can't be employed to find new isochronous system unlike our RG method. Both Calogero *et al*'s and Chouikha *et al*'s methods appeal to the mathematical side of dynamical systems theory whereas our method appeals more to the physics side of it.

In conclusion, we remark that much of the work in the field of isochronous oscillations involves finding a Hamiltonian description of the system and proceeding to find conditions for integrability or super-integrability. For instance, Calogero's method relies on finding a Hamiltonian description of the system to start with and the method also churns out systems which have a Hamiltonian description. However the technical difficulties involved have prevented a complete solution (*e.g.* for the Cherkas system) in one step. In addition many cases can't be immediately reduced to Hamiltonian systems and in most cases it proves rather difficult to find the first integrals when nonlinearities are involved. Our method, however, is not limited by such considerations and can be used effectively as a first probe for isochronous dynamical systems. It can be used to systematically identify the relevant parameter space for isochrony and to find the *necessary* and *sufficient* conditions for isochrony. The prime advantage of our RG approach lies in the fact that without actually solving the system we are able to find conditions under which a given system exhibits isochrony.

Chapter 5

Oscillators without linear restoring force

5.1 Introduction

It has been amply demonstrated in the previous chapter that the renormalization group method [85, 86, 88, 89] is a very effective tool in analyzing periodic solutions occurring in various nonlinear differential equations. However, being perturbative in nature, our method comes with the usual disadvantages of perturbation theory. Chief amongst these disadvantages is the problem of finding a suitable unperturbed system around which the perturbative treatment can proceed. Any perturbative approach for finding the solution to a nonlinear system demands that we find a linear system (for certain parameter values of the system), the solution to which is known and this solution can serve as the first approximation. Therefore, for any given 2-D system, our methodology depends on our ability to find a centre for a certain set of parameters. As an example, the Van der Pol oscillator has the equation of motion $\ddot{x} + \alpha\dot{x}(x^2 - 1) + \omega^2 x = 0$, where the part $\ddot{x} + \omega^2 x = 0$ serves as the linear system with a centre about which the perturbative calculation can be done. Dynamical systems where a centre-like solution is not directly apparent, can also be handled by intelligently expanding the parameter space to consider a more general system than the one in question. We have already discussed in Chapter 2 as how its possible to do so in some cases. On the other hand, for systems devoid of any linear terms, our methodology fails and it warrants further modification to the methodology. For instance consider the oscillator given by, $\ddot{x} + \lambda x^3 = 0$. This oscillator does not have a linear part about which perturbative calculation can be carried out. In this chapter we will use the RG method in conjunction with a self-consistent averaging technique in order to overcome this difficulty.

The trouble ensues from the fact that if we don't have a linear system for any combination of parameter values in a given problem we don't have a starting point for our method. However, a class of

these problems may be handled by making use of ideas from a popular technique called the *equivalent linearization* [4] method, to artificially create a linear system which can serve as a starting point. This brings us to the point where we are faced with the question— what ensures the validity of a such a approach? First of all, as is true with most examples handled in this thesis, the nonlinearity has to be weak. Further, in order for the perturbation theory to work it would be necessary that the artificially created linear state is close enough to the full nonlinear state and there has to be a way to ensure this. We will show that this can be achieved by keeping an undetermined dimensionless constant during the equivalent linearization process which can later be fixed order by order in a self-consistent manner ensuring the validity of the initial assumption. The method of equivalent linearization was originally developed by Krylov and Bogoliubov [179, 274, 275]. Jordan and Smith [3] give a very simple way to implement the method which we shall follow here. For a pedagogic review of the method see [141].

5.2 RG in the absence of linear terms

We have already discussed the anharmonic oscillator given by the equation,

$$\ddot{x} + \omega^2 x + \lambda x^3 = 0, \tag{5.1}$$

in quite some detail. We now ask the question: how does one proceed in the case where $\omega = 0$? The equation now reads,

$$\ddot{x} + \lambda x^3 = 0, \tag{5.2}$$

and there is no obvious way doing perturbation theory. A direct integration of the above equation allows us to obtain the oscillation frequency in this case which turns out to be

$$\Omega_0 = (2\sqrt{\pi})\frac{\Gamma(3/4)}{\Gamma(1/4)}(\lambda E)^{1/4} \simeq 1.18\,(\lambda E)^{1/4} \tag{5.3}$$

where E is total energy of the oscillating particle. Our aim is to modify the perturbative RG method used on Eq.(5.1) to handle the situation when $\omega = 0$. To achieve this, we rewrite Eq. (5.2) as follows:

$$\ddot{x} + \alpha\lambda\langle x^2\rangle x + \lambda\left(x^3 - \alpha\lambda\langle x^2\rangle x\right) = 0, \tag{5.4}$$

where $\langle x^2\rangle$ is the average of $x(t)^2$ over a cycle. In the above equation α is a dimensionless constant which we need to determine. If we now ignore the term in parenthesis, we have an equivalent linear oscillator with frequency

$$\Omega^2 = \alpha\lambda\langle x^2\rangle. \tag{5.5}$$

We assume that the frequency Ω is the frequency of the full dynamics. The dimensionless constant α, can be chosen arbitrarily to suit our purposes. Here we assume that the value of α is such that the term $\left(x^3 - \alpha\lambda\langle x^2\rangle x\right)$ is small enough to be treated as a perturbation. Clearly, our task is to find 'α' which can be done perturbatively. We proceed to do so by expanding x in powers of the small parameter λ as usual i.e. $x = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots$, plugging into the equation,

$$\ddot{x} + \Omega^2 x + \lambda\left(x^3 - \alpha\lambda\langle x^2\rangle x\right) = 0, \tag{5.6}$$

to obtain the equations for different powers of λ,

$$\begin{aligned}
\ddot{x}_0 + \Omega^2 x_0 &= 0 && (5.7)\\
\ddot{x}_1 + \Omega^2 x_1 &= -\left(-x_0^3 - \alpha\langle x_0^2\rangle x_0\right) && (5.8)\\
\ddot{x}_2 + \Omega^2 x_2 &= -\left(3x_0^2 x_1 - \alpha x_1\langle x_0^2\rangle - 2\alpha x_0\langle x_1 x_0\rangle\right) && (5.9)
\end{aligned}$$

We work with the initial condition $x = A_0$ and $\dot{x} = 0$ at $t = t_0$ and we have at the lowest order

$$x_0 = A_0\cos(\Omega t + \theta_0) \tag{5.10}$$

Plugging this into the equation at $\mathscr{O}(\lambda)$, we get

$$\ddot{x}_1 + \Omega^2 x_1 = -\left[\frac{3A_0^3}{4}\cos(\Omega t + \theta_0) + \frac{A_0^3}{4}\cos 3(\Omega t + \theta_0) - \frac{\alpha A_0^3}{2}\cos(\Omega t + \theta_0)\right], \tag{5.11}$$

leading to the solution,

$$x_1 = -\frac{A_0^3}{2\Omega}\left(\frac{3}{4} - \frac{\alpha}{2}\right) t\sin(\Omega t + \theta_0) + \frac{A_0^3}{32\Omega^2}\left[\cos(\Omega t + \theta_0) - \cos 3(\Omega t + \theta_0)\right]. \tag{5.12}$$

We can now write the perturbative solution upto $\mathscr{O}(\lambda)$,

$$\begin{aligned}
x = A_0\cos(\Omega t + \theta_0) + \lambda\Big[&\frac{A^3}{32\Omega^2}\left\{\cos(\Omega t + \theta_0) - \cos 3(\Omega t + \theta_0)\right\}\\
&-\frac{A^3}{2\Omega}\left(\frac{3}{4} - \frac{\alpha}{2}\right)\left(t + \frac{\theta_0}{\omega}\right)\sin(\Omega t + \theta_0)\Big] + \mathscr{O}(\lambda^2),
\end{aligned} \tag{5.13}$$

where $t_0 = -\theta/\omega$. The renormalization constants are introduced in an identical fashion as described in chapter 2 and after some straight-forward algebra we obtain the flow equations right upto $\mathscr{O}(\lambda)$:

$$\frac{dA}{d\tau} = 0 \tag{5.14}$$

$$\frac{d\theta}{d\tau} = \frac{\lambda}{2\Omega}\left(\frac{3}{4}-\frac{\alpha}{2}\right) \tag{5.15}$$

Since by definition Ω is the frequency of the full dynamics it has to remain unaffected. And this happens if Eq. (5.15) has a fixed point, which it has at $\alpha = 3/2$. This immediately fixes the value of α at $3/2$ upto this order. The solution $x(t)$ is simply $A\cos\Omega t + \mathscr{O}(\lambda)$ and hence $\langle x^2\rangle \cong A^2/2 = (E/\lambda)^{1/2}$. The frequency of the nonlinear oscillator at this order is found from Eq. (5.5) and is given by

$$\Omega = \left(\frac{3}{2}\right)^{1/2}(\lambda E)^{1/4} \tag{5.16}$$

This is within 3% of the the exact result shown in Eq. (5.3). Usually equivalent linearization forms the basis of self-consistent treatment of differential equations. Our assumption that the frequency of full dynamics, Ω, can be defined by as in Eq. (5.3) has no apriori basis. However, the fact that starting from such a definition we end up with an approximate result within 3% speaks for itself as to the efficacy of this approach. In some sense in this case the end justifies the means.

We will now demonstrate that the result can be in fact improved by including the contribution from the next order of perturbation. In the process, the method of carrying out higher order calculation becomes apparent. In order to do so we first write down the final form x_1,

$$x_1 = \frac{A^3}{32\Omega^2}\Big[\cos 3(\Omega t+\theta_0) - \cos(\Omega t+\theta_0)\Big] \tag{5.17}$$

At $\mathscr{O}(\lambda^2)$, we write Eq. (5.9) as

$$\begin{aligned}
\ddot{x}_2 + \Omega^2 x_2 &= -\Bigg[3\frac{A^2}{2}(1+\cos 2\Phi)\frac{A^3}{32\Omega^2}(\cos 3\Phi - \cos\Phi) - \alpha\frac{A^3}{32\Omega^2}(\cos 3\Phi - \cos\Phi) \\
&\quad -2\alpha A\cos\Phi\left\langle\frac{A^4}{32\Omega^2}(\cos 3\Phi - \cos\Phi)\right\rangle\Bigg] \\
&= -\left[-\frac{3A^5}{64\Omega^2}\cos\Phi + \left(\frac{\alpha A^5}{64\Omega^2}+\frac{\alpha A^5}{32\Omega^2}\right)\cos\Phi + \text{higher harmonics}\right] \\
&= \frac{3(1-\alpha)}{64}\frac{A^5}{\Omega^2}\cos\Phi + \text{higher harmonics}
\end{aligned} \tag{5.18}$$

where $\Phi = \Omega t + \theta$. It should be clear from the derivation of the amplitude equation in Eq.(5.14), only

the secular terms matter for determining the flow equation at any order and not the higher harmonics. Also noting that $\alpha\lambda A^2/2 = \Omega^2$ to the lowest order, we can write the solution to Eq. (5.18) as

$$x_2 = -\frac{3A_0^5}{2\Omega^3}\left(\frac{1-\alpha}{64}\right)\cos(\Omega t + \theta_0) \tag{5.19}$$

We can now re-calculate the flow equations including the second order corrections which now gives us

$$\frac{dA}{d\tau} = 0 \tag{5.20}$$

$$\frac{d\theta}{d\tau} = \frac{\lambda}{2\Omega}\left(\frac{3}{4}-\frac{\alpha}{2}\right) - \frac{3\lambda A_0^5}{2\Omega^5}\left(\frac{1-\alpha}{64}\right) \tag{5.21}$$

Note that we can use the expression

$$\alpha\lambda A^2/2 = \Omega^2 \tag{5.22}$$

to simplify the flow equation. Using the same criterion as before i.e. the frequency correction must vanish, we get the following equation for α,

$$\frac{3}{4}-\frac{\alpha}{2}-\frac{3}{32}\frac{1-\alpha}{\alpha} = 0 \tag{5.23}$$

which leads to the answer,

$$\alpha \simeq 2\left(\frac{3}{4}+\frac{1}{32}\right) = \frac{3}{2}\left(1+\frac{1}{24}\right) \tag{5.24}$$

The actual frequency to this order comes from

$$\begin{aligned}
\Omega^2 &= \alpha\lambda\langle x^2\rangle = \alpha\lambda\langle x_0^2 + 2\lambda x_0 x_1\rangle \\
&= \alpha\lambda\frac{A^2}{2} + 2\alpha\lambda^2\langle x_0 x_1\rangle \\
&= \alpha\lambda\frac{A^2}{2} - \frac{\alpha\lambda A^2}{32\Omega^2}\lambda A^2 \qquad \text{using Eqs.(5.10) and (5.17)} \\
&\simeq \alpha\lambda\frac{A^2}{2} - \frac{\lambda A^2}{16} \qquad \text{using Eq. (5.22)} \\
&\simeq \frac{\alpha\lambda A^2}{2}\left[1-\frac{1}{12}\right] \\
&= (\lambda E)^{1/2}\frac{3}{2}\left(1+\frac{1}{24}\right)\left(1-\frac{1}{12}\right) \\
&\simeq (\lambda E)^{1/2}\frac{3}{2}\left(1-\frac{1}{24}+\ldots\right)
\end{aligned} \tag{5.25}$$

leading to

$$\Omega = \sqrt{\frac{3}{2}}(\lambda E)^{1/4}\left(1 - \frac{1}{48} + \ldots\right) \tag{5.26}$$

which turns out to be within 1% of the actual result. This shows how our perturbation theory can be used to systematically improve the answer and how effective it is in giving reasonable answers.

5.3 Riccati equation

As the next example we consider the second-order differential equation given by the equation:

$$\ddot{x} + \beta x\dot{x} + \alpha x^3 = 0 \tag{5.27}$$

where α and β are dimensionless arbitrary parameters. This is the so called second order Riccati equation [276–278]. This differential equation, with slight variations has occurred widely in literature under several names. For example it arises in the study of modified Emden equation [279–281]. It has also been investigated as an oscillator of Liénard type by Chandrasekhar *et al* [189, 269]. Over the past three decades or so the symmetry, solvability and integrability properties of this equation alone has been studied by several authors [282–295]. This equation and its variant are well known to physicists and mathematicians for over a century [277,278,296] because it is known to arise in various contexts. Some of these are in modelling of the fusion of pellets [271], in thermodynamic study of equilibrium configuration of a spherical cloud of gas [279, 297], motion of a free particle is a space of constant curvature [298], spherically symmetric collapse of relativistically gravitating mass [299] etc. It also appears in the theory of univalent functions [300], in the reduction of the 2nd member of Burger's hierarchy [301], 1D analogue of Yang-Mills bosonic gauge theory [272, 302] and probably many others of which we have no knowledge. The equation with $\alpha = \beta^2/9$ is known to posses 8 Lie point symmetries and is completely integrable. The solution is not periodic but for values of α much lesser than $\beta^2/9$ it is known to have a periodic solution. We are interested in the parameter range where this oscillator has a periodic response.

As is obvious this oscillator having no linear part is not amenable to perturbative treatment. This oscillator displays some very striking properties. For $\beta = 3$ it has several symmetries and is exactly solvable [280]. It does not show oscillatory behavior at $\beta = 3$ rather there is single attractive fixed point at origin to which all trajectories are converge. For $\beta \ll 1$ it is known to have periodic solutions [3]. We now pose the question as one varies the β from values much less than 1 to 3 when and how does the oscillatory behavior vanish?

We will consider Eq.(5.27) in the form considered by Chisholm *et al* [272]

$$\ddot{x} + \beta\lambda x\dot{x} + \lambda^2 x^3 = 0. \tag{5.28}$$

where they refer to it as a generalization of the second order Riccati equation. Note that we have introduced a small parameter λ to keep track of the degree of nonlinearity. The second order Riccati equation has $\beta = 3$. When we say that we consider a generalization of the oscillator we simply mean that here the parameter β can take up any real value. We will simply refer to it as the *second order Riccati equation* or simply *Riccati equation*. The only fixed point of this equation is the origin.

We now use our modified RG method to predict that the oscillatory behavior seen for $\beta \ll 1$, disappears through a vanishing of the frequency of oscillation i.e. the time period diverges. This happens at a critical value, β_c, for which a second order calculation yields the value $\beta_c = \sqrt{\frac{75}{11}}$. A numerical determination of this critical value yields the result $\beta_c = 2.81$, in very good agreement with our predictions. We further show that for β close to β_c the time period diverges as $(\beta - \beta_c)^{-1/2}$. The numerical results bear out our analytical results.

We investigate if there is a possibility of this equation having periodic solution for any parameter range. In order to carry out the perturbation calculation as have detailed before we begin by naively expanding x in powers of the small parameter λ as,

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \lambda^4 x_4 + \ldots \tag{5.29}$$

Further in accordance to our method we re-write Eq. (5.28)

$$\begin{aligned} & \ddot{x} + \beta\lambda x\dot{x} + \lambda^2\alpha\langle x^2\rangle x + \lambda^2\left(x^3 - \alpha\langle x^2\rangle x\right) = 0 \\ \text{or} \quad & \ddot{x} + \Omega^2 x = -\beta\lambda x\dot{x} - \lambda^2\left(x^3 - \alpha\langle x^2\rangle x\right) \end{aligned} \tag{5.30}$$

where Ω is the frequency of the full dynamics and is given by the expression:

$$\Omega^2 = \lambda^2\alpha\langle x^2\rangle. \tag{5.31}$$

α is a arbitrary dimensionless positive constant which is as yet undetermined. As per our methodology α will fixed perturbatively and self-consistently as we do the calculation. Plugging in the perturbation series into Eq. (5.30), we find at different orders

$$\ddot{x}_0 + \Omega^2 x_0 = 0 \tag{5.32}$$
$$\ddot{x}_1 + \Omega^2 x_1 = -\beta x_0 \dot{x}_0 \tag{5.33}$$
$$\ddot{x}_2 + \Omega^2 x_2 = -\left[x_0^3 - \alpha\langle x_0^2\rangle x_0\right] - \beta x_1 \dot{x}_0 - \beta x_0 \dot{x}_1 \tag{5.34}$$
$$\ddot{x}_3 + \Omega^2 x_3 = -\left[3x_0^2 x_1 - \alpha x_1\langle x_0^2\rangle - 2\alpha x_0\langle x_1 x_0\rangle\right] - \beta(x_2\dot{x}_0 + \dot{x}_2 x_0 + \dot{x}_1 x_1) \tag{5.35}$$
$$\begin{aligned}\ddot{x}_4 + \Omega^2 x_4 &= \beta(x_0\dot{x}_3 + x_1\dot{x}_2 + x_2\dot{x}_1 + x_0\dot{x}_3) - 3x_0 x_1^2 - 3x_0^2 x_2 \\ &\quad + \alpha x_2\langle x_0^2\rangle + 2\alpha x_1\langle x_1 x_0\rangle + \alpha x_0\langle x_0 x_2\rangle\end{aligned} \tag{5.36}$$

We can now immediately write down the solutions at the first two orders

$$x_0 = A_0\cos(\Omega t + \theta_0), \tag{5.37}$$
$$x_1 = -\beta\frac{A_0^2}{6\Omega}[\sin 2(\Omega t + \theta_0) - 2\sin(\Omega t + \theta_0)]. \tag{5.38}$$

Notice that the solutions to the first two orders do not contain any secular terms. The divergence comes at the next order. In order to renormalize the perturbation series we need to carry out the calculation to at least until the next order. Using the solutions for x_0 and x_1 we write down the equation at $\mathscr{O}(\lambda^2)$

$$\ddot{x}_2 + \Omega^2 x_2 = A_0^3\left(-\frac{3}{4} + \frac{\alpha}{2} + \frac{\beta^2}{12}\right)\cos(\Omega t + \theta_0) + \text{non-resonating terms} \tag{5.39}$$

To order $\mathscr{O}(\lambda^2)$, we have

$$\begin{aligned}x &= A_0\cos(\Omega t + \theta_0) - \frac{\lambda\beta A_0^2}{6\Omega}[\sin 2(\Omega t + \theta_0) - 2\sin(\Omega t + \theta_0)] \\ &\quad + \frac{\lambda^2 A_0^3}{2\Omega}\left(-\frac{3}{4} + \frac{\alpha}{2} + \frac{\beta^2}{12}\right)\left(t + \frac{\theta_0}{\Omega}\right)\sin(\Omega t + \theta_0) + \cdots\end{aligned} \tag{5.40}$$

As before we define two renormalization constants $\mathscr{Z}_1$ and $\mathscr{Z}_2$ and following the usual steps of the renormalization procedure we find the flow equations as

$$\frac{dA}{d\tau} = 0 \tag{5.41}$$
$$\frac{d\theta}{d\tau} = \lambda\left(\frac{3}{4} - \frac{\alpha}{2} - \frac{\beta^2}{12}\right)\frac{A^2}{2\Omega} \tag{5.42}$$

The amplitude flow equation predicts that we have a centre at the origin. Further, as we had defined Ω as the frequency of the full dynamics the phase flow equation must vanish. The frequency remains at

Ω, provided,

$$\alpha = \frac{3}{2}\left(1 - \frac{\beta^2}{9}\right) \tag{5.43}$$

By definition $\alpha > 0$ and hence the for a periodic state to be possible we have the condition:

$$\beta < 3. \tag{5.44}$$

This is the result from the lowest non trivial order and simply gives a restriction which is already known from the exact solution for the equation at $\beta = 3$.

In order to find an improved result, we need to carry out the calculation until the next non-trivial order of perturbation which happens to be the 5th order in amplitude. The 4th order equation, Eq. (5.35), does not yield any secular terms and consequently doesn't give any further corrections. Eq. (5.35) can be easily solved by using the solutions from first three orders of perturbation. However, the calculation though pretty straightforward, is rather long and tedious. Without going into the details of the calculation we simply quote the answer here:

$$\begin{aligned} x_3 = \quad & \frac{\beta A^4}{36\Omega^3}\left(\alpha - 3 + 2\beta^2\right)\sin 2\Phi + \frac{\beta(1-\beta^2)A^4}{32\Omega^3}\sin 3\Phi \\ & + \frac{\beta A^4}{\Omega^3}\left(\frac{13\beta^2}{144\times 15} - \frac{1}{80}\right)\sin 4\Phi \end{aligned} \tag{5.45}$$

Now we turn to Eq. (5.36) and using the above solution along with results from previous three orders, it is immediately possible to find the coefficient of $\cos(\Omega t + \theta)$ which gives us the flow equation upto this order. Again we simply quote the result here without giving the steps of the calculations. The phase flow equation correct upto the fifth order of perturbation is,

$$\frac{d\theta}{d\tau} = \lambda\left(\frac{3}{4} - \frac{\alpha}{2} - \frac{\beta^2}{12}\right)\frac{A^2}{2\Omega} + \frac{\lambda^2 A^4}{2\Omega^3}\left[\frac{\alpha}{64}\left(3 + \frac{37\beta^2}{9} - \frac{(23\beta^2+9)(9-\beta^2)}{96\times 18}\right)\right]. \tag{5.46}$$

In the second term on the right hand side of the above equation, we insert the lowest order value of $\frac{\lambda A^2}{\omega^2}$ i.e. $2/\alpha$. With this substitution and making use of the fact that the lowest order result for α is $\frac{9-\beta^2}{6}$, we finally have the result:

$$\frac{d\theta}{d\tau} = \frac{\lambda A^2}{2\Omega}\left[\frac{9-\beta^2-6\alpha}{12} - \frac{23\beta^2+9}{144} - \frac{3}{32} + \frac{37\beta^2}{288}\right] \tag{5.47}$$

The above expression gives the critical value of β as

$$\beta_c = c = \sqrt{\frac{75}{11}} \simeq 2.61. \tag{5.48}$$

We thus conclude from a calculation good to the fifth order in amplitude, that a periodic state will be found in the second order Riccati equation for values of $\beta < \beta_c (= 2.61)$. Further it is evident that the time period of oscillation will become very large and approach *infinity* as β approaches the critical value β_c. From the fixed point of Eq. (5.47) we find that $\alpha = \frac{225-33\beta^2}{144}$ and hence the time period, T is given by

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{[\alpha\lambda\langle x^2\rangle]^{1/2}} \propto \left(\frac{75}{11} - \beta^2\right)^{-1/2} \tag{5.49}$$

which diverges in a characteristic fashion, $(\beta_c - \beta)^{-1/2}$, as β approaches β_c.

In order to further verify our results we have carried out numerical integration of Eq. (5.28) for varying values of β. We used a Runge-Kutta 4th order method with variable step size to integrate the Eq. (5.28). As the value of β is increased, the time period increases as well, as can be seen from Fig. 5.1. The time period becomes extremely large as β approaches 2.81, which is the numerically determined critical point. Fig. 5.2 shows the phase space evolution of the oscillator for five different values of β. We notice that smaller the value of β, larger is the amplitude of the periodic orbit. For $\dot{x} < 0$, i.e. the lower part of the orbit, has a convexity, the tangent to which approaches asymptotically towards the x-axis as the value of β approaches the critical value β_c. Which means, as the value of β approaches β_c that as the phase space trajectory nears the origin the velocity becomes slower and slower. If value the parameter β is close to the critical value yet remains slightly lower than the critical value, the trajectory crosses the y-axis with extremely small but non-zero velocity. This bottleneck while crossing the velocity-axis ultimately leads to the time period divergence and at the critical value β_c the trajectory simply falls into the fixed point at origin where the periodic solution vanishes altogether.

Leach *et al* [285] have considered the Riccati equation in the form $\ddot{x} + x\dot{x} + \alpha x^3 = 0$ and have carried out extensive studies on its solution for different values of α. They have used a self-similar transformation to obtain a different form of the equation of motion in a new phase space defined by: $\zeta = xt$; $\eta = \dot{x}t^2$, where the potential, $V(\zeta) = \alpha\zeta^4/4 - \zeta^3/3 + \zeta^2$, is much more transparent to analysis. It turns out that for value of $\alpha = 1/8$, $V(\zeta)$ is seen to have a stationary point of inflection which corresponds to the critical value where periodic solution vanishes. Recently, Chandrasekhar *et al* [294] have also found the general solutions to the second order Riccati equation, which they write as: $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$, for arbitrary values of α and β. Choosing suitable canonical transformations

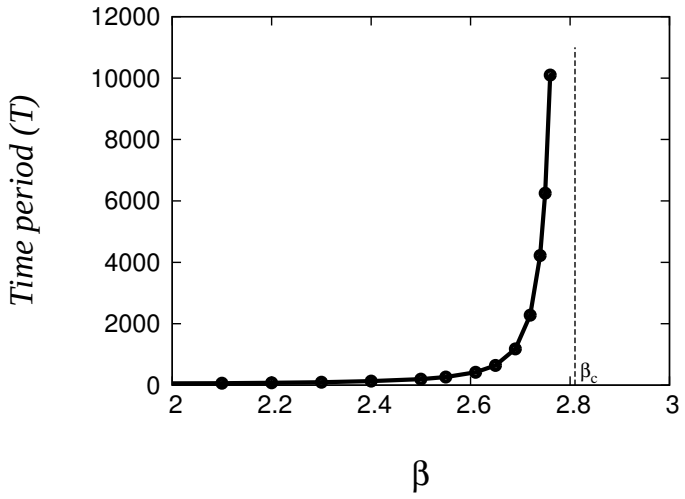

Figure 5.1: The dependence of time period (T) on the parameter β.

for each of the three cases: $\alpha^2 < 8\beta$, $\alpha^2 = 8\beta$, and $\alpha^2 > 8\beta$, they write down their respective general solutions. Although Chandrasekhar *et al* haven't focussed on the transition of the solution from a periodic to aperiodic state rather they show more interest in the underlying symmetry properties of the equation and integrability conditions. However, from their results as well, it is apparent that the transition from periodic to non-periodic solutions occur at $\alpha^2 = 8\beta$. Noting that α as defined by Leach *et al* is related to β defined here as $\beta = \sqrt{1/\alpha}$, we see that the critical value according to both Leach *et al* and Chandrasekhar *et al* is $\beta_c = \sqrt{8} = 2.83$. Noting that α as defined by Leach *et al* is related to β defined here as $\beta = \sqrt{1/\alpha}$, we see that the critical value according to them is $\beta_c = \sqrt{8} = 2.83$. Numerical simulations put the critical value of β at 2.82. Keeping in mind that our approach is perturbative, the value of $\beta_c = 2.61$ (which is within 7% of the correct result) obtained through perturbative RG upto $\mathscr{O}(\lambda^2)$ is fairly good result and in agreement with previous literature and our numerics.

5.4 Van der Pol-Duffing oscillator

Now we consider the so called Van der Pol-Duffing oscillator which is given by,

$$\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + \lambda x^3 + \omega^2 x = F\cos(\Omega t) \tag{5.50}$$

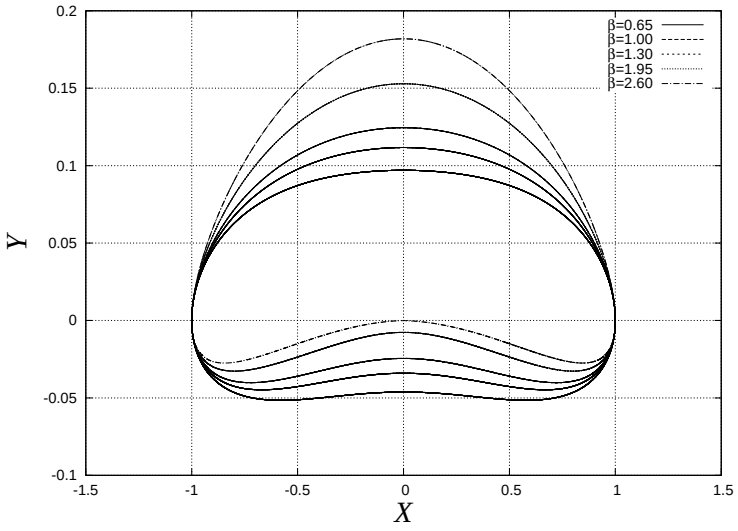

Figure 5.2: Solutions in phase space for different values of β.

where ε and λ are both small parameters. This oscillator is a popular choice for studying various nonlinear phenomena. Van der Pol-Duffing oscillator serves as an important mathematical model to emulate dynamical systems which have a single unstable fixed point and a single stable limit cycle. There have been several recent studies which investigate various aspects of this oscillator [303–306]. When we drop the forcing term and the linear term, we have yet another example of an oscillator devoid of any linear terms,

$$\ddot{x} + \varepsilon \dot{x}(x^2 - 1) + \lambda x^3 = 0 \tag{5.51}$$

We are also interested in this oscillator because its a purely non-linear differential equation which can execute self-sustained oscillations i.e. there is a limit cycle. We are interested in finding out how our tweaked methodology works in cases with limit cycles. As outlined above, to carry out perturbative RG we have to rewrite Eq. (5.51) as

$$\begin{aligned} & \ddot{x} + \varepsilon \dot{x}(x^2 - 1) + \alpha\lambda \langle x^2 \rangle x + \lambda \left(x^3 - \alpha \langle x^2 \rangle x \right) \\ \text{or} \qquad & \ddot{x} + \Omega^2 x = -\varepsilon \dot{x}(x^2 - 1) - \lambda \left(x^3 - \alpha \langle x^2 \rangle x \right) \end{aligned} \tag{5.52}$$

where Ω is the frequency of the full dynamics and is defines as,

$$\Omega^2 = \lambda \alpha \langle x^2 \rangle \tag{5.53}$$

We expand x as usual in powers of the small parameters ε and λ and obtain at different orders the following equations:

$$\begin{aligned}
\mathscr{O}(\varepsilon^0\lambda^0): \qquad \ddot{x}_0+\Omega^2 x_0 &= 0 && (5.54)\\
\mathscr{O}(\varepsilon^1\lambda^0): \qquad \ddot{x}_1+\Omega^2 x_1 &= -\dot{x}_0(x_0^2-1) && (5.55)\\
\mathscr{O}(\varepsilon^0\lambda^1): \qquad \ddot{x}_2+\Omega^2 x_2 &= -\left[x_0^3-\alpha\langle x_0^2\rangle x_0\right] && (5.56)
\end{aligned}$$

Now we can solve the above set of equations recursively and obtain the perturbation series. The solution to the above set of equations upto first order in small parameters is given by,

$$\begin{aligned}
x(t) &= A_0\cos\Omega t-\varepsilon\left[\frac{A_0^3}{32\Omega}(\sin 3\Omega t-3\sin\Omega t)+\frac{A_0}{2}\left(\frac{A_0^2}{4}-1\right)\right]\\
&\quad +\lambda\left[\frac{A_0^3}{32\Omega^2}(\cos 3\Omega t-3\cos\Omega t)-\frac{A_0^3}{2\Omega}\left(\frac{3}{4}-\frac{\alpha}{2}\right)t\sin\Omega t\right] \qquad (5.57)
\end{aligned}$$

From the above solution we can obtain the RG flow equations right upto first order in the perturbation parameters ε and λ, which is given by

$$\begin{aligned}
\frac{dA}{d\tau} &= -\varepsilon\frac{A_0}{2}\left(\frac{A_0^2}{4}-1\right) && (5.58)\\
\frac{d\theta}{d\tau} &= \lambda\frac{A_0^3}{2\Omega}\left(\frac{3}{4}-\frac{\alpha}{2}\right) && (5.59)
\end{aligned}$$

The amplitude flow given by Eq. (5.58), suggests that the oscillator exhibits limit cycle oscillations with radius $A=2$. By definition the frequency remains unchanged at Ω, which can happen provided the RHS of the flow equation, Eq. (5.59), vanishes. This condition fixes the value of α at $\alpha=3/2$. Making use of the relation, Eq.(5.53), we finally have

$$\Omega=\sqrt{\lambda\alpha(A^2/2)}=\sqrt{3\lambda}. \qquad (5.60)$$

The above result suggests that the oscillator frequency, Ω (consequently the time-period $T(=2\pi/\Omega)$), has a dependence on the parameter λ. We carried out numerical integration of the Van der pol-Duffing oscillator via Runge-Kutta 4th order method to check if our predictions hold true. We have plotted the time-period (T) with varying values of the small parameter λ in Fig.(5.3). As is clear the numerically obtained result compares extremely well with the 1st order perturbative RG result. We again emphasize that although doing perturbative RG in systems with linear terms is obvious its not so in cases without any linear terms. We have borrowed ideas from equivalent linearization and combined with

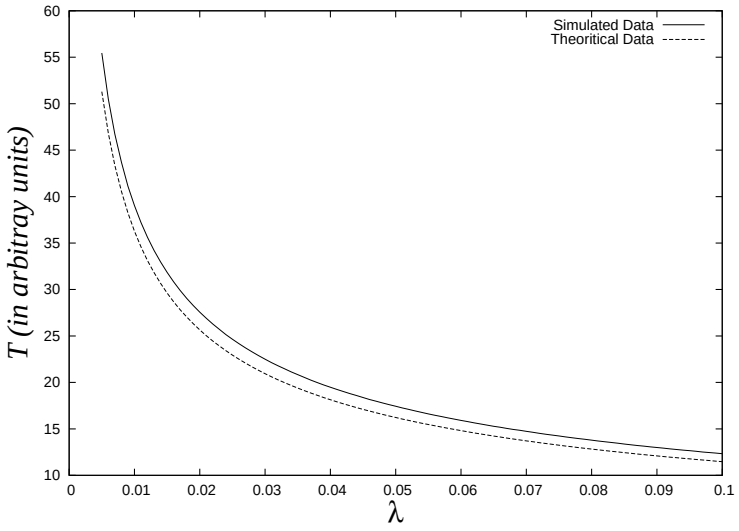

Figure 5.3: Dependence of time period on λ. Comparison between RG result and numerical results.

RG approach to successfully reproduce results in few such cases. As we have shown the disappearance of the periodic solution to the Riccati equation is satisfactorily explained with results obtained from RG analysis. We also capture the correct behavior of the limit cycle appearing in force free Van der Pol-Duffing oscillator. Here it can be said that this technique of doing perturbative RG in absence of linear terms may find applications in wide range of problems where absence of linear terms make them inaccessible to perturbative techniques.

5.5 Generalized Riccati equation

The second order Riccati equation as we have shown here presents some very striking features. For instance although the solution is initial-condition-dependent the value of the critical parameter β_c does not have any initial-condition-dependence. Further the oscillator's time-period diverges in a characteristic fashion when β approaches β_c. It prompted us to further investigate this particular oscillator. In our efforts to study this oscillator we noticed a queer thing. If we write the nonlinearities — '$x\dot{x}$' and 'x^3' — of the Riccati equation (Eq. (5.28)) in a more generalized form — '$x^{2n+1}\dot{x}$' and 'x^{4n+3}' — respectively, where $(n = 0, 1, 2, \cdots)$, some of the general features of the oscillator still persist. For instance, we found that the oscillator underwent a transition from being oscillatory to non-oscillatory depending on a critical parameter for all values of n. In this section we will discuss the dynamics of this generalized Riccati equation.

The generalized second-order Riccati equation is given by

$$\ddot{x} + (2n+3)x^{2n+1}\dot{x} + x^{4n+3} = 0, \tag{5.61}$$

which can also be written as

$$\left(\frac{d}{dt} + x^{2n+1}\right)^2 x = 0. \tag{5.62}$$

The general system, i.e. with arbitrary n, has been briefly investigated in [289,307] has several interesting symmetries. The number of Lie point symmetries of the above equation varies with the value of n. For example, the equation with $n = 0$ is obviously linear and has seven symmetries. When $n = -3/2$, we have the Kummer–Schwarz equation with two $sl(2,R)$ symmetries. In the case that $n = -3/4$ there are again seven symmetries and the equation is linearizable by means of an interchange of dependent and independent variables. In order to understand the dynamical properties of this rather interesting oscillator we will carry out an extensive study of this system using not only RG method but a few other techniques.

5.5.1 Lagrangian analysis

When $n = 0$, Eq. (5.61) is easily transformed into a linear third-order equation by the transformation,

$$x = 3(\dot{u}(t)/u(t)). \tag{5.63}$$

For general n the corresponding transformation was suggested by Leach and co-authors [287,288], and is given by,

$$x = \left[\left(\frac{\dot{u}}{u}\right)\frac{1}{2n+1}\right]^{\frac{1}{2n+1}}. \tag{5.64}$$

The above transformation reduces Eq. (5.61) into a third-order equation,

$$\frac{\dddot{u}}{\ddot{u}} = \frac{2n}{2n+1}\frac{\ddot{u}}{\dot{u}} \tag{5.65}$$

which is in a readily integrable form. Integrating we have the solution which has the structure,

$$u = (c_1 t + c_2)^{2n+2} + c_3, \tag{5.66}$$

where c_1, c_2 and c_3 are constants of integration. This allows us to find $x(t)$ and in the asymptotic limit, (i.e. $t \to \infty$), we see that

$$x(t) \approx t^{-\left(\frac{1}{2n+1}\right)}. \tag{5.67}$$

When we write down the dynamical system corresponding to Eq. (5.61), namely

$$\dot{x} = y, \tag{5.68}$$
$$\dot{y} = -(2n+3)x^{2n+1}y - x^{4n+3}, \tag{5.69}$$

we see that the solution heads for a fixed point at the origin. The origin is thus a stable node of the system.

The Lagrangian structure underlying Eq. (5.61) can be found using Jacobi's last multiplier [308–310] which we denote as $\mathscr{M}$. We proceed by writing $\mathscr{M} = u^{1/\zeta}$, where

$$\frac{1}{\zeta}\frac{\dot{u}}{u} = (2n+3)x^{2n+1}. \tag{5.70}$$

We substitute $\dot{x} = u + \mathscr{W}(x)$, where $\mathscr{W}(x)$ needs to be determined, into Eq. (5.61) and we have

$$-(2n+3)\dot{x}x^{2n+1} - x^{4n+3} = \ddot{x} = \zeta(2n+3)\Big(\dot{x} - \mathscr{W}(x)\Big)x^{2n+1} + \dot{x}\mathscr{W}'(x). \tag{5.71}$$

The terms containing $\dot{x}$ and those without it, have to be the same. Equating the coefficients of $\dot{x}$ and x separately, we obtain the equations:

$$\mathscr{W}'(x) + (2n+3)(\zeta+1)x^{2n+1} = 0, \tag{5.72}$$
$$(2n+3)\zeta\mathscr{W}(x) - x^{2n+2} = 0. \tag{5.73}$$

Eliminating $\mathscr{W}(x)$ from the above two equations, leads to the expression

$$\zeta(\zeta+1)(2n+3)^2 + 2n + 2 = 0, \tag{5.74}$$

which yields two possible values of ζ,

$$\zeta = -\frac{1}{2n+3} \quad \text{and} \quad -\frac{2n+2}{2n+3} \tag{5.75}$$

and correspondingly, two possible last multipliers $\mathscr{M}_1$ and $\mathscr{M}_2$ given by

$$\mathscr{M}_1 = \left(\dot{x} + x^{2n+2}\right)^{-(2n+3)} \quad \text{and} \quad \mathscr{M}_2 = \left(\dot{x} + x^{2n+2}\right)^{-\left(\frac{2n+3}{2n+2}\right)}. \tag{5.76}$$

Since $\mathscr{M} = \partial^2 L/\partial \dot{x}^2$, this leads to two possible Lagrangians, namely

$$\mathscr{L}_1 = \frac{\left(\dot{x} + x^{2n+2}\right)^{-(2n+1)}}{(2n+2)(2n+1)} \quad \text{and}$$
$$\mathscr{L}_2 = \frac{4(n+1)^2}{2n+1}\left(\dot{x} + \frac{x^{2(n+1)}}{2(n+1)}\right)^{1-\frac{1}{2(n+1)}} \tag{5.77}$$

in which the two constants of integration can be set at zero to recover (5.61). Thus the system given by Eq. (5.61) has a bi-lagrangian structure. The corresponding Hamiltonians can easily be found as well.

5.5.2 RG for generalized Riccati equation

We now apply our RG method to the generalized version of Eq. (5.61) which we write as

$$\ddot{x} + \beta x^{2n+1}\dot{x} + x^{4n+3} = 0. \tag{5.78}$$

We call the above equation a generalization of Eq. (5.61) in the sense that whereas in Eq. (5.61) β could take up only the integral value – '$2n+3$', in Eq. (5.78) β can be any positive real number. As we will find out in this section, varying the parameter β can significantly alter the dynamics of the system. For the $n = 0$ case we have already demonstrated that there exists a critical value of β such that, when $\beta < \beta_c$, the dynamics is periodic, whereas for $\beta > \beta_c$, the dynamics evolves towards the fixed point at the origin [89, 285]. Numerical studies indicate that the general system given by equation (5.78) behaves in a similar manner with an n-dependent β_c. As we have already discussed, the critical value of β for $n = 0$ can be found by different methods. However, the method which is most readily generalizable to arbitrary n is the perturbative renormalization group technique [86, 89]. We will indicate through our calculations how a critical β_c could be obtained from this procedure. We begin by rewriting Eq. (5.78) as

$$\ddot{x} + \alpha\langle x^{4n+2}\rangle x = -\beta\dot{x}x^{2n+1} - \left[x^{4n+3} - \alpha\langle x^{4n+2}\rangle x\right], \tag{5.79}$$

where $\langle x^{4n+2}\rangle$ is the time average over one cycle. As usual we define the frequency Ω of the full dynamics as

$$\Omega^2 = \alpha\langle x^{4n+2}\rangle. \tag{5.80}$$

A perturbation expansion in powers of the amplitude leads to the following equations at different orders of perturbation:

$$\ddot{x}_0 + \Omega^2 x_0 = 0, \tag{5.81}$$

$$\ddot{x}_1 + \Omega^2 x_1 = -\beta \dot{x}_0 x_0^{2n+1}, \tag{5.82}$$

$$\ddot{x}_2 + \Omega^2 x_2 = -\beta \dot{x}_1 x_0^{2n+1} - (2n+1)\beta \dot{x}_0 x_1 x_0^{2n} - \left[x_0^{4n+3} - \alpha \langle x_0^{4n+2} \rangle x_0\right]. \tag{5.83}$$

We use the initial conditions $x(t=0) = A_0$ and $\dot{x}(t=0) = 0$. We recollect that the renormalization group prescription is to calculate the flow equations for the amplitude and phase in terms of which we can express the periodic solution. The flows are obtained as $\dot{A} = f(A)$ and $\dot{\theta} = g(A)$. The functions $f(A)$ and $g(A)$ are obtained perturbatively from the coefficients of $\sin\Omega t$ and $\cos\Omega t$, respectively, appearing in the right hand side of Eqs. (5.81)–(5.83). Straightforward algebra leads us to the solutions

$$x_0 = A_0 \cos\Omega t \quad \text{and} \tag{5.84}$$

$$x_1 = \frac{\beta A_0^{2n+2}}{2\Omega} \sum_{m=0}^{n} \frac{\Lambda_m^{2n+1} - \Lambda_{m+1}^{2n+1}}{1 - 4(m+1)^2} \Big[\sin 2(m+1)\Omega t - 2(m+1)\sin\Omega t\Big], \tag{5.85}$$

where

$$\Lambda_m^{2n+1} = \frac{1}{2^{2n}} \frac{(2n+1)!}{(n-m)!(n+m+1)!}. \tag{5.86}$$

The secular term first appears on the right hand side of the equation for x_2, which is given by

$$\ddot{x}_2 + \Omega^2 x_2 = f(\alpha,\beta)\cos\Omega t + \cdots, \tag{5.87}$$

where

$$\begin{aligned} f(\alpha,\beta) = & \left[\frac{\beta^2}{4} \sum_{m=0}^{n-1} \frac{\left(\Lambda_m^{2n+1} - \Lambda_{m+1}^{2n+1}\right)^2}{4(m+1)^2 - 1} + \frac{\beta^2}{4} \frac{\Lambda_n^{2n+1}\left(\Lambda_n^{2n+1} - \Lambda_{n+1}^{2n+1}\right)}{4(n+1)^2 - 1}\right. \\ & \left. - \frac{1}{\pi} \frac{\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}{(n+1)\Gamma(2n+2)} + \frac{\alpha}{4} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}{(n+1)\Gamma(2n+2)}\right]. \end{aligned} \tag{5.88}$$

From the above expression one can immediately write down the flow equations,

$$\frac{d\theta}{dt} = f(\alpha,\beta) \quad \text{and} \tag{5.89}$$

$$\frac{dA}{dt} = 0, \tag{5.90}$$

at this order of calculation. As per our RG prescription [86], the fact that $dA/dt = 0$ suggests that the origin, $x = \dot{x} = 0$, is a centre. So irrespective of the value of the integer n the origin is surrounded by a family of initial-condition-dependent orbits. The perturbative correction to the frequency, on the other hand, is obtained from the phase flow. But since by definition, Ω gives the correct frequency for the dynamics, the phase flow must vanish, $d\theta/d\tau = 0$, which simply fixes the value of α. Now the question that one can ask here is for what range of the parameter β, is the fixed point a centre? Equating $f(\alpha,\beta)$ to zero we get

$$\begin{aligned}\frac{\beta}{4}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma(2n+2)(n+1)} &= -\frac{\alpha^2}{4}\left[\sum_{m=0}^{n-1}\frac{\left(\Lambda_m^{2n+1}-\Lambda_{m+1}^{2n+1}\right)^2}{4(m+1)^2-1}+\frac{\Lambda_n^{2n+1}\left(\Lambda_n^{2n+1}-\Lambda_{n+1}^{2n+1}\right)}{4(n+1)^2-1}\right] \\ &+ \frac{1}{\pi}\frac{\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}{(n+1)\Gamma(2n+2)}. \end{aligned} \tag{5.91}$$

Since Ω^2 is always positive and by definition we have $\alpha > 0$, from the above equation we have $\beta < \beta_c$, where

$$\frac{\beta_c}{4}\left[S_n+\frac{\Lambda_n^{2n+1}\left(\Lambda_n^{2n+1}-\Lambda_{n+1}^{2n+1}\right)}{4(n+1)^2-1}\right] = \frac{1}{\pi}\times\frac{\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}\right)}{(n+1)\Gamma(2n+2)} \tag{5.92}$$

with

$$S_n = \sum_{m=0}^{n-1}\frac{\Lambda_m^{2n+1}\left(\Lambda_m^{2n+1}-\Lambda_{m+1}^{2n+1}\right)^2}{4(m+1)^2-1}. \tag{5.93}$$

Evaluation of the above sum yields $\alpha_c = 2n+3$. As we have already shown in Section 5.3, in the $n = 0$ case, the perturbation theory calculation to the next order yields $\alpha_c \simeq 2.60$, while the exact answer [285] is $2\sqrt{2} \simeq 2.828$ and the numerically obtained result [89] is $\alpha_c = 2.81$. For $n = 1$, the correction to the lowest-order answer of $\alpha_c = 5$, turns out to be an even larger departure from the numerically obtained result. This prompts us to look for a different technique for finding the critical β when $n >> 1$. The reason RG method is not as effective in finding out the critical value for $n = 1$ and higher as it was in $n = 0$ might be due to it perturbative nature. When the nonlinearity becomes stronger with increasing n the validity of the perturbative answer comes under question. However, RG still furnishes the upper limit ($\beta_c = 2n+3$) for the parameter β beyond which one can't have periodic solution. In fact its possible to write down an exact solution to Eq. (5.61) for $\beta = 2n+3$.

5.5.3 Connection with an isochronous oscillator

Before changing techniques we take a short detour and point out a very interesting feature of this dynamical system. Note that when β takes up the integral value of $2n+3$, we have $f(\alpha,\beta)=0$. Thus the phase flow vanishes, i.e. $d\theta/dt=0$ for $\beta=2n+3$. There are two nonlinear terms in the system and the fact that at $\beta=2n+3$ both flow equations vanish suggest that the "divergence"-causing contributions from the two nonlinearities cancel out each other exactly. We already know that the Eq. (5.61) doesn't have an oscillatory behaviour for the parameter value $\beta=2n+3$. However, if we add a linear term '$\omega^2 x$' to Eq. (5.61) we have an oscillator,

$$\ddot{x}+\beta x^{2n+1}\dot{x}+x^{4n+3}+\omega^2 x=0, \tag{5.94}$$

which exhibits isochronous oscillations when $\alpha=2n+3$. Such a conclusion follows because the term added, '$\omega^2 x$', does not contribute to the 'secular term' at any order of perturbation. Furthermore at $\beta=2n+3$ the amplitude-dependent frequency, Ω^2 as defined by us vanishes and hence the oscillator can execute oscillations only at the linear frequency ω which is amplitude-independent. Hence, the two competing nonlinear terms are exactly equal at $\beta=2n+3$, thereby giving an isochronous oscillator. Treating the nonlinear terms as perturbations and finding the RG flow equations to the second order we notice that both dA/dt and $d\theta/dt$ vanish, further indicating that we have an isochronous system.

We carried out numerical integration of Eq. (5.94) to verify our conjecture about it being isochronous and indeed it turns out to be true. To the best of our knowledge the general system given by Eq. (5.94) hasn't been identified as an isochronous system before. Please note that this oscillator is in fact a special case of the oscillator already discussed in Chapter 4, where we demonstrate how RG can be used to systematically pin down new isochronous systems. Chandrasekhar *et al* [269] have studied the '$n=0$' case of Eq. (5.94) and interpret the result as an "unusual oscillator of Liénard type with properties of a linear harmonic oscillator", highlighting the amplitude-independence of the oscillation's time period, which essentially means that the oscillations are isochronous in nature.

5.5.4 Understanding the dynamics

Next we take a closer look at the mechanisms of this oscillator to better comprehend its dynamics. We do so by examining the phase-space of the family of periodic orbits surrounding the fixed point at origin. We are interested in the periodic dynamics only, and thus assume that the value of β is lower

than β_c for the sake of this analysis. We begin by looking at the isoclines for the dynamical system,

$$\dot{x} = v, \tag{5.95}$$
$$\dot{v} = -\beta x^{2n+1} v - x^{4n+3}. \tag{5.96}$$

The isoclines for the above system are basically given by

$$x = 0, \tag{5.97}$$
$$v = 0, \tag{5.98}$$
$$\beta v = -x^{2n+2}. \tag{5.99}$$

The curve given by Eq. (5.99) is shown in Fig. 5.4 and it has flat region (for small value of $|x|$) which is an asymptote to the x-axis and then the curve turns away (for larger values of $|x|$) from the x-axis. The sharpness of this turn and the steepness of the slope of the curve thereafter becomes increasingly large as n varies from 0 to higher integral values. From Eq. (5.96) one can easily see that the acceleration, $\dot{v}$, vanishes for either when $v = -x^{2n+2}/\beta$ or for $x = 0$. Further for when x is less than 1, the acceleration is very small, i.e. $\dot{v} << 1$. The isocline and a typical orbit are shown in Fig. 5.4. In order for the orbit not to become a closed one, the velocity v of the particle has to vanish as $x \to 0$. Lets take a closer look at what happens to the velocity as x varies between $x = (-\beta v)^{1/(2n+2)}$ and $x = 0$. From Eq. (5.96) one can conclude that for $v < 0$ (as it must be in this region) the first nonlinear term increases v, which would mean it would slow the leftward motion (in phase space) and on the other hand the second term helps the particle to move to the left. The passage from a point (x_0, v_0) on the isocline ($\beta v = -x^{2n+2}$) to the axis $x = 0$ is the crucial step in the whole process which determines whether the trajectory continues leftward and closes the orbit or simply falls into the fixed point at origin. It is useful to look at the differential equation for the phase path, which can be obtained from Eqs. (5.95) and (5.96). The equation reads,

$$\frac{dv}{dx} = -\beta x^{2n+1} - \frac{x^{4n+3}}{v}. \tag{5.100}$$

Lets try to find an approximate solution to the above equation between (x_0, v_0) and $(0, v)$. For β values close to β_c, as a first approximation, we can set $v = v_0$ as dv/dt is very small in this range and we obtain

$$v + \frac{\beta}{2(n+1)} x^{2n+2} + \frac{x^{4n+4}}{4(n+1)v} = C, \tag{5.101}$$

where C is a constant of integration.

In addition we have performed extensive numerical studies in order to understand the dynamics of the system given by Eqs. (5.95) and (5.96) by numerically integrating the equations using Runge-Kutta

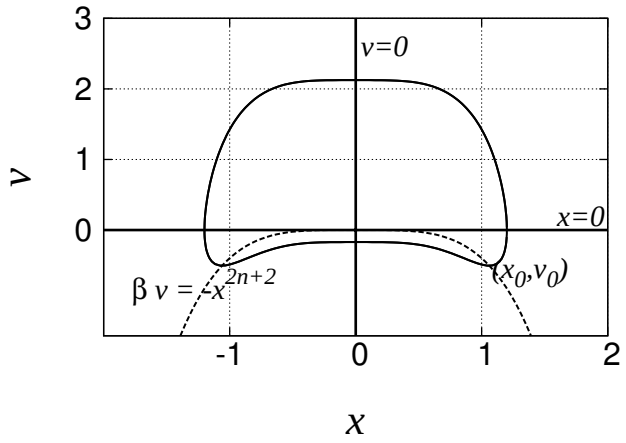

Figure 5.4: The above figure displays a typical orbit of the dynamical system given by Eqs. (5.95) and (5.96) and the isoclines for the dynamics. The orbit and isoclines shown above are for $n = 1$ and $\alpha = 2.5$.

4th order method in various parameter ranges. A numerical determination of the critical parameter, β_c, was done for different values of n and the results are tabulated in Table 5.1.

Table 5.1: Comparison between theoretical and numerical results.

n	β_c(numerical)	β_c (theoretical)
0	2.81	2.82
1	3.98	4.00
2	4.86	4.89
3	5.62	5.66
4	6.26	6.32
5	6.84	6.93
6	7.37	7.48
7	7.86	8.00
8	8.31	8.49
9	8.72	8.94
10	9.13	9.38

Based upon the numerical data and the fact that $\sqrt{n+1}$ turns out to be an ever-present number in the analytic expressions associated with the dynamics of the system, we make the conjecture that critical β has the form

$$\beta_c = 2\sqrt{2}(n+1)^{1/2}. \tag{5.102}$$

For $\beta < \beta_c$ the solution to the dynamical system, Eqs. (5.95) and (5.96), is a centre, i.e. a family of periodic orbits dependent upon the initial conditions. Yet the critical β value has no dependence upon the initial conditions whatsoever. This fact suggests that there must be some universal property associated with the family of orbits. Keeping this in mind we plotted the numerical solutions to the

system (Eqs. (5.95) and (5.96)), with $n = 1$ and $\beta = 3.9$, for different initial conditions. For simplicity we took initial conditions of the form $x_i = \kappa(const.)$ and $y_i = 0.0$. Next, we attempt to collapse all the orbits onto the one trajectory corresponding to the initial condition $x_i = 1.0$ and $y_i = 0.0$. It turns out that, if x is scaled as x/κ and y as y/κ^4, all the trajectories collapse onto a single orbit. In general for arbitrary n the trajectories collapse onto one orbit when x is scaled as x/κ and y as y/κ^{2n+2}. Fig. 5.5 shows the orbits for five different initial conditions for generalized Riccati equation with $n = 1$. And Fig. 5.6 shows how all of them collapse onto a single trajectory. It is also possible to collapse trajectories for initial conditions of the form $x_i = \kappa_1$ and $y_i = \kappa_2$ onto the orbit for $x_i = 1.0$ and $x_i = 0.0$. However, then x is not necessarily scaled by $1/\kappa_1$. Suppose in that case that x is scaled as some x/γ. Then y is automatically scaled as y/γ^{2n+2}.

We can analytically obtain this scaling behaviour from Eq. (5.100). We re-write Eq. (5.100) by substituting x as and y as $x = \gamma x_s$ and y as $y = \gamma^l y_s$

$$\gamma^{l-1}\frac{dv_s}{dx_s} = -\beta\gamma^{2n+1}x_s^{2n+1} - \gamma^{4n+3-l}\frac{x_s^{4n+3}}{v_s}. \tag{5.103}$$

From the above equation it is clear that in order to keep Eq. (5.100) invariant the value of l must be $2n+2$. Thus, if x scales as x/γ and y scales as y/γ^{2n+2}, the trajectory remains invariant. This universal behaviour is the reason why the critical α_c does not have any dependence upon the initial conditions.

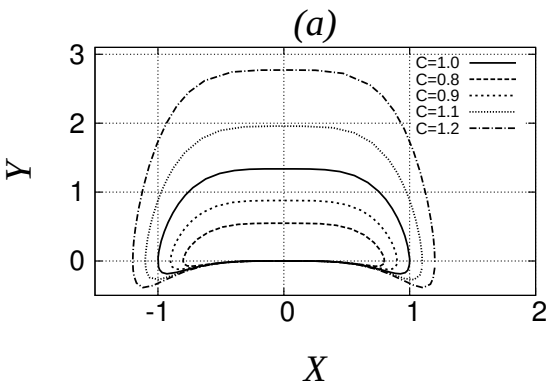

Figure 5.5: Periodic orbits corresponding to various initial conditions: $x_i = C$ and $y_i = 0$.

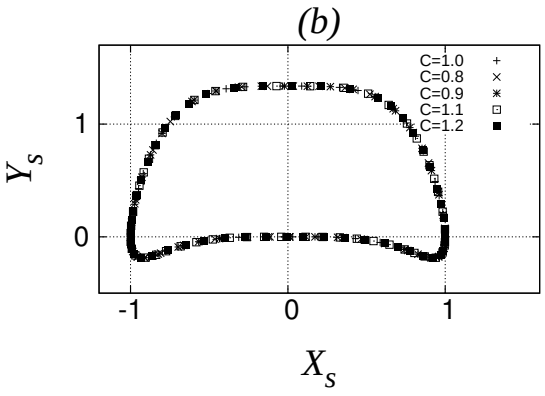

Figure 5.6: All the data points have collapsed onto a single orbit as x and y are scaled respectively as $x_s = x/C$ and $y_s = y/C^4$.

5.5.5 Hidden-symmetry approach

We now endeavour to establish the answer analytically. For this purpose we will utilize the symmetries of the differential equation by an approach called 'hidden-symmetry reduction', proposed by

Abraham-Shrauner [311]. The basic idea behind this approach can be summed up as follows. The first step is to increase the order of the given differential equation via a non-local transformation. Next, from any of the symmetries of the equation, we make use of an appropriate one to reduce the order of this equation and obtain an equation which is of the same order as the original one. In the process one hopes that the ensuing equation has more symmetries than the original one and hence might help in understand the dynamics better. For this purpose we rewrite the generalized Riccati equation in the form given by Bouquet *et al* (1991), also referenced as modified Painlevé-Ince equation in the literature,

$$\ddot{x} + x^m \dot{x} + k x^{2m+1} = 0, \tag{5.104}$$

where k is related to β of (5.61) by the relation $k = 1/\beta^2$ and m is related to n as $m = 2n+1$. The first transformation we use here is a slight generalization of the Riccati transformation and is given by

$$x = \left(1 + \frac{2}{m}\right)^{1/m} \left(\frac{\dot{u}}{u}\right)^{1/m}. \tag{5.105}$$

Using the above transformation, the resultant third-order equation is

$$m u^2 \dot{u} \dddot{u} + (1-m) u^2 \ddot{u}^2 + \left((m+2)^2 k - 1\right) \dot{u}^4 = 0. \tag{5.106}$$

If one analyzes the above equation for symmetry, ∂_t turns out to be one of the symmetries. We use this symmetry and its invariants to obtain new variables in order to achieve the next reduction of order. The invariants associated with ∂_t are simply u and $\dot{u}$. We define the new variables and consequent derivatives as

$$\begin{aligned} u(t) \to s,\ \dot{u}(t) \to w(s),\ \ddot{u}(t) \to w'(s) w(s) \\ \text{and } \dddot{u}(t) \to w''(s) w(s)^2 + w(s) w'(s)^2. \end{aligned} \tag{5.107}$$

Using the above relations we reduce Eq.(5.108) to a second-order equation, given by

$$(km^2 + 4km + 4k - 1) w(s)^4 + s^2 w(s)^2 w'^2 + m s^2 w(s)^3 w'' = 0 \tag{5.108}$$

Note that the Eq. (5.108) has the structure of an Euler equation in s. Finally we make yet another transformation:

$$s \to e^r,\ w(s) \to v(r),\ w'(s) \to \frac{v'(r)}{s} \text{ and } w''(s) \to \frac{v''(r)}{s^2} - \frac{v'(r)}{s^2} \tag{5.109}$$

so that (5.108) takes the autonomous form

$$(km^2+4km+4k-1)v(r)^4 - mv(r)^3v' + v(r)^2v'^2 + mv(r)^3v'' = 0. \tag{5.110}$$

The final two terms in Eq. (5.110) do suggest the presence of the second derivative of some fractional power of $v(r)$. We divide Eq. (5.110) by v^2 and m and multiply by (some so far unknown) p to obtain

$$(4k - \frac{p}{m} + \frac{4kp}{m} + kmp)v^2 - pvv' + \frac{p}{m}v'^2 + pvv'' = 0. \tag{5.111}$$

For the last two terms to be a second derivative we need to solve

$$\frac{p}{m} = p(p-1) \tag{5.112}$$

for which the nontrivial solution is

$$p = \frac{1+m}{m}. \tag{5.113}$$

Then Eq.(5.111) becomes, after multiplication by $v^{-1+1/m}$ and the substitution $v \to q^{m/(m+1)}$ followed by some simplification,

$$(5k - \frac{1}{m^2} + \frac{4k}{m^2} - \frac{1}{m} + km)q - q' + q'' = 0. \tag{5.114}$$

We can immediately write down the solution of Eq. (5.114) as

$$w = A_1 e^{\lambda_1 r} + A_2 e^{\lambda_2 r}, \tag{5.115}$$

where

$$\lambda_{1,2} = \frac{m}{2} \pm \delta; \qquad \delta = \sqrt{m^2 - 4(1+m)\left[k(m+2)^2 - 1\right]}, \tag{5.116}$$

except in the case $\delta = 0$. Equation (5.114) admits two different kinds of solution depending upon the value of the parameter k. The point where δ vanishes is the critical point for which the solution transits from being that of one kind to that of the other. In terms of the original variable, x, this is the point where the solution changes from being periodic to aperiodic. So the critical value for k is $k_c = \frac{1}{4(m+1)}$ and this k_c corresponds to

$$\alpha_c = 2\sqrt{2}(n+1)^{1/2} \tag{5.117}$$

which agrees exactly with our conjectured answer and the numerical data.

5.6 Concluding remarks

We have successfully tweaked the methodology developed in Chapter 2, to make it applicable to a class of nonlinear oscillators which weren't originally accessible to perturbative RG analysis. We have used the technique of equivalent linearization to artificially create an equivalent linear system for oscillators without any linear terms. The general form of the oscillators considered here can be written as

$$\ddot{x} + \varepsilon f(x,\dot{x}) = 0 \tag{5.118}$$

where $f(x,\dot{x})$ is a purely nonlinear function. The way we have implemented the method is by writing $f(x,\dot{x})$ as the product of x and another function $g(x,\dot{x})$ such that its average, $< g(x,\dot{x}) >$, over a cycle is non-vanishing. For this to happen $g(x,\dot{x})$ has to be an even-function in x and $\dot{x}$. When $< g(x,\dot{x}) >$ is non-vanishing it is possible to rewrite Eq. (5.118) as

$$\ddot{x} + \Omega^2 x = \varepsilon\left[(f(x),\dot{x}) - \alpha x < g(x,\dot{x}) >\right] \tag{5.119}$$

where $\Omega^2 = \varepsilon\alpha < g(x,\dot{x}) >$ and α is arbitrary constant. The introduction of the term $\lambda\alpha\langle g(x,\dot{x})\rangle x$, has the advantage of making the parameter 'α' dimensionless. Further it makes the term $\lambda\left(x^3 - \alpha\lambda\langle x^2\rangle x\right)$ "small" in the sense that a certain portion of x^3 is subtracted out. Since the choice of α is in our hands this trick allows us to treat the RHS of Eq. (5.119) as a perturbation. In most cases equivalent linearization forms the basis for self-consistent treatment, however, here we use it for a RG based perturbation theory. We have considered just a few cases where this method can be implemented. There are numerous oscillators where this approach will work. For instance the oscillator given by

$$\ddot{x} + \lambda x^2 + \varepsilon x^3 = 0 \tag{5.120}$$

can be treated along the same lines. The choice of how to create a equivalent linear system will though vary from case to case. For instance in the above case one can't use 'x^2'$(x < x >= 0)$ to create the nonlinearity rather one has to use 'x^3'$(x < x^2 >\neq 0)$ and the perturbation calculation carried out until the 2nd order in amplitude gives a good approximate result for frequency correction with contributions from both nonlinearities. In this case the solution is a centre.

On the other hand consider the well known paradigm for a limit cycle given by the equation

$$\dot{x} = \lambda(x^2+y^2)\left(x - y - \gamma(x^2+y^2)\right) \tag{5.121}$$

$$\dot{y} = \lambda(x^2+y^2)\left(x + y - \gamma(x^2+y^2)\right). \tag{5.122}$$

The way to proceed here is by defining Ω as $\Omega = \lambda\alpha < x^2+y^2 >$. Next using the term '$-y(x^2+y^2)$' in Eq. (5.121) and the term '$x(x^2+y^2)$' in Eq. (5.122) to create the linear system about which to perturb. Doing which one promptly arrive at the conclusion that the above equation gives a limit cycle of radius $1/\sqrt{\gamma}$. The choice of which term to use for the linearization is crucial for the success of this method. It can safely be concluded that this technique of doing perturbative RG in absence of linear terms may find applications in wide range of problems where absence of linear terms make them inaccessible to perturbative techniques

The other highlight of this chapter is our investigations of the Riccati equation which presents a very rich variety of dynamical properties. Depending on a parameter, β, the system can be either periodic or non-periodic. When the parameter is gradually changed from a small value to larger ones there is a critical value where the periodic solution vanishes altogether. We have used RG to find the critical value of the parameter where this transition takes place. Further this critical parameter doesn't have any initial-condition-dependence though solutions are initial-condition-dependent. We have shown through some scaling arguments that this can be attributed to certain universal nature of the solutions. We have also presented a hitherto unreported hierarchy of non-linear equations (with increasingly stronger nonlinearities) all of which present similar behaviour. We have found the critical parameter for this hierarchy of equations both analytically and numerically. It suffices to say we have presented a class of oscillators which display interesting dynamical and transformation properties and future work on this system will certainly be rewarding in understanding nonlinear phenomena.

Chapter 6

Renormalization Group near resonance

6.1 Introduction

Oscillations and resonances are amongst the most oft-encountered phenomena in our daily lives. The discovery of the phenomenon of resonance is definitely one of the most important developments in physics. For instance, the simple situation of a child playing on a swing with a person pushing it is an example of forced oscillations and resonance is what makes the swing go higher and higher; When one tunes a radio one is using the resonance. Due to the role "*resonance*" plays in various oscillatory systems, it is the underlying concept behind numerous technological applications. Various electronic devices such as different kinds of filters, NMR (nuclear magnetic resonance) imaging, ESR (electron spin resonance) based technologies all explore the properties of resonance to mention just a few. When speaking of resonances one usually means *the phenomenon of an anamolously sharp increase in the amplitudes of oscillations in response to a (small) external perturbation*. There are two ways to drive a classical oscillator by a small periodic force, one is by an external force independent of the oscillator state, in which case it is simply termed as *resonance* [64]. A textbook example of such phenomenon is the case of a simple undamped harmonic oscillator driven by an external force. In this simple example, its easily shown that when the natural frequency equals the driving frequency, there exist unbounded linearly increasing terms in the solution leading to what one knows as *resonance*. The other slightly more complicated example of resonant behaviour is the so called *parametric resonance* [64], which occurs when a relevant parameter of an oscillatory system is modulated to achieve resonance.

Whether it is a forced nonlinear oscillator or a parametric oscillator the equation of motions are non-autonomous (has explicit time dependence) in nature. Few of the usual analytical tools one employs for analysis of periodic solutions to such dynamical systems are multiple scale analysis, Lindstedt-

Poincaré method, harmonic balance method, Bogoliubov-Krylov method etc [3, 4, 6, 7]. We have shown that the renormalization group (RG) technique developed by Goldenfeld et al [84, 95] is very useful in the analysis of asymptotic behaviour of nonlinear differential equations. Specifically, RG-method has proved quite handy in the analysis of periodic solutions to nonlinear differential equations in case of *autonomous* systems. The question whether it can be used as effectively for non-autonomous systems. It turns that RG indeed can be used for non-autonomous system rather effectively [86]. We had briefly considered a non-autonomous system in Chapter 2. In this chapter we will consider the question of oscillatory behaviour near various kind of resonances. Hence, its imperative that we establish the methodology for *non-autonomous* systems on a firm footing which we proceed to do first in the next section. After establishing the methodology for autonomous systems we will show that suitably applied, one can use RG for analyzing periodic response for not only general and parametric resonances but also to study the subharmonic responses of forced nonlinear oscillators, quasiperiodicity, autoparametric resonance etc.

6.2 RG for non-autonomous systems

For sake of completion we now show how RG can be used for study of non-autonomous systems. Non-autonomous systems are characterized by the presence of explicit time dependence in the ODE describing a dynamical system. Non-autonomous systems are of great importance, since in general a dynamical system may have external inputs which have explicit time dependences, for e.g. a system acting under external periodic forcing. Conceptually non-autonomous systems are in some ways very different from autonomous systems although there are also ways in which they are alike. Let's first look at a few of these issues. Consider a 2nd order ODE which has the general form

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = f(x,\dot{x},t). \tag{6.1}$$

First thing one must notice here is that the above system can have no 'true' equilibrium or fixed points. To understand this lets assume $f(x,\dot{x},t) = -x + a\sin t$. It is of course possible to find $(x^*,\dot{x}^*,t^*)$ so that $f(x^*,\dot{x}^*,t^*) = 0$, but time is changing continuously and thus dx/dt is never really zero for more than an instant. Further for non-autonomous systems $x-\dot{x}$ is no longer the proper phase space for this system. Non-autonomous equations generate an infinite number of phase paths through a given $(x^*,\dot{x}^*,t^*)$ in the $x-\dot{x}$ phase space; a different path, in principle for every t. This is why the nature of the system at a given point in the state space depends on the time at which that point was reached and this essentially means that the system has an extra dimension. One can in fact write a d-dimensional autonomous

system and as a $d+1$-dimensional one. For example we can write Eq. (6.1) as

$$\frac{dx}{d\tau} = y \,, \tag{6.2a}$$
$$\frac{dy}{d\tau} = -k\frac{dx}{dt} - f(x,\dot{x},t) \,, \tag{6.2b}$$
$$\frac{dt}{d\tau} = 1 \,. \tag{6.2c}$$

The above form of writing the equations doesn't provide any technical advantages but conceptually it highlights the fact that a d-dimensional non-autonomous system is in fact equivalent in a sense to a $d+1$-dimensional autonomous system. However, the evolution of the time variable is trivial and so strictly speaking the above system isn't really equivalent to a $d+1$-dimensional autonomous system. One needs to keep these things in mind while doing analysis of non-autonomous systems. In the next part we take the example of a nonlinear oscillator with external forcing and illustrate how RG may be applied to non-autonomous systems.

6.2.1 A damped driven oscillator

We consider the Duffing oscillator again actually, but this time with a non-zero external forcing, given by

$$\ddot{x} + \omega^2 x + k\dot{x} + \beta x^3 = \Gamma \cos \Omega t \,. \tag{6.3}$$

Since this is a case where there is external sinusoidal forcing and we will illustrate how RG can be used to study the asymptotic behavior of a nonlinear oscillator near resonance. We will show via this example that the RG methodology can be used to study not only the primary resonances but also the sub- and super-harmonic ones as well. We begin by writing a naive perturbative expansion for x as, $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$. We also rewrite the Eq. (6.3) as

$$\ddot{x} + \Omega^2 x = -k\dot{x} - \beta x^3 + \Gamma \cos \Omega t + (\Omega^2 - \omega^2)x \,. \tag{6.4}$$

We work in the regime where the forcing frequency is close to natural frequency. We thus treat k, β, Γ and $\Omega^2 - \omega^2$ as small and perturb about the centre given by: $k = \beta = \Gamma = \Omega^2 - \omega^2 = 0$. At the lowest order we have

$$\ddot{x_0} + \Omega^2 x_0 = 0 \,, \tag{6.5}$$

to which we can immediately write down the solution, $x_0 = A_0\cos(\Omega t + \theta_0)$. At the next order we have,

$$\ddot{x_1} + \Omega^2 x_1 = -k\dot{x}_0 - \beta x_0^3 + \Gamma\cos\Omega t + (\Omega^2 - \omega^2)x_0 \,. \tag{6.6}$$

Plugging in the zeroth order solution into the above equation we have,

$$\begin{aligned} \ddot{x_1} + \Omega^2 x_1 &= kA_0\sin\varphi - \frac{\beta A_0^3}{4}(3\cos\varphi + \cos 3\varphi) + \Gamma\cos\Omega t + (\Omega^2 - \omega^2)A_0\cos\varphi \\ &= (kA_0 + \Gamma\sin\theta)\sin\varphi - \left(\frac{3\beta A_0^3}{8} + \Gamma\cos\theta - (\Omega^2 - \omega^2)\right)\cos\varphi + \cdots \,, \end{aligned} \tag{6.7}$$

where $\varphi = \Omega t + \theta_0$. We are interested here only in the solutions with period 2π, of the forcing term. Here we must also mention that there exist periodic solutions with periods other than 2π with which we shall deal later. For now we proceed in accordance with the already established scheme and to the first order in all these small parameters, we obtain the RG flow equations:

$$\frac{dA}{d\tau} = -\frac{kA}{2} - \frac{\Gamma\sin\theta}{2\Omega}; \quad \frac{d\theta}{d\tau} = -\frac{3\beta A^2}{8} - \frac{\Gamma\cos\theta}{2\Omega A} + \Delta\omega \tag{6.8}$$

where $\Delta\omega \equiv \omega - \Omega$. Since, Ω is maintained externally, it cannot change, implying $d\theta/d\tau = 0$. The existence of a stable periodic response requires $dA/d\tau = 0$ for some (A, θ). Thus the fixed point of the above set of equations characterizes the stable periodic response of the forced oscillator. Equating both the flow equations to zero we obtain the following cubic equation in A^2,

$$A^6 - \frac{8}{3}\frac{\Delta\omega}{\beta\Omega}A^4 + \frac{16}{\beta^2}\left(k^2 + \Delta\omega^2\right)A^2 - \frac{16\Gamma}{9\Omega^2\beta^2} = 0 \,. \tag{6.9}$$

From the above equation one can find the amplitude *response* of the oscillator as one brings the forcing frequency, Ω, close to the natural one, ω. This result is exactly in accordance with the literature of forced oscillators [3]. It has also been shown in [312] that RG can be used to analyze the jump phenomenon for the forced Duffing oscillator.

6.2.2 Systems with delay

We move away from the topic of resonances for a bit and address a somewhat different class of problems — ODEs with time delays. In various naturally occurring dynamical systems time delayed feedbacks occur naturally. For instance, considering buying and selling of stocks in a stock market; This is usually done depending on forecasts about future trends; Stocks are thus bought or sold based on expectations of profit or loss and this essentially introduces a delay in the system. Any model trying

to emulate dynamics of a stock market thus has to include time-delayed feedbacks. In recent times there has been an increasing interest amongst researchers in systems with time delays [313,314]. Time delayed systems have also received much attention because of the role they play in controlling various complex dynamics [315–317].

First let us consider a simple linear oscillator with time delay which satisfies the differential equation:

$$\ddot{x}(t) + \omega^2 x(t) + \varepsilon x(t - t_d) = 0 \tag{6.10}$$

Here we treat ε as small and consider the perturbation around the centre ($\varepsilon = 0$). We work with an expansion of x in powers of ε ($x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$) and at various orders of perturbation we obtain,

$$\ddot{x}_0 + \omega^2 x_0 = 0 \tag{6.11a}$$

$$\ddot{x}_1 + \omega^2 x_1 = -x_0(t - t_d) \tag{6.11b}$$

At the lowest order we have the solution $x_0 = A_0 \cos(\omega t + \theta_0)$. Plugging this solution into the Eq. (6.11b) and solving we obtain the RG flow equations upto $\mathscr{O}(\varepsilon)$, to be:

$$\frac{dA}{d\tau} = \varepsilon \frac{\sin \omega t_d}{2\omega}; \quad \frac{d\theta}{d\tau} = \varepsilon \frac{\cos \omega t_d}{2\omega} \tag{6.12}$$

For a periodic orbit (centre), $dA/d\tau$ should be trivially zero; and this yields $\omega t_d = \pi$. This result can be compared with the exact result, *i.e.* Eq. (6.10) exhibits oscillatory solution — $A \exp(it\sqrt{\omega^2 - \varepsilon}) + c.c.$ when $t_d = \pi/\sqrt{(\omega^2 - \varepsilon)}$. The frequency of the periodic orbit is seen to be $\omega - \varepsilon/2\omega + \mathscr{O}(\varepsilon^2)$ in accordance with the expansion of the exact answer of Eq. (6.10).

Next, we consider a system with a weakly nonlinear time delayed system [318] which has a limit cycle. We show that RG can be successfully implemented in such nonlinear time-delayed systems. The system is given by

$$\frac{d^2x(t)}{dt^2} + \alpha x(t) + \beta x(t - t_d) = \lambda \left(x(t) - x^3(t) \right) \tag{6.13}$$

where α, β λ and are constants, λ is the small parameter. The LHS of Eq.(6.13) constitutes the unperturbed system and the nonlinear terms in RHS will be treated as the perturbation. We proceed as usual with an expansion of the form $x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \cdots$. At zeroth order, we have

$$\frac{d^2x_0(t)}{dt^2} + \alpha x_0(t) + \beta x_0(t - t_d) = 0 \tag{6.14}$$

It is easy to see analytically that the above equation has an oscillatory solution, when the following condition is satisfied,

$$t_d = \frac{\cos^{-1}(\alpha/\beta)}{\sqrt{\beta^2-\alpha^2}}. \tag{6.15}$$

We restrict ourselves to cases where the above condition holds and we find at the lowest order,

$$x_0(t) = A_0 \cos \omega t \tag{6.16}$$

where $\omega = \sqrt{\beta^2-\alpha^2}$; $\beta > \alpha$. Further we find: $\beta \sin \omega r = \omega$ and $\beta \cos \omega r = -\alpha$. At order $\mathscr{O}(\lambda^1)$, using the solution (6.16) we have

$$\begin{aligned} \frac{d^2x_1(t)}{dt^2} + \alpha x_1(t) + \beta x_1(t-t_d) &= x_0(t) - x_0^3(t) \\ &= \left(A_0 - \frac{3A_0^3}{4}\right)\cos \omega t - \frac{A_0^3}{4}\cos 3\omega t \end{aligned} \tag{6.17}$$

A little bit of algebra yields the solution to the above equation and $x_1(t)$ is given by,

$$x_1(t) = \frac{1+\alpha t_d}{(1+\alpha t_d)^2 + (\omega t_d)^2} t \cos \omega t + \frac{\omega t_d}{(1+\alpha t_d)^2 + (\omega t_d)^2} t \sin \omega t \tag{6.18}$$

To $\mathscr{O}(\lambda^1)$, thus, we have

$$x(t) = A_0 \cos \omega t + \lambda \frac{1+\alpha t_d}{(1+\alpha t_d)^2 + (\omega t_d)^2} t \cos \omega t + \lambda \frac{\omega t_d}{(1+\alpha t_d)^2 + (\omega t_d)^2} t \sin \omega t \tag{6.19}$$

From this point onwards proceeding as described earlier we arrive at the RG flow equation valid upto $\mathscr{O}(\lambda^1)$, given by

$$\frac{dA}{d\tau} = \frac{\lambda A(1+\alpha t_d)}{(1+\alpha t_d)^2 + (\omega t_d)^2}\left[1 - \frac{3}{4}A^2\right] \tag{6.20a}$$

$$\frac{d\theta}{d\tau} = \frac{\lambda \omega t_d}{(1+\alpha t_d)^2 + (\omega t_d)^2}\left[1 - \frac{3}{4}A^2\right] \tag{6.20b}$$

In accordance to our classification scheme we can immediately conclude that the system given by Eq. (6.13) exhibits limit cycle oscillations. Amplitude of the limit cycle is given by the stable fixed point $A^2 = 4/3$. The validity of our results can be ascertained by numerically integrating the system.

6.3 Subharmonic response

We have established that RG can be used for non-autonomous systems as effectively as we have done for autonomous systems. In case of forced Duffing oscillator we have shown that one can compute the response function for normal harmonic resonance i.e. periodic response with period 2π of the forcing frequency. We show here that one can even study subharmonic response for the Duffing oscillator. For this we take the undamped case ($k = 0$),

$$\ddot{x} + \omega^2 x + \beta x^3 = \Gamma \cos \Omega t., \tag{6.21}$$

We intend to study the $1/3rd$-subharmonic response of the Duffing oscillator using our RG method. In order to do so we expand x in the usual manner as $x = x_0 + \beta x_1 + \beta^2 x_2 + \cdots$ and but the frequency ω is expanded around $\Omega/3$ because we are looking for $2\pi/3$-periodic solutions. For convenience, we make the substitution $\Omega t = \tau$ and rewrite (6.21) as

$$\begin{aligned} \Omega^2 x'' + \omega_0^2 x + \beta x^3 &= \Gamma \cos \tau \\ \Rightarrow x'' + \frac{\omega_0^2}{\Omega^2} x + \frac{\beta}{\Omega^2} x^3 &= \frac{\Gamma}{\Omega^2} \cos \tau. \end{aligned} \tag{6.22}$$

We can immediately rewrite the above equation in a simpler form

$$\ddot{x} + \omega_r^2 x + \lambda x^3 = F \cos \tau, \tag{6.23}$$

where $\omega_r^2 = \omega_0^2/\Omega^2$, $\lambda = \beta/\Omega^2$, $F = \Gamma/\Omega^2$ and the overdots mean differentiation with respect to τ. We expand ω_r around $1/3$ and x as follows:

$$\omega_r = \frac{1}{3} + \phi\omega + \cdots \tag{6.24a}$$

$$x = x_0 + \lambda x_{1\lambda} + \phi x_{1\phi} + \cdots \tag{6.24b}$$

At the lowest order we have

$$\mathcal{O}(\lambda^0 \phi^0): \qquad \ddot{x}_0 + \frac{1}{9} x_0 = F \cos \tau. \tag{6.25}$$

Solving the Eq. (6.25) we have

$$x_0 = a \cos\left(\frac{\tau}{3} + \theta_0\right) - \frac{9F}{8} \cos \tau. \tag{6.26}$$

At the next order of expansions we have the equations,

$$\begin{aligned}
\mathcal{O}(\lambda^0\phi^1): \qquad \ddot{x}_{1\phi}+\frac{1}{9}x_{1\phi} &= -\omega x_0 \\
&= -\omega a\cos\left(\frac{\tau}{3}+\theta_0\right)+\frac{9F\omega}{8}\cos\tau \qquad (6.27)\\
\mathcal{O}(\lambda^1\phi^0): \qquad \ddot{x}_{1\lambda}+\frac{1}{9}x_{1\lambda} &= -x_0^3 \\
&= \left[\frac{3a^3}{4}+\frac{243aF^2}{128}\right]\cos\left(\frac{\tau}{3}+\theta_0\right)-\frac{27a^2F}{32}\cos 3\theta_0\cos\left(\frac{\tau}{3}+\theta_0\right) \\
&\quad + \text{ higher harmonics} \qquad (6.28)
\end{aligned}$$

Using the expressions for x_0, $x_{1\phi}$ and $x_{1\lambda}$ we can proceed with the renormalization process to finally get the flow equations:

$$\frac{da}{d\mu} = \frac{3\lambda}{2}\frac{27a^2F}{32}\sin 3\theta, \qquad (6.29a)$$

$$\frac{d\theta}{d\mu} = \frac{3}{2}\left[\phi\delta\omega-\lambda\left\{\frac{3a^2}{4}-\frac{27aF}{32}\cos 3\theta+\frac{243F^2}{128}\right\}\right], \qquad (6.29b)$$

where $\delta\omega=(9\omega_0^2-\Omega^2)/\phi$ is the frequency shift. Now for stable periodic solution both $da/d\mu$ and $d\theta/d\mu$ must vanish. $da/d\mu=0$ implies that $\sin 3\theta=0$, i.e. $\theta=n\pi/3$. Using this and from the criterion that Eq. (6.29b) must also be zero, we obtain a quadratic equation in a,

$$a^2-\frac{9F}{8}a+\frac{81F^2}{32}-\frac{4}{3}\delta\omega=0 \qquad (6.30)$$

Yet again the above equation is in accordance with the standard result for $1/3rd$-subharmonic response [3]. There are either two real solutions or none to the above equation depending on whether

$$F^2 > \text{or} < \frac{1024}{189\phi}\left(\omega_0^2-\frac{1}{9}\Omega^2\right), \qquad (6.31)$$

respectively. From the Eq. (6.31) and the Eqs. (6.29) all information about stable subharmonic responses of the Duffing oscillator can be gleaned. The Eqs. (6.29) are equivalent to the flow equations for a, b in the Van der Pol plane that one might obtain by assuming a solution of the form $a\cos(\frac{1}{3}\omega_r t)+b\sin(\frac{1}{3}\omega_r t)$. The number of fixed points of the Eqs. (6.29) represent the number of solution and the stability of a particular fixed point of the Eqs. (6.29) gives the stability of the corresponding periodic response. One might also look for other *subharmonic* responses or *superharmonic* responses for any given oscillator using this approach.

6.4 Parametric oscillators

For many naturally occurring periodic systems the only interaction with the surroundings is often mediated via a modification of one of the parameters of the system. The modification might come as a simple periodic variation in the value of a the said parameter. This kind of excitation is called *parametric excitation* and can also lead to resonances. The phenomenon of parametric resonances has been a major area of interest in physics since its first discovery in 1831 by Faraday. Parametrically excited oscillations arise in many branches of physics and engineering for e.g. in forced motion of a swing [319], stability of ships [320], Faraday surface wave patterns on water [321] etc. In recent times parametric oscillators have been important in context of micro- and nano-mechanical-electrical systems (NEMS and MEMS) devices [322, 323, 335, 336].

The first mathematical description of parametric excitations was due to Mathieu way back in 1868 when he derived his famous second-order periodic differential equation to study the vibrations of elliptic membranes. The topic of parametric excitations and the associated literature has been quite will described by Nayfeh and Mook [124]. A simple harmonic oscillator can be described by a differential equation of the form

$$x + \omega^2 x = 0, \tag{6.32}$$

which is a 2nd order ODE with constant coefficients. The moment one assumes that the coefficients need not be constant and make them time dependent, (*parametric*), the dynamics of such an oscillator no longer remains simple and unlike normal resonance the response of such systems does not remain restricted to the forcing frequency. There can be in fact excitations of the natural frequencies and certain integral fractions or multiples of the primary resonant frequency.

6.4.1 Hill's equation

We begin with the most general equation of motion for a parametrically excited oscillator, the so called Hill's equation given by

$$\ddot{u} + 2\mu\dot{u} + \left[\delta + \sum_{m=1}^{\infty} \varepsilon^m f_m(t)\right] u = 0. \tag{6.33}$$

where the function f_m is such that $f_m(t+\pi) = f_m(t)$. The parameters ε and μ are both small and $\mu > 0$. We use this case to illustrate in detail the RG methodology for studying asymptotic behaviour

of a parametric oscillator. Without any loss of generality one can always assume that

$$\int_0^{\pi} f_m(t)dt = 0, \tag{6.34}$$

such that f_m can be expressed as a Fourier series in the following manner:

$$f_m = \sum_{n=1}^{\infty} (a_{mn}\cos 2nt + b_{mn}\sin 2nt) \tag{6.35}$$

We intend to determine the transition curves for the system governed by Eq. (6.33). For this purpose we assume that u and δ can be expanded in the following manner:

$$u(t,\varepsilon) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \tag{6.36}$$

$$\delta(\varepsilon) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots \tag{6.37}$$

We further assume that $\mu = \varepsilon\mu_0$. Substituting Eqs. (6.36) and (6.37) into Eq.(6.33) we obtain the following equation at each order of expansion:

$$\ddot{u}_0 + \delta_0 u_0 = 0\,, \tag{6.38a}$$

$$\ddot{u}_1 + \delta_0 u_1 = -(\delta_1 + f_1)u_0 - 2\mu_0\dot{u}_0\,, \tag{6.38b}$$

$$\ddot{u}_2 + \delta_0 u_2 = -(\delta_1 + f_1)u_1 - (\delta_2 + f_2)u_0 - 2\mu_0\dot{u}_1\,. \tag{6.38c}$$

According to Floquet theory, u has either a period of π or 2π along a transition curve. Thus at the lowest order we write the solution in the form

$$u_0 = A_0\cos(nt + \theta_0), \tag{6.39}$$

where n is a non-zero integer related to δ_0 by the following relation:

$$\delta_0 = n^2\,. \tag{6.40}$$

A_0 and θ_0 are arbitrary constants. Now plugging in the zeroth order solution into Eq. (6.38b) we have

$$\begin{aligned}
\ddot{u}_1 + \delta_0 u_1 &= -\delta_1 A_0 \cos(nt+\theta_0) + 2\mu_0 n A_0 \sin(nt+\theta_0) \\
&\quad -A_0 \cos(nt+\theta_0) \sum_{m=1}^{\infty} [a_{1m}\cos 2mt + b_{1m}\sin 2mt] \\
&= -A_0 \left[\delta_1 + \frac{a_{1n}}{2}\cos 2\theta_0 - \frac{b_{1n}}{2}\sin 2\theta_0\right]\cos(nt+\theta_0) \\
&\quad +A_0 \left[2\mu_0 n + \frac{a_{1n}}{2}\sin 2\theta_0 + \frac{b_{1n}}{2}\cos 2\theta_0\right]\sin(nt+\theta_0) \\
&\quad +\frac{A_0}{2}\sum_{m=1,m\neq n}^{\infty}\left[a_{1m}\cos\big((2m+n)t+\theta_0\big) + b_{1m}\sin\big((2m+n)t+\theta_0\big)\right] \\
&\quad +\frac{A_0}{2}\sum_{m=1,m\neq n}^{\infty}\left[a_{1m}\cos\big((2m-n)t-\theta_0\big) + b_{1m}\sin\big((2m-n)t-\theta_0\big)\right]. \qquad (6.41)
\end{aligned}$$

So upto this order we can write down u as

$$\begin{aligned}
u = A_0\cos(nt+\theta_0) - \frac{\varepsilon A_0}{2n}\left[\delta_1 + \frac{a_{1n}}{2}\cos 2\theta_0 - \frac{b_{1n}}{2}\sin 2\theta_0\right](t-t_0)\sin(nt+\theta_0) \\
-\frac{\varepsilon A_0}{2n}\left[2\mu_0 n + \frac{a_{1n}}{2}\sin 2\theta_0 + \frac{b_{1n}}{2}\cos 2\theta_0\right](t-t_0)\cos(nt+\theta_0) + \text{higher harmonics} \qquad (6.42)
\end{aligned}$$

As usual, we now introduce two renormalization constants such that,

$$\begin{aligned}
A_0 &= \mathscr{Z}_1(t_0,\tau)A(\tau), &(6.43a)\\
\theta_0 &= \mathscr{Z}_2(t_0,\tau) + \theta(\tau), &(6.43b)
\end{aligned}$$

and where $\mathscr{Z}_1$ and $\mathscr{Z}_2$ have the following expansions,

$$\begin{aligned}
\mathscr{Z}_1 &= 1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots, &(6.44a)\\
\mathscr{Z}_2 &= \varepsilon\beta_1 + \varepsilon^2\beta_2 + \cdot. &(6.44b)
\end{aligned}$$

We now proceed to split the time interval $t-t_0$ as $t-\tau+\tau-t_0$ and insert the renormalization constants into Eq. (6.42) and rewrite it in the following way:

$$\begin{aligned} u(t,\tau) &= A_0(1+\varepsilon\alpha_1)\cos(nt+\theta_0+\varepsilon\beta_1) \\ &\quad -\frac{\varepsilon A_0}{2n}\left[\delta_1+\frac{a_{1n}}{2}\cos 2\theta_0-\frac{b_{1n}}{2}\sin 2\theta_0\right](t-\tau+\tau-t_0)\sin(nt+\theta_0) \\ &\quad -\frac{\varepsilon A_0}{2n}\left[2\mu_0 n+\frac{a_{1n}}{2}\sin 2\theta_0+\frac{b_{1n}}{2}\cos 2\theta_0\right](t-\tau+\tau-t_0)\cos(nt+\theta_0)+\cdots \end{aligned} \tag{6.45}$$

Choosing appropriately the values of α_1 and β_1 we can get rid of divergence upto this order. The proper choices being

$$\alpha_1 = -\frac{1}{2n}\left[2\mu_0 n+\frac{a_{1n}}{2}\sin 2\theta_0+\frac{b_{1n}}{2}\cos 2\theta_0\right](\tau-t_0) \tag{6.46a}$$

$$\beta_1 = \frac{1}{2n}\left[\delta_1+\frac{a_{1n}}{2}\cos 2\theta_0-\frac{b_{1n}}{2}\sin 2\theta_0\right](\tau-t_0) \tag{6.46b}$$

And now we are left with,

$$\begin{aligned} u(t) &= A_0\cos(nt+\theta_0)-\frac{\varepsilon A_0}{2n}\left[\delta_1+\frac{a_{1n}}{2}\cos 2\theta_0-\frac{b_{1n}}{2}\sin 2\theta_0\right](t-\tau)\sin(nt+\theta_0) \\ &\quad -\frac{\varepsilon A_0}{2n}\left[2\mu_0 n+\frac{a_{1n}}{2}\sin 2\theta_0+\frac{b_{1n}}{2}\cos 2\theta_0\right](t-\tau)\cos(nt+\theta_0)+\cdots. \end{aligned} \tag{6.47}$$

In order to obtain the RG flow equation upto this order we impose the condition,

$$\left(\frac{\partial u}{\partial \tau}\right)_t = 0, \tag{6.48}$$

which essentially means that the final answer must be independent of the arbitrary time τ. Using this above condition we obtain the RG flow equations:

$$\frac{dA}{d\tau} = -\frac{A}{2n}\left[2\mu_0 n+\frac{a_{1n}}{2}\sin 2\theta+\frac{b_{1n}}{2}\cos 2\theta\right] \tag{6.49a}$$

$$\frac{d\theta}{d\tau} = \frac{1}{2n}\left[\delta_1+\frac{a_{1n}}{2}\cos 2\theta-\frac{b_{1n}}{2}\sin 2\theta\right] \tag{6.49b}$$

Now for solutions with period π or 2π, i.e., on the transition curves both the flow equations must vanish. Solving the Eqs. (6.49) simultaneously we derive the following relations:

$$\delta_1^2+4\mu_0^2 n^2 = \frac{1}{4}\left(a_{1n}^2+b_{1n}^2\right). \tag{6.50}$$

Consequently, the transition curves separating the stable regions from the unstable ones are given by the equation:

$$\delta = n^2 \pm \frac{1}{2}\left[(a_{1n}^2 + b_{1n}^2)\varepsilon^2 - 16\mu^2 n^2\right]^{1/2} \tag{6.51}$$

Therefore, upto 1st order the viscous term facilitates complete stabilization of the solution as long as $\mu > \mu_c = \frac{\varepsilon}{4n}(a_{1n}^2 + b_{1n}^2)$. Otherwise there is always a region of instability which intersperses the regions of stable solution. However, as the viscous damping increases it decreases the unstable region by lifting it from the δ-axis and narrowing its boundaries in $\varepsilon - \delta$ plane. When viscous damping is absent, we simply have $\delta_1 = \pm\frac{1}{4}\left(a_{1n}^2 + b_{1n}^2\right)$.

6.4.2 Mathieu equation

Now let us consider the so called Mathieu equation, which is a special case of the Hill's equation (with just a single lowest harmonic *cosine* term),

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon\cos t)x = 0. \tag{6.52}$$

The above equation can be investigated using Floquet's Theory [3] which predicts that in the (δ, ε) plane, there are regions where the solution to Eq. (6.52) is stable and remains bounded whereas for the rest of the plane the solution is unstable. For sufficiently small ε and for $\delta > 0$ and $\delta \neq n^2/4$ (n being an integer) its known to have stable solutions. Without loss of generality we intend to study the stability of the solutions around $\delta = 1/4$ and $\varepsilon = 0$ in order to find the stability boundary in δ-ε space. We treat δ as a function of ε and expand both δ and x in powers of ε as follows:

$$\delta(\varepsilon) = \frac{1}{4} + \delta_1\varepsilon + \delta_2\varepsilon^2 + \cdots, \tag{6.53}$$

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots. \tag{6.54}$$

Its our intention to perturbatively determine the values of δ_1, $\delta_2, \cdots$. Normal perturbation theory fails to solve this problem and usually one has to resort to multiple scales analysis. However, a simple introduction of multiple scales of the form $\tau_1 = \varepsilon t$, $\tau_2 = \varepsilon^2 t, \cdots$ does not furnish the correct δ_2 even after setting the 1st order coefficient $\delta_1 = 1$. Careful analysis reveals that a time scale of $\varepsilon^{3/2}t$ is involved and in order to get the right answer, the perturbative expansion has to be done in powers of $\varepsilon^{1/2}$ instead of ε. Using such a multiple scale analysis and the correct coefficients turn out to be $\delta_1 = 1$ and $\delta_2 = -1/2$. The great advantage of RG comes from the fact that even for this case it does not require apriori knowledge of hidden time scales involved. A naive perturbative expansion of the form (6.53) is enough and the unexpected time scales like $\varepsilon^{3/2}t$ emerge automatically on implementation of

RG. Inserting the expansions (6.53) and (6.54) into Eq. (6.52) we get at different orders of ε,

$$\begin{aligned}
\frac{d^2x_0}{dt^2}+\frac{1}{4}x_0 &= 0, && \text{(6.55a)}\\
\frac{d^2x_1}{dt^2}+\frac{1}{4}x_1 &= -(\delta_1+\cos t)\,x_0, && \text{(6.55b)}\\
\frac{d^2x_2}{dt^2}+\frac{1}{4}x_2 &= -\delta_2 y_0-(\delta_1+\cos t)\,x_1. && \text{(6.55c)}
\end{aligned}$$

The solution at the zeroth order is immediately given by, $x_0 = A_0\cos\left(\frac{t}{2}+\theta_0\right)$. Plugging this solution into Eq. (6.55b), we have

$$\begin{aligned}
\frac{d^2x_1}{dt^2}+\frac{1}{4}x_1 &= -(\delta_1+\cos t)A_0\cos\left(\frac{t}{2}+\theta_0\right)\\
&= -\delta_1 A_0\cos\left(\frac{t}{2}+\theta_0\right)-\frac{A_0}{2}\cos\left(\frac{3t}{2}+\theta_0\right)-\frac{A_0}{2}\cos\left(\frac{t}{2}-\theta_0\right) && \text{(6.56)}
\end{aligned}$$

Solving Eq. (6.56), we have

$$\begin{aligned}
x_1 &= -\frac{\delta_1 A_0}{2\Omega}(t-t_0)\sin(\Omega t+\theta_0)-\frac{A_0}{4\Omega}(t-t_0)\sin(\Omega t-\theta_0)\\
&\quad+\frac{A_0}{16\Omega^2}\left[\cos(3\Omega t+\theta_0)-\cos(\Omega t+\theta_0)\right], && \text{(6.57)}
\end{aligned}$$

where $\Omega^2 = 1/4$. The solution $x(t)$ reads

$$\begin{aligned}
x(t) &= A_0\cos(\Omega t+\theta_0)+\frac{\varepsilon A_0}{16\Omega^2}\left[\cos(3\Omega t+\theta_0)-\cos(\Omega t+\theta_0)\right]\\
&\quad-\frac{\varepsilon\delta_1 A_0}{2\Omega}(t-t_0)\sin(\Omega t+\theta_0)-\frac{\varepsilon A_0}{4\Omega}(t-t_0)\sin(\Omega t-\theta_0)+\mathscr{O}(\varepsilon^2). && \text{(6.58)}
\end{aligned}$$

As usual we define two renormalization constants,

$$\begin{aligned}
A_0 &= \mathscr{Z}_1(t_0,\tau)A(\tau), && \text{(6.59a)}\\
\theta_0 &= \mathscr{Z}_2(t_0,\tau)+\theta(\tau), && \text{(6.59b)}
\end{aligned}$$

where $\mathscr{Z}_1$ and $\mathscr{Z}_2$ have the following expansions,

$$\begin{aligned}
\mathscr{Z}_1 &= 1+\varepsilon\alpha_1+\varepsilon^2\alpha_2+\cdots, && \text{(6.60a)}\\
\mathscr{Z}_2 &= \varepsilon\beta_1+\varepsilon^2\beta_2+\cdot. && \text{(6.60b)}
\end{aligned}$$

We now proceed to split the time interval $t-t_0$ as $t-\tau+\tau-t_0$ and insert the renormalization constants into Eq. (6.58), so that we have

$$\begin{aligned} x(t) &= A(1+\varepsilon\alpha_1)\cos(\Omega t+\theta+\varepsilon\beta_1)+\cdots-\frac{\varepsilon\delta_1 A_0}{2\Omega}(t-\tau+\tau-t_0)\sin(\Omega t+\theta_0) \\ &\quad -\frac{\varepsilon A_0}{4\Omega}(t-\tau+\tau-t_0)\sin(\Omega t-\theta_0). \end{aligned} \tag{6.61}$$

The choice $\alpha_1=\frac{\sin 2\theta}{4\Omega}(\tau-t_0)$ and $\beta_1=-(\tau-t_0)\left(\frac{\delta_1}{2\Omega}+\frac{\cos 2\theta}{2\Omega}\right)$ removes the divergence causing terms upto this order and we are left with

$$\begin{aligned} x(t) &= A\cos(\Omega t+\theta)+\frac{\varepsilon A}{16\Omega^2}\left[\cos(3\Omega t+\theta)-\cos(\Omega t+\theta)\right] \\ &\quad -\frac{\varepsilon\delta_1 A}{2\Omega}(t-\tau)\sin(\Omega t+\theta)-\frac{\varepsilon A}{4\Omega}(t-\tau)\sin(\Omega t-\theta) \end{aligned} \tag{6.62}$$

Now we impose the condition $(\partial x/\partial\tau)_t=0$ to obtain the RG flow equations upto this order:

$$\frac{dA}{d\tau} = \frac{\varepsilon A}{2}\sin 2\theta, \tag{6.63a}$$

$$\frac{d\theta}{d\tau} = \varepsilon(\delta_1+\frac{1}{2}\cos 2\theta). \tag{6.63b}$$

One must note here that the above flow equation could have easily been derived from the flow equations (Eqs.(6.49)) obtained for Hill's equation. Coming back to the problem at hand, according to our prescription, the stability boundary is obtained where both $dA/d\tau$ and $d\theta/d\tau$ vanish and on the stability boundary the solution is periodic with frequency Ω. While on one side ($\delta^2<1/2$) of this boundary the solution remains bounded and stable on the other side ($\delta^2>1/2$) it grows exponentially. Solving we have $\cos 2\theta=-2\delta_1$, where $\delta_1=\pm 1/2$ and thus $\theta=0,\pi/2,\cdots$. At this point we can easily make the connection between this result and the result obtained for Hill's equation. Comparing with the Hill's equation, we can immediately see that $\delta_1=\pm 1/2$ as in this case $\mu_0=b_{1n}=0$ and $a_{1n}=1$. So from Eq. (6.50) we have the answer,

$$\delta_1^2=\frac{1}{4}. \tag{6.64}$$

In order to find δ_2 we need to do the calculation for next order of perturbation. Setting $\delta_1=1/2$ (*i.e.* $\theta=\pi/2$), we proceed to the next order to determine δ_2. Plugging in x_0 and x_1 into Eq. (6.55c) and

solving we find y_2 as follows

$$\begin{aligned} x_2 &= -A_0\left(\delta_2 - \frac{1}{8}\cos 2\theta_0\right)(t-t_0)\sin\left(\frac{t}{2}+\theta_0\right) - \frac{A_0}{4}(t-t_0)\sin 2\theta_0 \cos\left(\frac{t}{2}+\theta_0\right) \\ &\quad + \frac{A_0}{48}\left[\cos\left(\frac{5t}{2}+\theta_0\right) - \cos\left(\frac{t}{2}+\theta_0\right)\right] \end{aligned} \tag{6.65}$$

Going through the renormalization procedure we find that properly choosing the α_2 and β_2 we are able to remove all divergences at this order too. Yet again this shows that the original assumption of perturbative renormalizability holds. A little algebra, in which we need only determine the coefficients of the lowest harmonics, yields the flow equations,

$$\frac{dA}{d\tau} = \frac{\varepsilon A}{2}\sin 2\theta + \mathscr{O}(\varepsilon^3), \tag{6.66a}$$

$$\frac{d\theta}{d\tau} = \frac{\varepsilon}{2}\left(\frac{1}{2}+\cos 2\theta\right) + \varepsilon^2\left(\delta_2 - \frac{1}{8}\cos 2\theta\right) + \mathscr{O}(\varepsilon^3). \tag{6.66b}$$

As per our criterion, the flow equations must vanish on the boundary and thus, we immediately get the answer $\delta_2 = -1/8$ (as θ values was fixed at $\pi/2$) from the above flow equations. Thus, one of the transition curves passing through $1/4$ is given by,

$$\delta(\varepsilon) = \frac{1}{4} + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \mathscr{O}(\varepsilon^3). \tag{6.67}$$

From the solution to the flow equations (6.66) it can also be concluded that the solutions are stable for $\delta_2 > -1/8$ and unstable for $\delta_2 < -1/8$. The other transition curve can be obtained, if instead of fixing $\delta_!$ at $1/2$, we had proceeded with $\delta_1 = -1/2$. In that case, the corresponding δ_2 is again found to be $\delta_2 = -1/8$, giving us the expression for the other transition curve,

$$\delta(\varepsilon) = \frac{1}{4} - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \mathscr{O}(\varepsilon^3). \tag{6.68}$$

In this manner one can find the δ_i's which determine the transition curves between stable and unstable regions not only for $\delta_0 = 1/4$ but also for $\delta_0 = 1, 9/4, 4, \cdots$.

6.5 Effect of nonlinearity on parametric excitations

For a truly linear parametrically excited system, the amplitude of oscillations grows indefinitely with time, unless there is some damping in the system, in which case the damping usually brings in a stabilizing effect. In real world however, most oscillators have some degree of nonlinearity. Near

resonance when the amplitude of oscillations becomes appreciable enough with time, the nonlinearity begins playing an increasingly greater role and modifies the response. We will use the following example to illustrate the fact that RG fares equally well when weak non-linearities are involved alongwith parametric excitations

6.5.1 Van der Pol-Duffing-Mathieu equation

We now turn to an example of parametric oscillator with nonlinear terms in it. The particular model [337–339] is given by,

$$\ddot{x} - (\alpha_1 - \beta_1 x^2)\dot{x} + (\delta - \mu_1 \cos\Omega t)x + \gamma x^3 = 0. \tag{6.69}$$

Notice that the model consists of nonlinear Van der Pol, Duffing and linear Mathieu terms simultaneously. This particular model has been used by Lifshitz and Cross [323] to provide a theoretical explanation for physical behaviour of parametrically-excited micro- and nano-mechanical (MEMS and NEMS) systems. In doing so they had to, of course, consider sets of coupled oscillators. But for now let us concentrate on the uncoupled model and examine how RG method fares in comparison to secular perturbation theory used by Cross et al.

We introduce a small parameter $\varepsilon \ll 1$ into Eq.(6.69), such that $\alpha_1 = \varepsilon\alpha$, $\beta_1 = \varepsilon\beta$, $\mu_1 = \varepsilon\mu$ and $\gamma_1 = \varepsilon\gamma$ and rewrite Eq.(6.69) as,

$$\ddot{x} + \delta x = \varepsilon\left[\left(\alpha - \beta x^2\right)\dot{x} + \sigma x + \mu x\cos\Omega t - \gamma x^3\right] . \tag{6.70}$$

We investigate the main parametric resonance of this oscillator for $\delta = 1$ ($\Omega \approx 2$). We define a detuning parameter σ such that $\Omega^2/4 = 1 + \varepsilon\sigma$. Using this we have

$$\ddot{x} + \delta x = \varepsilon\left[\left(\alpha - \beta x^2\right)\dot{x} + \mu x\cos\Omega t - \gamma x^3\right] . \tag{6.71}$$

We expand x in powers of ε as $x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, and obtain at successive orders of perturbation,

$$\ddot{x}_0 + \frac{1}{4}\Omega^2 x_0 = 0 , \tag{6.72a}$$

$$\ddot{x}_1 + \frac{1}{4}\Omega^2 x_1 = \left[\left(\alpha - \beta x_0^2\right)\dot{x}_0 + \sigma x_0 + \mu x\cos\Omega t - \gamma x_0^3\right] , \tag{6.72b}$$

and so on. We can immediately write down the solution to Eq.(6.72a) as $x_0 = A_0\cos\left((\Omega/2)t + \theta_0\right)$

plugging which into the next order and with a little manipulation we have,

$$\begin{aligned}\ddot{x}_1+\frac{\Omega^2}{4}x_1 &= \frac{A_0}{2}\left[\Omega\left(\frac{\beta A_0^2}{4}-\alpha\right)+\mu\sin 2\theta_0\right]\sin\varphi \\ &\quad +A_0\left[\sigma-\frac{3\gamma A_0^2}{4}+\frac{\mu}{2}\cos 2\theta_0\right]\cos\varphi+\cdots\,, \end{aligned} \tag{6.73}$$

where $\varphi=(\Omega/2)t+\theta_0$. At this point we can write down the RG flow equations upto $\mathscr{O}(\varepsilon)$ by simply reading of the coefficient of the secular terms,

$$\frac{dA}{d\tau} = -\frac{\varepsilon A}{2\Omega}\left[\Omega\left(\frac{\beta A^2}{4}-\alpha\right)+\mu\sin 2\theta\right]\,, \tag{6.74a}$$

$$\frac{d\theta}{d\tau} = -\frac{\varepsilon A}{\Omega}\left[\sigma-\frac{3\gamma A^2}{4}+\frac{\mu}{2}\cos 2\theta\right]\,. \tag{6.74b}$$

For stable periodic response we must look at the equilibrium points of the above set of equations i.e. points where both flow equations vanish

$$\Omega\left(\frac{\beta A^2}{4}-\alpha\right)+\mu\sin 2\theta = 0 \tag{6.75a}$$

$$\sigma-\frac{3\gamma A^2}{4}+\frac{\mu}{2}\cos 2\theta = 0 \tag{6.75b}$$

and then eliminating θ between the above expressions one obtains the frequency response function,

$$\mu^2=\Omega^2\left(\frac{\beta A^2}{4}-\alpha\right)^2+\left(2\sigma-\frac{3\gamma A^2}{2}\right)^2 \tag{6.76}$$

A comparison of our result and Cross et al's result reveals that the results are actually identical. However, note that in order to calculate the correction to linear response by secular perturbation theory one needs to begin by writing a perturbation expansion in terms of $\varepsilon^{1/2}$. This apriori knowledge is unnecessary while doing the RG calculation. The case of N such coupled oscillators can also be treated by RG method and in fact extending the RG method for more than one-degree-of-freedom systems is quite straightforward. We discuss one such case with multiple degrees of freedom later in this chapter.

6.5.2 Ermakov-Pinney

We next turn our attention to another rather interesting example: the Ermakov-Pinney (EP) equation. The EP equation has arisen in number of physical systems starting from scalar field cosmolo-

gies [340, 341] to modelling of oscillations in Bose-Einstein condensate [342] states. The EP equation is given by

$$\ddot{X} + \lambda(t)X = \frac{1}{X^3} \,. \tag{6.77}$$

The solution to this equation is quite well known [343] and is given by

$$X(t) = \sqrt{u^2(t) + \frac{v^2(t)}{W^2}} \,, \tag{6.78}$$

where $u(t)$ and $v(t)$ are two independent solutions to the equation

$$\ddot{U} + \lambda(t)U = 0 \,, \tag{6.79}$$

satisfying the relations $u(t_0) = X(t_0)$, $\dot{u}(t_0) = X'(t_0)$, $v(t_0) = 0$, $\dot{v}(t_0) \neq 0$ and $W = u\dot{v} - \dot{u}v$. The Eq. (6.77) is like a nonlinear Hill's equation and the solutions are thus closely related to the Hill's equation. The problem is very well studied [344] and for specific choices of $\lambda(t)$, such as $\lambda(t) = 1 + \varepsilon\gamma\cos\omega t$, it is explicitly solvable [343, 345]. The thing to notice here is that it appears that there is no direct contribution of the nonlinear terms to the solution. A RG calculation of the Eq. (6.77) with $\lambda(t) =$ *constant*, reveals a rather curious fact: at every order of perturbation both the amplitude and phase flow equations are identically zero. Incidentally the equation

$$\ddot{X} + X = \frac{1}{X^3} \,, \tag{6.80}$$

is an isochronous oscillator, as we have already discussed in Chapter 4. For isochronous oscillators both the RG phase and amplitude flow equations must be identically zero at every order of perturbation [88]. Keeping in mind this fact, we proceed to do the RG calculations for Eq. (6.77). We note that for $\lambda(t) = 1$ the system has a fixed point at $(X, \dot{X}) = (1, 0)$, which is a centre. We seek to study the behavior of the oscillator for small oscillations around this fixed point (1,0). The next step is thus to shift the coordinates such that origin is at (1,0). So, the new coordinate is given by $x = X + 1$. In terms of new coordinates we have,

$$\begin{aligned} & \ddot{x} + (\delta + \varepsilon\gamma\cos\omega t)(x+1) = \frac{1}{(1+x)^3} \\ \Rightarrow \quad & \ddot{x} + (\delta + \varepsilon\gamma\cos\omega t)x = -\delta - \varepsilon\gamma\cos\omega t + 1 - 3x + 6x^2 - 10x^3 + \cdots \,, \end{aligned} \tag{6.81}$$

where we have assumed that the displacement x from the origin is small enough allowing us to Taylor expand $(1+x)^{-3}$. Notice that the LHS of Eq. (6.81) is Mathieu-like and on the RHS there is a forcing term $\gamma\cos\omega t$ in addition to a series of nonlinear terms. We assume the expansions, $\delta = 1 + \varepsilon\delta_1 + \varepsilon\delta_2 +$

$\cdots$ and $x = x_0 + \varepsilon x_1 + \varepsilon x_2 + \cdots$. Using these expansions in Eq. (6.81) we have for different orders of ε,

$$\begin{aligned}
\ddot{x}_0 + 4x_0 &= 0\,, && \text{(6.82a)}\\
\ddot{x}_1 + 4x_1 &= -\gamma\cos(\omega t)x_0 - \delta_1 x_0 + \gamma\cos(\omega t) + 6x_0^2\,, && \text{(6.82b)}\\
\ddot{x}_2 + 4x_2 &= -\gamma\cos(\omega t)x_1 - \delta_2 x_0 - \delta_1 x_1 + 6x_0x_1 - 10x_0^3\,. && \text{(6.82c)}
\end{aligned}$$

and so on. For the lowest order we can immediately write down the solution $x_0 = A_0\cos(2t+\theta_0)$. Similar to Mathieu equation the instability regions, in this case, start at frequencies $\omega = 2, 1, 1/2, ..., 2/n^2, ...$ [344]. Let us concentrate on the case where $\omega = 1$. We know that the RG flow equation should have contributions from the Mathieu-term, the forcing term and the nonlinear terms. We have already discussed that when $\lambda(t) = constant$, the resulting oscillator is an isochronous one,*i.e.* contributions from the nonlinear terms to both the phase and amplitude flow equations must be identically zero at every order of perturbation. It would suffice therefore to calculate the contributions coming from Mathieu- and forcing-terms only. Plugging in the zeroth order solution into Eq. (6.82b), we have

$$\begin{aligned}
\ddot{x}_1 + 4x_1 &= -\gamma\cos t - \gamma\cos t\cdot x_0 + \delta_1 x_0 + 6x_0^2\\
&= -\gamma\cos t - \frac{\gamma A_0}{2}\left[\cos(3t+\theta_0) + \cos(t-\theta_0)\right]\\
&\quad + \delta_1 A_0\cos(2t+\theta_0) + \text{irrelevant terms.} \qquad \text{(6.83)}
\end{aligned}$$

We have referred the terms due to nonlinear parts as irrelevant in the above equation. At this order there are no *secular terms*, apart from δ_1 containing term, which means there is no correction to δ from this order *i.e.* $\delta_1 = 0$. But as we write down the solution to x_1 it becomes clear that there will be secular terms at the next order of perturbation. We write down the solution to x_1 next, but only the relevant terms (terms which will contribute to the flow equations):

$$x_1 = \frac{\gamma}{3}\cos t + \frac{\gamma A_0}{10}\cos(3t+\theta_0) + \frac{\gamma A_0}{6}\cos(t-\theta_0) + \cdots\,. \qquad \text{(6.84)}$$

Using the above expression and solution for x_0 in Eq. (6.82c), we have

$$\begin{aligned}
\ddot{x}_2 + x_2 &= -\gamma\cos t\cdot x_1 + \delta_2 x_0 + \text{irrelevant terms}\\
&= A_0\left[\delta_2 - \frac{\gamma^2}{20} + \frac{\gamma^2}{6A_0}\cos\theta_0 + \frac{\gamma^2}{12}\cos 2\theta_0\right]\cos(2t+\theta_0)\\
&\quad + A_0\left[\frac{\gamma^2}{6A_0}\sin\theta_0 + \frac{\gamma^2}{12}\sin 2\theta_0\right]\sin(2t+\theta_0) + \cdots\,. \qquad \text{(6.85)}
\end{aligned}$$

From the above equation we can immediately write down the flow equations,

$$\frac{dA}{d\tau} = -\frac{\varepsilon A_0}{4}\left[\frac{\gamma^2}{6A_0}\sin\theta_0 + \frac{\gamma^2}{12}\sin 2\theta_0\right] + \mathscr{O}(\varepsilon^3)\,, \tag{6.86a}$$

$$\frac{d\theta}{d\tau} = -\frac{\varepsilon}{4}\left[\delta_2 - \frac{\gamma^2}{20} + \frac{\gamma^2}{6A_0}\cos\theta_0 + \frac{\gamma^2}{12}\cos 2\theta_0\right] + \mathscr{O}(\varepsilon^3)\,, \tag{6.86b}$$

solving which one can easily obtain the correction, δ_2, at this order.

6.6 Autoresonance

We have already discussed what parametric resonance is. *Autoresonance* can occur in nonlinear systems driven by a small almost periodic external force. Typically when the driving frequency is exactly periodic the slow amplitude growth of a nonlinear oscillator can be countered by the nonlinearity in the system although the system remains phase-locked. However if the driving frequency is slowly varying in time it can lead a slow growth of the oscillator amplitude while remaining phase-locked with external drive. So the driven system self-adjusts to stay in resonance with the drive and thus can lead to continuous resonant excitations. Consequently autoresonance has found numerous applications e.g. in relativistic particle acceleration [324–327], quasiclassical excitations of atoms and molecules [328,329], excitation of nonlinear waves [330,331] and solitons [332,333]. But for the most part autoresonance is externally driven. However, we will consider a problem where autoresonance in parametrically driven. Problems where autoresonance is externally driven can be handled in a similar manner as we handle parametric autoresonance.

6.6.1 Parametric autoresonance

Parametric autoresonance is a combination of two different kinds excitations *parametric* and *autoresonance*. We will consider a case where the auto-resonant drive is parametric. Consider the equation given by

$$\ddot{x} + 2k\dot{x} + (1 + \varepsilon\cos\phi)x - \lambda x^3 = 0, \tag{6.87}$$

where $\phi = \omega(t)$. The frequency $\omega(t)$ is assumed to be a slowly varying function given by

$$\omega(t) = 2 + \delta - 2\gamma t, \tag{6.88}$$

where $\gamma \ll 1$ is called the chirp rate. Initially we will treat the case where the chirp rate $\gamma = 0$. So, we have an oscillator given by

$$\ddot{x} + 2k\dot{x} + (1 + \varepsilon\cos(2+\delta)t)x - \lambda x^3 = 0 \tag{6.89}$$

We work in the regime where both the damping coefficient k and λ are small. We need not explicitly solve this problem because we have already dealt with parametrically excited Van der Pol-Duffing oscillator. Note that if in Eq. (6.69) $\alpha_1 = \beta_1 = 0$ and $\gamma = -beta$ the above equation is recovered. Proceeding with RG calculations in a manner identical as for the Van der Pol-Duffing oscillator we obtain the flow equations,

$$\frac{dA}{d\tau} = -kA + \frac{\varepsilon A}{4}\sin 2\theta \,, \tag{6.90a}$$

$$\frac{d\theta}{d\tau} = -\frac{\delta}{2} - \frac{3\lambda A^2}{8} + \frac{\varepsilon}{4}\cos 2\theta \,. \tag{6.90b}$$

The equations obtained above are equivalent to the ones if one would have done an averaging *a la* Bogoliubov-Krylov. Now what to do in cases where the chirp rate γ is non-zero. Typically, one just treats the quantity $\delta - 2\gamma t$ as almost a constant. Any result obtained by such a treatment will obviously be valid as long as $\gamma t \ll 1$. The RG flow equations thus in case of $\gamma \neq 0$ are given by,

$$\frac{dA}{d\tau} = -kA + \frac{\varepsilon A}{4}\sin 2\theta \tag{6.91a}$$

$$\frac{d\theta}{d\tau} = -\frac{\delta}{2} + \gamma t - \frac{3\lambda A^2}{8} + \frac{\varepsilon}{4}\cos 2\theta \tag{6.91b}$$

This is the correct result for parametric autoresonance for this case [334]. The γt in Eq. (6.91b) when small enough and has the correct sign it can lead to continuous phase locking and autoresonance.

6.7 Quasiperiodicity

Renormalization group method can also be used to for probing whether a system displays quasiperiodicity or not. The example we chose to illustrate how one might study quasiperiodicity using RG, is the Van der Pol- Mathieu equation. We have already treated the Van der Pol-Duffing-Mathieu equation and for the present purposes we drop the Duffing term and treat the equation in the form,

$$\ddot{x} - (\alpha - \beta x^2)\dot{x} + \omega^2(1 + h\cos\gamma t) = 0, \tag{6.92}$$

where $\alpha = \varepsilon\alpha_0$, $\beta = \varepsilon\beta_0$ and $h = \varepsilon h_0$. We expand x and ω as : $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ and $\omega^2 = \omega_0^2 + \varepsilon\omega_1^2 + \varepsilon^2\omega_2^2 + \cdots$ respectively. We expand ω around $\gamma/2$ i.e., $2\omega_0 = \gamma$. Going through the renormalization procedure again we finally arrive at the flow equations,

$$\frac{dA}{d\tau} = \varepsilon\left[\frac{A}{2}\left(\alpha_0 - \frac{\beta_0 A^2}{4}\right) + \frac{Ah_0\omega_0}{4}\sin 2\theta\right] \tag{6.93a}$$

$$\frac{d\theta}{d\tau} = \varepsilon\left[\frac{\omega_1^2}{2\omega_0} + \frac{h_0\omega_0}{4}\cos 2\theta\right] \tag{6.93b}$$

In order to probe whether or not the Van der Pol-Mathieu, Eq. (6.92), displays quasiperiodicity, we now investigate the above amplitude flow Eq. (6.93a). Since, both flow equations must vanish for the periodic response, we use this fact to eliminate θ between the equations by setting $\frac{dA}{d\tau} = 0$ and $\frac{d\theta}{d\tau} = 0$. Then we split the amplitude flow equation into a set of equation in the following manner:

1. We make the substitution $A = \chi\cos\tau + \psi\sin\tau$ in the flow equation.
2. We plug in the above into the flow equation to find a set of differential equations giving the evolution of χ and ψ.

From Eq. (6.93b) we have

$$\cos 2\theta = -\frac{2\omega_1^2}{h_0\omega_0^2} \Rightarrow \sin 2\theta = \sqrt{1 - \frac{4\omega_1^4}{h_0^2\omega_0^4}} \tag{6.94}$$

Differentiating $A = \chi\cos\tau + \psi\sin\tau$ w.r.t τ we obtain,

$$\frac{dA}{d\tau} = \frac{d\chi}{d\tau}\cos\tau - \chi\sin\tau + \frac{d\psi}{d\tau}\sin\tau + \psi\sin\tau \tag{6.95}$$

Then substituting the expression obtained in Eq. (6.94) into RHS of Eq. (6.93a) we have

$$\begin{aligned}\frac{dA}{d\tau} = {} & \frac{\varepsilon}{2}\left(\alpha_0 + \frac{\sqrt{h_0^2\omega_0^4 - \omega_1^4}}{2\omega_0}\right)(\chi\cos\tau + \psi\sin\tau) - \frac{\varepsilon\beta_0}{32}\psi\left(\chi^2 + \psi^2\right)\cos\tau \\ & - \frac{\varepsilon\beta_0}{32}\chi\left(\chi^2 + \psi^2\right)\sin\tau + \text{higher harmonics}\end{aligned} \tag{6.96}$$

Equating LHS and RHS and collecting coefficients of the lowest harmonics we have the evolution equations, right upto $\mathscr{O}(\varepsilon)$,

$$\dot{\chi} = -\psi + \varepsilon a\chi - \varepsilon b\chi(3\chi^2+\psi^2), \tag{6.97a}$$
$$\dot{\psi} = \chi + \varepsilon a\psi - \varepsilon b\psi(\chi^2+3\psi^2), \tag{6.97b}$$

where $a = (\alpha_0/2 + h_0\omega_0 r/4)$ and $b = \beta_0/32$. Our intention is to investigate the nature of the oscillatory behaviour, if any, for the Eqs. (6.97). For the above set of equations an oscillatory solution (a centre) of frequency $\Omega = 1$ exists. One can do perturbative RG around this centre. We expand χ and ψ as $\chi = \chi_0 + \varepsilon\chi_1 + \cdots$ and $\psi = \psi_0 + \varepsilon\psi_1 + \cdots$ respectively and proceed as before. At the lowest order we have,

$$\dot{\chi}_0 = -\psi_0\,, \tag{6.98a}$$
$$\dot{\psi}_0 = \chi_0\,, \tag{6.98b}$$

to which we can immediately write the solution as $\chi_0 = \Lambda_0\cos(\tau+\phi_0)$ and $\psi_0 = \Lambda_0\sin(\tau+\phi_0)$. The equations at the next order are

$$\dot{\chi}_1 = -\psi_1 + a\chi_0 - b\chi_0(3\chi_0^2+\psi_0^2)\,, \tag{6.99a}$$
$$\dot{\psi}_1 = \chi_1 + a\psi_0 - b\psi_0(\chi_0^2+3\psi_0^2)\,. \tag{6.99b}$$

Plugging in the zeroth order solution and with some algebraic manipulation we obtain,

$$\ddot{\chi}_1 + \chi_1 = -\Lambda_0\left(2a - 5b\Lambda_0^2\right)\sin\phi_0 + \cdots\,. \tag{6.100}$$

Performing the RG procedure at this juncture, with μ serving as the arbitrary running parameter, we have the flow equations

$$\frac{d\Lambda}{d\mu} = \frac{\varepsilon}{2}\Lambda\left(2a - 5b\Lambda^2\right) \tag{6.101a}$$
$$\frac{d\phi}{d\mu} = 0 \tag{6.101b}$$

The amplitude flow given by Eq. (6.101a) suggests that the system given by Eqs. (6.97) have a limit cycle solution i.e. a self sustained periodic state. Now to $\mathscr{O}(\varepsilon)$ there is no correction to the frequency. If the frequency of the original system (Eqs.(6.92)) and the derived system (6.97) are incommensurate then it indicates that we have quasiperiodicity in the system. The frequency of the derived system is

unity to the first leading order in ε and all one needs for quasiperiodicity in the system (6.97) is that the frequency ω is irrational. The fact this condition is enough for existence of quasiperiodicity in the system is easily verified numerically.

6.8 System with multiple degrees of freedom

Until now we have considered cases with only one degree of freedom. The RG method however, can easily be extended to a systems with more than one degrees of freedom and as a test case we take the interesting example of the spring pendulum. It's well known that [180, 346] if the length and the mass of the spring pendulum are so chosen that the frequency of spring-type oscillation is nearly twice that of the pendulum-type oscillations then the motion switches back and forth between the two modes. Such a type of resonance is often referred to as an *internal resonance*.

This is a two dimensional mechanical problem to which Olsson [347] obtains a set of coupled equations. We begin with the coupled equation of motion derived by Olsson, but with added damping terms:

$$\ddot{u}_1 + \omega_1 u_1 = -2\mu_1 \dot{u}_1 + \alpha_1 u_1 u_2, \tag{6.102a}$$

$$\ddot{u}_2 + \omega_2 u_2 = -2\mu_2 \dot{u}_2 + \alpha_2 u_1^2. \tag{6.102b}$$

Far from resonance behaviour of systems having quadratic nonlinearities is the same as linear behaviour *i.e.* the frequencies remain independent of amplitude and oscillation modes do not couple. We are interested in the near resonance case, *i.e.* $\omega_2 \approx 2\omega_1$, where an extra term linking u_1 and u_2 comes in leading to internal resonance [124]. As usual we begin with simple perturbative expansions of the form: $u_1 = u_{10} + \varepsilon u_{11} + +\varepsilon^2 u_{12} + \cdots$ and $u_1 = u_{10} + \varepsilon u_{11} + +\varepsilon^2 u_{12} + \cdots$. Further, we define a detuning parameter σ as: $\omega_2 = 2\omega_1 + \sigma$. So at the lowest order we have the equations

$$\ddot{u}_{10} + \omega_1^2 u_{10} = 0, \tag{6.103a}$$

$$\ddot{u}_{20} + \omega_2^2 u_{20} = 0, \tag{6.103b}$$

to which solutions are $u_{10} = A_{1i}\cos(\omega_1 t + \theta_{1i})$ and $u_{20} = A_{2i}\cos(\omega_2 t + \theta_{2i})$, where A_{1i}, θ_{1i}, A_{2i} and θ_{2i} are constants determined from the initial conditions of the two oscillators. We work with the standard initial conditions we have worked with throughout this thesis: $u_1(0) = A_{1i}$, $\dot{u}_1(0) = 0$; $u_2(0) =$

$A_{2i},\ \dot{u}_2(0) = 0$. At the next order we have the equations,

$$\ddot{u}_{11} + \omega_1^2 u_{11} = \sigma u_{10} - 2\mu_1 \dot{u}_{10} + \alpha_1 u_{10} u_{20}, \tag{6.104a}$$

$$\ddot{u}_{21} + \omega_2^2 u_{21} = -4\sigma u_{20} - 2\mu_2 \dot{u}_{20} + \alpha_2 u_{10}^2. \tag{6.104b}$$

Plugging in the zeroth order solutions and little algebra, renders the above equations in the following form:

$$\begin{aligned} \ddot{u}_{11} + \omega_1^2 u_{11} &= A_{1i}\left[2\mu_1\omega_1 - \frac{\alpha_1 A_{2i}^2}{2}\sin\gamma\right]\sin\varphi_1 \\ &\quad + A_{1i}\left[\sigma + \frac{\alpha_1 A_{2i}^2}{2}\cos\gamma\right]\cos\varphi_1 + \cdots \end{aligned} \tag{6.105a}$$

$$\begin{aligned} \ddot{u}_{21} + \omega_2^2 u_{21} &= \left[2\mu_2 A_{2i}\omega_2 + \frac{\alpha_2 A_{1i}^2}{2}\cos\gamma\right]\sin\varphi_2 \\ &\quad + \left[-4\sigma A_{2i} + \frac{\alpha_2 A_{1i}^2}{2}\cos\gamma\right]\cos\varphi_2 + \cdots \end{aligned} \tag{6.105b}$$

where $\varphi_1 = \omega_1 t + \theta_{1i}$, $\varphi_2 = \omega_2 t + \theta_{2i}$ and $\gamma = \theta_{2i} - 2\theta_{1i}$. Following, the now very familiar scheme, we write down the RG flow equations (upto $\mathscr{O}(\varepsilon)$) from the above expressions,

$$\frac{dA_1}{d\tau} = -\frac{\varepsilon A_1}{2\omega_1}\left[2\mu_1\omega_1 - \frac{\alpha_1 A_2^2}{2}\sin\gamma\right], \tag{6.106a}$$

$$\frac{d\theta_1}{d\tau} = -\frac{\varepsilon}{2\omega_1}\left[\sigma + \frac{\alpha_1 A_2^2}{2}\cos\gamma\right]; \tag{6.106b}$$

$$\frac{dA_2}{d\tau} = -\frac{\varepsilon A_2}{2\omega_2}\left[2\mu_2\omega_2 - \frac{\alpha_2 A_1^2}{2}\sin\gamma\right], \tag{6.106c}$$

$$\frac{d\theta_2}{d\tau} = -\frac{\varepsilon}{2\omega_2}\left[-4\sigma A_2 + \frac{\alpha_2 A_1^2}{2}\cos\gamma\right]. \tag{6.106d}$$

where $A_1,\ A_2,\ \theta_1$ and θ_2 are the renormalized amplitudes and phases. Note that being a two-degree-of-freedom system, here we have a pair of amplitude and phase flow equations corresponding to each degree of freedom. From the amplitude equations we can easily eliminate γ leading to the expression,

$$A_1^2 + \frac{\mu_2\omega_2\alpha_1}{\mu_1\omega_1\alpha_2}A_2^2 = 0 \tag{6.107}$$

which is the standard result for this system [124]. For oscillations to exist (i.e. positive A_1 and A_2), α_1 and α_2 must be of opposite signs (since ω_i's and μ_i's are positive by definition). So, if α_1 and α_2 are

of opposite signs then there is internal resonance which drives self sustained oscillations in the system in spite of non-zero damping.

6.9 Conclusion

We have shown that RG can be an effective tool for probing behavior of oscillators near various kinds of resonance phenomena. Be it the usual external forcing induced resonance or resonance due to parametric excitations or for that matter more exotic ones such as autoresonance or internal resonance. Not only this RG can be used to probe subharmonic and superharmonic responses of oscillators under external forcing. Its clear from the various examples we have considered here that RG can be used as a global asymptotic analysis tool for oscillators near various kinds of resonances. Typically, one uses averaging techniques or various singular perturbation techniques for such analysis but RG can replace all these methods effectively. The advantage is that one works with minimal assumptions and there is no need for apriori knowledge of various scales involved. We can derive the slow dynamics of a given system using RG without having to assume slow variation of amplitudes and phases as one needs to in case of Bogoliubov-Krylov averaging method. From the last example we have considered, its also evident that the RG method is easily generalizable to multiple degrees of freedom. For instance, systems of N-coupled oscillators can be easily analyzed using perturbative RG approach. In conclusion, it can be said that renormalization group approach is definitely much simpler and more efficient analytic tool for analyzing non-autonomous systems in general and in particular for analyzing periodic response of oscillators near resonance, compared to the standard techniques in use.

Chapter 7

Parametrically excited nonlinear Van der Pol oscillator

7.1 Introduction

Van der Pol oscillator was first given as a nonlinear model for explaining triode oscillations [133, 197–199] way back in the 1920s. Van der Pol, had experimented with oscillations in a vacuum tube triode circuit and found that all initial conditions converged to the same periodic orbit of finite amplitude. This behavior was radically different from the behavior of solutions of linear oscillator and to explain this Van der Pol proposed a nonlinear differential equation which is given by

$$\ddot{x} + \alpha\dot{x}(x^2 - 1) + \omega^2 x = 0, \tag{7.1}$$

Following the discovery of this oscillator, which admits a self-sustained oscillatory solution, there have been numerous research studies on it and there is a rich body of literature dealing with various aspects of this oscillator. For instance, in studying the case $\alpha \gg 1$, Van der Pol discovered the importance of what has now become known as *relaxation oscillations* [200]. Based on Van der Pols work, Cartwright and Littlewood carried out their seminal research on this topic [348, 349]. Cartwright and Littlewood carried out studies [350–353] on the forced van der Pol equation and showed that it has bistable parameter regimes and that there does not exist a smooth boundary between the basins of attraction of the stable periodic orbits. In addition they discovered chaotic dynamics in the system as well. Following these developments there have been many studies on the Van der Pol oscillator such as on its strong nonlinearity behaviour [354], chaotic behavior [355–357], bifurcation properties [358, 359], synchronization properties [360] and so on. Over the years the Van der Pol equation has been used as basic

model for systems exhibiting self-sustained oscillations appearing across disciplines such as physics, electronics, biology, neurology, sociology and economics [361–365]. In particular, synchronization phenomena and appearance of dynamical chaos has received special attention (see e.g., [357]). Van der Pol himself carried out experiments with electronic circuits trying to emulate the human heart to study the stability of dynamics of mammalian heart [366]. These experiments of Van der Pol oscillator circuits with an external driving signal is analogous to the situation in which mammalian heart is driven by a pacemaker. Lot of work has gone into study of entrainment properties of Van der Pol oscillator and its modifications [367]. This is motivated by the question how to stabilize a heart's irregular beating or "arrhythmias".

In recent times Van der Pol oscillator has found numerous applications. For instance, in biology, the van der Pol equation has been used to model coupled neurons [368, 369]. The FitzHugh-Nagumo equation [370, 371] otherwise known as Bonhoeffer-Nagumo model, extends the Van der Pol equation as a model for neurons. In seismology, Van der Pol equation has been used to model the interaction of two plates in a geological fault [372]. Apart from these Van der Pol oscillator has been used to model many other phenomena such as pulsating motion of stars [373], sudden and periodic outbreak of epidemics [374], periodic bursts of steam in geysers [375], periodic occurrence of economical crises [376, 376] etc. It suffices to say that the Van der Pol oscillator is one of the most important widely studied models in nonlinear dynamics. It was a natural choice therefore for us to pick this system to study the effect a periodically varying 'coefficient of nonlinearity', can have on the behaviour of a nonlinear oscillator.

Van der Pol equation has been studied over wide parameter regimes, starting from perturbations of harmonic motion to relaxation oscillations. Much attention has been given to investigations of the Van der Pol oscillator behaviour under the influence of both external and parametric periodic (sinusoidal) forcing. When parametrically forced, the parameter with periodic variation is usually the one multiplying the linear part.

$$\ddot{x} + \alpha\dot{x}(x^2 - 1) + (a + b\cos\Omega t)x = 0 \tag{7.2}$$

We have dealt with such systems in the previous chapter. We now pose the question what if the parameter with periodic variation is the one multiplying the nonlinearity? How would an oscillator behave if the parametric excitation is introduced via the nonlinear part?

7.2 The model

Resonances in oscillators are of great importance particularly because of its practical applicability. For instance, resonant systems can be utilized to generate vibrations of a specific frequency (e.g.

signal generators, musical instruments), or to pick out specific frequencies from a complex excitation and vibrate at those frequencies, essentially "filtering out" other frequencies present in the excitation (e.g. filters) etc. Typically an oscillator is said to be at *resonance* when it oscillates with far greater amplitude at certain frequencies (resonant frequencies) than at others in the presence of an external driving periodic force which not necessarily be large. A very simple example of resonance from daily life is a child on a swing being pushed by somebody. In this case the periodic "pushes" are the external forcing in the system. However, there can be oscillatory systems whose interaction with the external world is via an periodic variation of its parameters. Such systems are called parametrically excited systems and can display some very novel phenomena including resonance. The phenomenon of parametric resonance [124] is well studied in the context of Mathieu equation or its more general case the damped Hill equation, given by

$$\ddot{x} + k\dot{x} + [\omega^2 + p(t)]x = 0. \tag{7.3}$$

where, ω is the natural frequency of the oscillator, $p(t)$ is a periodic function and k the damping coefficient. Typically, $p(t)$ has the form: $p(t) = a_0 + 2\sum_{n=0}^{\infty} a_n \cos 2n\omega t$. We have already dealt with this system in the previous chapter. Here, we consider a nonlinear parametric equation which is far more general and complex having the form [379],

$$\ddot{x} + k\dot{x} + [\omega_0^2 + p(t)]\left(x + \sum_{i=1}^{N} \alpha_i x^{1+i}\right) = 0. \tag{7.4}$$

For the damped Hill equation, which is recovered when $\alpha_i = 0$, the behaviour under parametric excitations is quite well known. When favourable, these parametric excitations, set up *instabilities* in the center-type oscillations. However, the case where the parameter multiplying the nonlinearity has a periodic dependence hasn't been studied to the best of our knowledge.

Its our intention here to study the effect of a parametrically excited nonlinearity on the behaviour of oscillators. As a starting point we will see what effect can such excitation have on the self-sustained oscillations i.e. limit cycle. Van der Pol oscillator is such a widely studied model because of its practical importance in various physical, chemical and biological sciences [380–382] as well as in economics [383], and of late, even in seismology [384]. People have thus studied the effect of parametric forcing on this oscillator in great detail; See for example, studies on the Mathieu-Van-der-Pol equation [385–388]. However, in all these studies the parameter in front of the non-linear term has been taken as a constant. Here, we relax this constraint and consider the equation,

$$\ddot{x} + \alpha p(t)(x^2 - 1)\dot{x} + \omega^2 x = 0\ , \tag{7.5}$$

where $p(t)$ is a 2π-periodic function and can be written in terms of Fourier components as:

$$p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t). \tag{7.6}$$

We will call this above model as the "**p**arametrically **e**xcited **n**onlinearity in **V**an der Pol **o**csillator" (referred to as "**PENVO**" hereafter). We will show that when the nonlinearity is parametrically excited, a plethora of hitherto unexplored phenomena for example an unconventional resonance and anti-resonance phenomena come to the fore. Here, it must be emphasized that anti-resonance [389], occurring in multi-degree-of-freedom systems is considered important in the context of seismic protection of structures. There are other practical applications of anti-resonance, for instance, driving piezoelectric motor at anti-resonant frequencies, has certain practical advantages [390]. This part of thesis is still a work in progress and here we report only the initial findings. We will concentrate solely on a simplified PENVO with a truncated $p(t)$ having a single harmonic mode, *i.e.*, having the form: $p(t) = a_0 + a_1 \cos \Omega t$.

7.3 Numerical study

Even in the simplified form PENVO presents some very interesting results, that form the core of our study presented here. The *simplified PENVO* can be written in the form:

$$\ddot{x} + \alpha[1 + \gamma\cos(\Omega t)](x^2 - 1)\dot{x} + x = 0. \tag{7.7}$$

One can always write $p(t) = a_0 + a_1 \cos \Omega t$ as $1 + \gamma\cos(\Omega t)$ without any loss of generality just to reduce the number of free parameters in the system. Similarly, we have rescaled time t and parameter α in order to fix $\omega = 1$ and $a_0 = 2$ without any loss of generality. The nonlinear term '$(x^2 - 1)\dot{x}$' is a nonlinear damping term and it is due to this term self-sustaining oscillations occur. Its well known that the limit cycle of Van der Pol oscillator (Eq. (7.1)) is stable only when the damping coefficient α is positive and unstable for negative α. What do we expect to happen when there is periodic variation of the damping coefficient. For $|\gamma| < 1$ the damping coefficient remains positive for all values and thus we can expect limit cycle behaviour to persist although we can't make an off-hand guess about the amplitude and frequency. However, for situations where $|\gamma| > 1$ the damping coefficient may become negative periodically and this can lead to instabilities creeping in. For $|\gamma| > 2$ the damping coefficient can take up relatively large [$\mathscr{O}(1)$] negative values for half of a cycle and the self-sustained dynamics is no more feasible. So, we limit our study to values of $|\gamma|$ between 0 and 2. Further, for most part we have limited ourselves to the weak nonlinearity regime $\alpha \ll 1$. We indeed found that for different

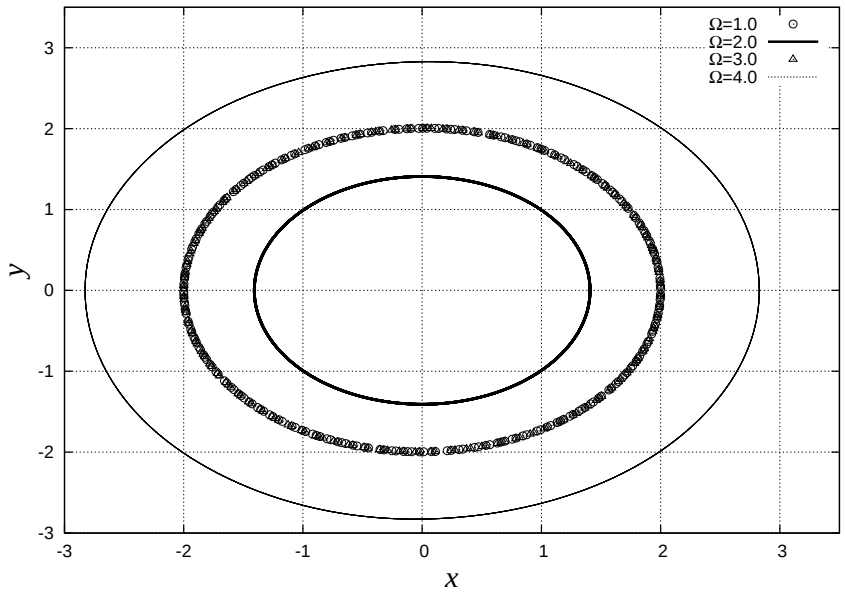

Figure 7.1: PENVO: Limit cycles for $\Omega = 1,\ 3$ coincide with the usual Van der Pol solution whereas limit cycle for $\Omega = 2$ is smaller and that for $\Omega = 4$ larger than the usual limit cycle.

values of excitation frequency Ω and the parameter γ, limit cycle behavior persists for PENVO, but it is affected by value of γ and Ω. We have then systematically studied the dependence of the limit cycle on these two quantities. Before, embarking on the weak nonlinear limit study we must mention here that the limit cycle behavior actually persists even for moderately strong and strong nonlinear limits. Although the behavior of the limit cycle trajectories is very complicated requiring extremely careful study alongwith some novel analytic methods and is outside the scope of the present study. In this study we work exclusively in the weak non-linear limit: $0 < \alpha \ll 1$, where the limit cycle of PENVO [Eq. (7.7) with $\gamma = 0$] is not much deformed from a circle.

The first study we carry out in the weak nonlinear regime is to see what happens to the limit cycle as we vary the values of parametric excitation frequency Ω. We take the value of $\alpha = 0.001$ and henceforth unless otherwise specified, α has been fixed at 0.001. We took, $\gamma = 1.0$ and numerically integrated the Eq. (7.7) for 4 different Ω values viz. $\Omega = 1,2,3,4$. We have plotted the limit cycles for these 4 different values in Fig. 7.1. It turns out that the limit cycle radius (R) remains unaffected at the odd frequencies 1 and 3 and remains at $R = 2$ (the usual weak nonlinearity limit for Van der Pol oscillator). However, for $\Omega = 2$ the limit cycle is less than the usual value where as for $\Omega = 4$ its larger than the usual value. Since we found unusual oscillatory behavior for $\Omega = 2$ and $\Omega = 4$ values, we ran numerics for different values of γ at fixed Ω values. In Fig. 7.2 we have plotted limit cycle solutions for Ω fixed at 2 & different γ values and in Fig. 7.3 frequency $\Omega = 4$ and different γ. In fact varying

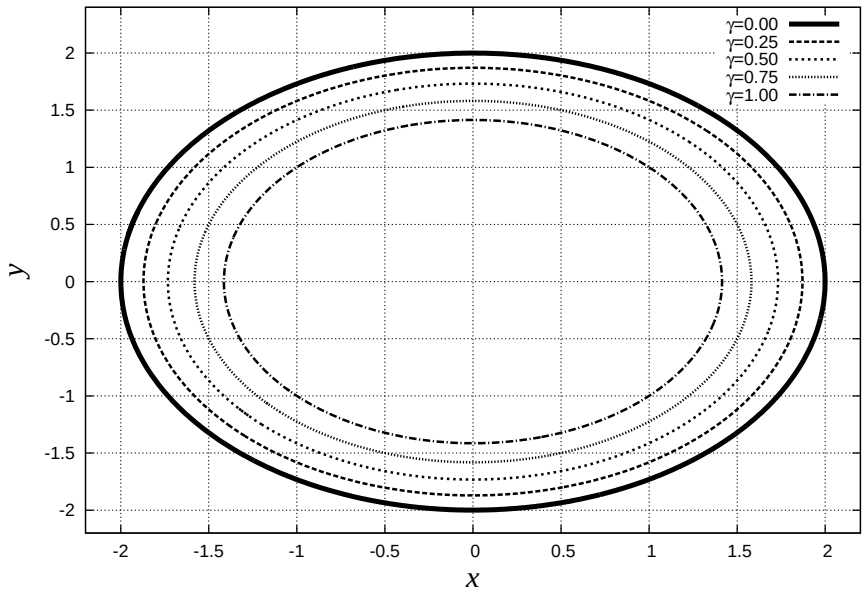

Figure 7.2: Simplified PENVO: Limit cycles for $\Omega = 2$ with varying values of γ; $\gamma = 0.0,\ 0.25,\ 0.50,\ 0.75\ , 1.0$; The radius of the limit cycle reduces with increasing γ and is always less than 2 for non-zero gamma.

the values of Ω and γ continuously and using double precision numerics to integrate Eq. (7.7), the following facts were deduced:

- For any $\Omega \neq 2$ or 4, PENVO admits a stable limit cycle of radius 2 — accurate up to 5% of relative deviation.
- When Ω is irrational, as one might guess, quasiperiodicity can also be seen in the stable oscillations.
- For $\Omega = 2$, as the value of γ is increased from 0 to 2 the radius decreases until $|\gamma| = 1$ and for $|\gamma| \geq 1$ the radius of limit cycle remains fixed at $\sqrt{2}$.
- For $\Omega = 2$, as the value of $|\gamma|$ is increased from 0 to 2 the radius of limit cycle increases continuously.

The dependence of the radius of the limit cycle on the parameter γ is plotted in Fig 7.4. For $\Omega = 2$, we observe the amplitude of the limit cycle oscillations reduces linearly with $|\gamma|$ until $|\gamma|$ is unity, after which amplitude saturates at a value $\sqrt{2}(< 2)$ (See Fig. 7.4). This can be considered as an *anti-resonance* phenomenon. On the other hand, at $\Omega = 4$, the simplified PENVO shows resonance (increase of amplitude): as $|\gamma|$ is increased from 0 to 2 the oscillations' amplitude increases monotonically

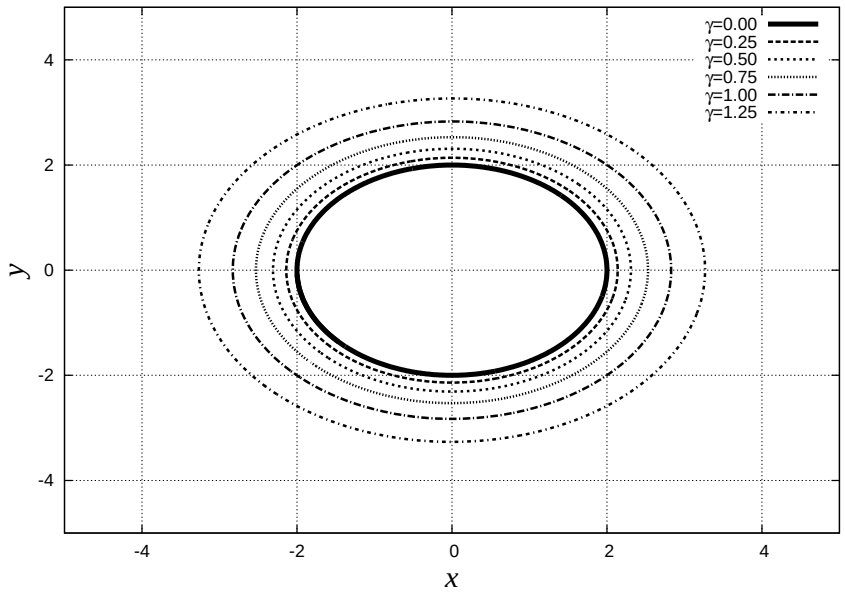

Figure 7.3: Simplified PENVO: Limit cycles for $\Omega = 4$ with varying values of γ; *gammma* = 0.0, 0.25, 0.50, 0.75 ,1.0, 1.25; The radius of the limit cycle increases with increasing γ and is always more than 2 for non-zero gamma.

and always remain *greater than* 2 (See Fig. 7.5). For $\Omega = 4$, the sharp increase in amplitude of oscillation particularly as $|\gamma| \to 2$ can be obviously termed as a kind of resonance but its unlike any usually encountered in literature. This is unlike the resonance due to external forcing where resonance occurs due to proximity of forcing frequency with the natural ones. Here the increase of amplitude is associated with varying of the strength of parametric excitation and one doesn't need an external (non-parametric) driving force. This is also unlike the parametric resonance seen in the Mathieu oscillator in at least in two respects: (i) the resonance in simplified PENVO is **not** an instability phenomenon where amplitude of the oscillations increases with *time*, and (ii) for resonance the parametric periodic forcing has a period *one-quarter* of the natural period of the simplified PENVO where the angular frequency of the limiting oscillations for all Ω is 1 up to the first subleading order in α.

7.4 Analytical results

In order to explain the numerical results, we resort to an averaging method based on the harmonic balance technique [3]. This method was originally developed by Bogoliubov and Krylov [391] and is essentially a singular perturbation technique. We begin by assuming a truncated Fourier series

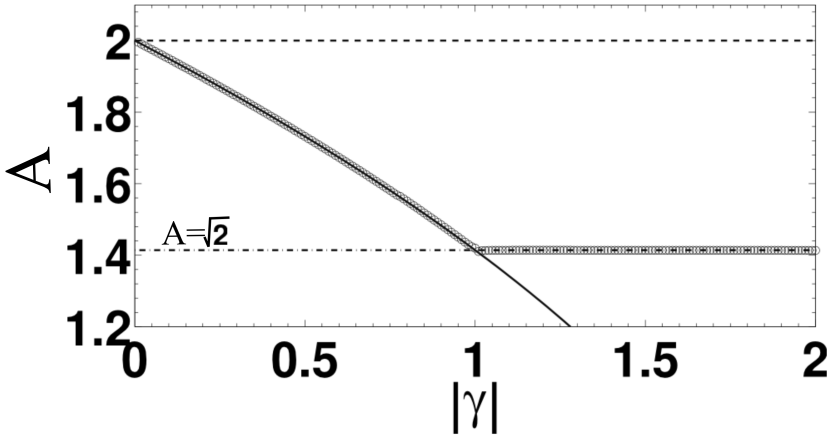

Figure 7.4: Anti-resonance for $\Omega = 2$. The solid line in the figure is given by $\sqrt{(4-2|\gamma|)}$. The dot-dashed line has the constant values of $\sqrt{2}$. Circular markers which collectively appear as dense thick lines are the numerical data points in excellent fit with theoretical predictions.

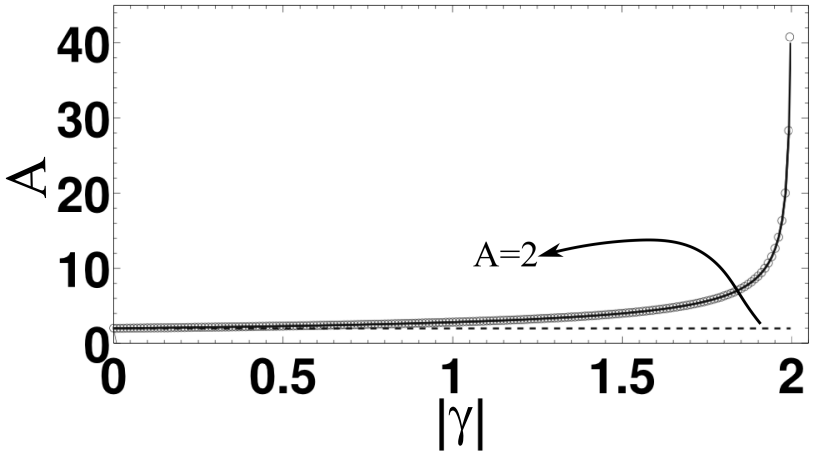

Figure 7.5: Resonance in simplified PENVO: $\Omega = 4$ (*right*). The solid line in the figure is given by $\sqrt{8/(2-|\gamma|)}$ in the right figure. The dashed line has a the constant value of 2. Circular points are the numerical data points which again are in excellent agreement with theoretical predictions.

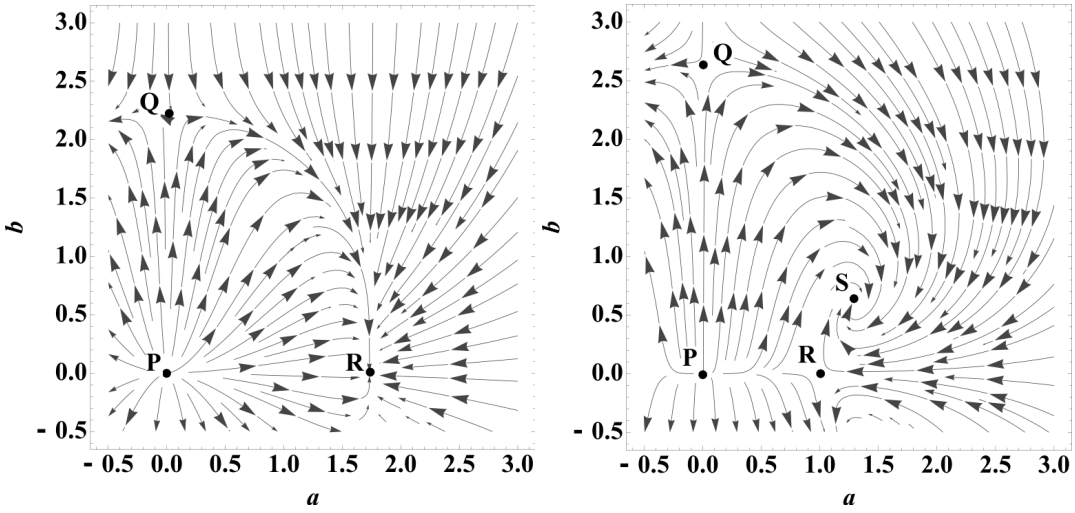

Figure 7.6: Phase plots for simplified PENVO ($\Omega = 2$) in the Van der Pol plane — $\gamma = 0.5$ (*left*) and $\gamma = 1.5$ (*right*): The fixed point S is absent for $0 < \gamma < 1$ when R is the stable fixed point but shows up as the sole stable fixed point in the positive quadrant when $\gamma > 1$.

expansion of the periodic solutions under question, of the form:

$$x(t) = a(t)\cos(t) + b(t)\sin(t)\ , \tag{7.8}$$

where we assume that a and b are not constants but variables "slowly varying" in time. The "slow variation" of the parameters is such that their derivatives $\dot{a}$ and $\dot{b}$ are small i.e. $\mathscr{O}(\alpha)$ terms. In practice we can ignore $\ddot{a}, \ddot{b}, \alpha\dot{a}$ and $\alpha\dot{b}$ ($|\alpha| << 1$) w.r.t. $\dot{a}$ and $\dot{b}$. Differentiating x w.r.t t we have the following expressions,

$$\dot{x} = \dot{a}\cos t - a\sin t + \dot{b}\sin t + b\cos t, \tag{7.9}$$

$$\ddot{x} \simeq -2\dot{a}\sin t - a\cos t + 2\dot{b}\cos t - b\sin t. \tag{7.10}$$

We substitute the expressions (7.8), (7.9) and (7.10) in the simplified PENVO [Eq. (7.7)] and equate the coefficients of $\cos(t)$ and $\sin(t)$ to zero. In doing so we ignore the $\mathscr{O}(\alpha^2)$ terms i.e. $\ddot{a}, \ddot{b}, \alpha\dot{a}$ and $\alpha\dot{b}$. After a little algebra we arrive at the flow equations for a and b. The trajectories of the flow equations are best visualized in the so-called Van der Pol plane i.e. the phase plane for a and b. It is convenient to consider three distinct cases.

We first focus on the interesting case of $\Omega = 2$ in details. The flow equations in this case turn out

to be

$$\dot{a} = \frac{\alpha a}{2}\left[\left(1-\frac{\gamma}{2}\right) - \frac{a^2}{4} + \left(\frac{\gamma}{2}-\frac{1}{4}\right)b^2\right] \tag{7.11a}$$

$$\dot{b} = \frac{\alpha b}{2}\left[\left(1+\frac{\gamma}{2}\right) - \frac{b^2}{4} - \left(\frac{\gamma}{2}+\frac{1}{4}\right)a^2\right]. \tag{7.11b}$$

An analysis of the above flow equations reveal the nature of the original periodic solution to the simplified PENVO. The system given by Eqs. (7.11) has nine fixed points, which are given by:

$$(a^2,b^2) = (0,0);\ (0,4+2\gamma);\ (4-2\gamma,0) \text{ and } (1+1/\gamma, 1-1/\gamma)$$

We can linearize the system about all these fixed points and glean the nature of the flow around it. The pair of eigenvalues of the Jacobian matrix obtained by linearizing Eqs. (7.11) about the above fixed points respectively are:

$$\begin{aligned}\lambda_{1,2} \;=\; & \frac{\alpha}{4}(2\pm\gamma);\ \frac{\alpha}{2}\gamma(\gamma+1), -\frac{\alpha}{2}(\gamma+2);\ \frac{\alpha}{2}\gamma(\gamma-1), \frac{\alpha}{2}(\gamma-2) \\ & \text{and } -\frac{\alpha}{4} \pm \frac{\alpha}{4}(5-4\gamma^2)^{1/2}.\end{aligned}$$

The parameters a and b characterize the periodic solution to the Eq. (7.7). The flow equations of the parameters a and b provide information as to whether a particular assumed solution is stable or not. Since, we are concerned about periodic responses we are interested only in the first (positive) quadrant of the $a-b$ phase plane. In this quadrant, the fixed points for $|\gamma| < 1$ are (cf. Fig. 7.6) the three points — P, Q and R given by $(0,0)$, $(0,+(4+2\gamma)^{1/2})$ and $(+(4-2\gamma)^{1/2},0)$ respectively. Linear stability analysis reveals that P is an unstable node, whereas Q is a saddle point and R is the only stable fixed point. Thus the asymptotic nature of the periodic solution will be such that a and b approach R as $t \to \infty$. The amplitude (A) of oscillations corresponding to R turns out to be ,

$$A = (4-2\gamma)^{1/2} \tag{7.12}$$

which we have used in Fig.7.4 to fit the numerical data. Here, it must also be mentioned that when $\gamma < 0$, instead of the fixed point R, the point Q becomes the only stable fixed point in the positive quadrant; but the corresponding stable oscillation's amplitude still remains at $(4-2|\gamma|)^{1/2}$. However, the fitting breaks down for $|\gamma| > 1$. So, the question remains what happens at $|\gamma| = 1$? Actually it so happens that at $\gamma = 1$, another real fixed point, S, given by $((1+1/\gamma)^{1/2}, (1-1/\gamma)^{1/2})$ emerges and it coincides with R. As the value of γ exceeds unity, if we compute the corresponding eigenvalues, it turns out that among the fixed points P, Q, R and S, now only S is a stable fixed point in the positive quadrant.

The emergence of this new stable fixed point actually fixes the amplitude of the stable oscillations at

$$A = (a^2 + b^2)^{1/2} \,(\text{at } S) = (1 + 1/\gamma + 1 - 1/\gamma)^{1/2} = \sqrt{2}(\ \forall\ \gamma \geq 1). \tag{7.13}$$

The argument for $\gamma \leq -1$ goes along similar lines. This explains the variation of the amplitude of the limit cycle with changing values of $|\gamma|$.

The γ-dependence of the limit cycle radius in case of $\Omega = 4$ can be explained similarly. When $\Omega = 4$, the flow equations for a and b in the Van der Pol plane are computed as,

$$\dot{a} = \frac{\alpha a}{16}\left[8 - (2+3\gamma)b^2 + (\gamma - 2)a^2\right] \tag{7.14a}$$

$$\dot{b} = \frac{\alpha b}{16}\left[8 - (2+3\gamma)a^2 + (\gamma - 2)b^2\right]. \tag{7.14b}$$

Yet again, there are nine fixed points for Eqs. (7.14) which are given by

$$(a^2, b^2) = (0,0);\ \left(0, \frac{8}{2-\gamma}\right);\ \left(\frac{8}{2-\gamma}, 0\right) \text{ and } \left(\frac{4}{2+\gamma}, \frac{4}{2+\gamma}\right).$$

The eigenvalues of the Jacobian obtained by linearizing Eqs. (7.14) about the above fixed points respectively are:

$$\lambda_{1,2} = \frac{\alpha}{2}, \frac{\alpha}{2};\ -2\alpha, \frac{2\gamma\alpha}{(\gamma-2)};\ -2\alpha, \frac{2\gamma\alpha}{(\gamma-2)};\ \text{and}\ -\alpha, \gamma\alpha + \gamma.$$

With the help of the above data one can deduce that the amplitude of the periodic oscillations in this case is given by

$$A = 2\sqrt{2}/(2 - |\gamma|)^{1/2} \tag{7.15}$$

We have used the above function to fit the numerical data we have obtained for $\Omega = 4$ and as can bee seen from Fig. 7.5, the analytical result fits well with the numerical data. Here one must note that, for $|\gamma| > 2$, there is no stable fixed point of Eqs. (7.14) and this also matches with the numerical observations. We found that the trajectory for $\gamma > 2$ becomes quite erratic. Further analysis is required to understand the behavior of PENVO beyond this point.

Finally consider the case: When Ω is an integer but $\Omega \neq 2, 4$. In this case the flow equations in the Van der Pol plane are calculated to be,

$$\dot{a} = \frac{\alpha a}{2}\left(1 - \frac{a^2}{4} - \frac{b^2}{4}\right); \dot{b} = \frac{\alpha b}{2}\left(1 - \frac{a^2}{4} - \frac{b^2}{4}\right). \tag{7.16}$$

For the above equations the origin $(a,b) = (0,0)$ is a fixed point. The eigenvalues of the characteristic matrix obtained by linearizing Eq. (7.16) about the origin are: $\alpha/2, \alpha/2$. Thus the origin is unstable for positive α. However, careful analysis (e.g. using RG method) of the above equations reveal that encircling this unstable origin, there exists a stable limit cycle in the Van der Pol plane having the radius: $a^2 + b^2 = 4$. This explains the fixed amplitude of 2 for the limit cycles in the simplified PENVO when Ω takes any integral values other than 2 and 4. For any non-integral value of Ω, the harmonic balance procedure adopted here conceptually breaks down. However, one can apply renormalization group method [84, 86] to see that the corresponding limit cycle's radius remains 2.

7.5 Renormalization group applied to PENVO

The harmonic balance technique used in the previous section readily explains the anomalous γ-dependence of the limit cycle in case $\Omega = 2$ or 4. However, it breaks down for nonintegral values of the excitation frequency. Apart from that the method proceeds by assuming a slow variation of the parameters a and b. Renormalization group method on the other hand is much more straightforward to apply in this case and the result so obtained remains valid even for non-integral values of Ω. Keeping in philosophy of this thesis its only apt to show how RG can be used successfully in analyzing this very interesting oscillator and how the anomalous γ-dependence of the limit-cycle radius naturally comes out of the RG flow equation. Unlike, the harmonic balance method RG doesn't need any *ad hoc* assumptions such as assuming slow variation of the parameters instead, we begin as usual with a naive expansion of x in powers of the small parameter α,

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots. \tag{7.17}$$

Inserting the above series into the Eq. (7.7) one obtains at different orders of perturbation,

$$\ddot{x}_0 + x_0 = 0, \tag{7.18}$$

$$\ddot{x}_1 + x_1 = -(1 + \gamma\cos\Omega t)\dot{x}_0\left(x_0^2 - 1\right). \tag{7.19}$$

The zeroth order solution is $x_0 = A_0\cos(t+\theta_0) = A_0\cos\phi$ where $\phi = (t+\theta_0)$, which we substitute into the Eq. (7.19) to obtain

$$\begin{aligned}\ddot{x}_1 + x_1 &= A_0\left(\frac{A_0^2}{4}-1\right)\sin\phi + \frac{A_0^3}{4}\sin 3\phi \\ &\quad + \frac{\gamma A}{2}\left(\frac{A_0^2}{4}-1\right)\left[\sin\Big((1+\Omega)t+\theta_0\Big) + \sin\Big((1-\Omega)t+\theta_0\Big)\right] \\ &\quad + \frac{\gamma A_0^3}{8}\left[\sin\Big((3+\Omega)t+3\theta_0\Big) + \sin\Big((3-\Omega)t+3\theta_0\Big)\right]. \end{aligned} \tag{7.20}$$

We have already established that the asymptotic nature of the oscillatory solution in a given system can be obtained from the RG flow equations and that the coefficients of the secular terms generated at each order of perturbation are the only ones contributing to the RG flow. An careful inspection of the terms appearing in the above equation (Eq. (7.20)) shows that apart from the γ-independent first term which is secular, other secular terms can be generated only when $\Omega = 2$ or 4. For instance, lets take the case when $\Omega = 2$, in which case Eq. (7.20) becomes

$$\begin{aligned}\ddot{x}_1 + x_1 &= A_0\left(\frac{A_0^2}{4}-1\right)\sin\phi + \frac{A_0^3}{4}\sin 3\phi \\ &\quad + \frac{\gamma A}{2}\left(\frac{A_0^2}{4}-1\right)\left[\sin(3t+\theta_0) - \sin(t-\theta_0)\right] \\ &\quad + \frac{\gamma A_0^3}{8}\left[\sin(5t+3\theta_0) + \sin(t+3\theta_0)\right]. \\ &= \left[A_0\left(\frac{A_0^2}{4}-1\right) + \frac{\gamma A_0}{2}\cos 2\theta_0\right]\sin\phi \\ &\quad + \frac{\gamma A_0}{2}\left(\frac{A_0^2}{2}-1\right)\sin 2\theta_0\cos\phi + \text{higher harmonics} \end{aligned} \tag{7.21}$$

This immediately leads to the RG flow equations correct upto $\mathscr{O}(\alpha)$

$$\frac{dA}{d\tau} = -\frac{\alpha A}{2}\left[\left(\frac{A^2}{4}-1\right) + \frac{\gamma}{2}\cos 2\theta\right], \tag{7.22a}$$

$$\frac{d\theta}{d\tau} = -\frac{\alpha\gamma}{4}\left(\frac{A^2}{2}-1\right)\sin 2\theta\ . \tag{7.22b}$$

In accordance with our prescription for dealing with non-autonomous systems, the stable fixed points of the above equations determine the stable oscillatory response of the system. In the above case we have two equations have 2 possible situations where both the amplitude and phase flows vanish. The following two situation arise:

- $\sin 2\theta = 0$ implying $\theta = 0$ and thus $\cos 2\theta = 1$. This fixes $A^2 = (4-2\gamma)$.
- $A^2 = 2$, $d\theta/d\tau$ becomes zero and we have $\cos 2\theta = 1/\gamma$ which is valid only when $\gamma > 1$.

The above two situations are mutually exclusive in the sense that, in the first scenario the amplitude is given by

$$A = \sqrt{4-2\gamma} \text{ for } \gamma < 1 \tag{7.23}$$

and this equilibrium is stable only when $\gamma < 1$ and on the other hand the equilibrium corresponding to the second scenario is stable for $\gamma \geq 1$ and the corresponding amplitude is given by

$$A = \sqrt{2} \text{ for } \gamma \geq 1. \tag{7.24}$$

is stable for $\gamma \geq 1$. The above result matches exactly with the result obtained via *harmonic balance* method.

Now lets consider the case of $\Omega = 4$. In this case the Eq. (7.20) reduces to

$$\begin{aligned} \ddot{x}_1 + x_1 &= \left[A_0\left(\frac{A_0^2}{4} - 1\right) + \frac{\gamma A_0^3}{8}\cos 2\theta_0\right]\sin\phi \\ &\quad + \frac{\gamma A_0^3}{8}\sin 2\theta_0 \cos\phi + \text{higher harmonics}, \end{aligned} \tag{7.25}$$

which leads to the flow equations

$$\frac{dA}{d\tau} = -\frac{\alpha A}{2}\left[\left(\frac{A^2}{4} - 1\right) + \frac{\gamma A_0^2}{8}\cos 2\theta\right], \tag{7.26a}$$

$$\frac{d\theta}{d\tau} = -\frac{\alpha\gamma A_0^2}{16}\sin 2\theta\ . \tag{7.26b}$$

Equating the above flow equations to zero we obtain $\sin 2\theta = 0$ which leads to two possible answers $A^2 = 8/(2+\gamma)$ or $8/(2-\gamma)$ of which only the latter is stable. Yet again the RG result matches with that obtained by harmonic balance method.

Now lets turn our attention to values of Ω when $\Omega \neq 2$ or 4. its clear from Eq. (7.20), as we have already pointed out, that when $\Omega \neq 2$ or 4, no secular terms apart from the first one i.e. '$A_0\left(\frac{A_0^2}{4} - 1\right)\sin\phi$' are generated. The first term has no dependence on γ hence for all other values of the excitation frequency the amplitude remains fixed at the usual value of 2.

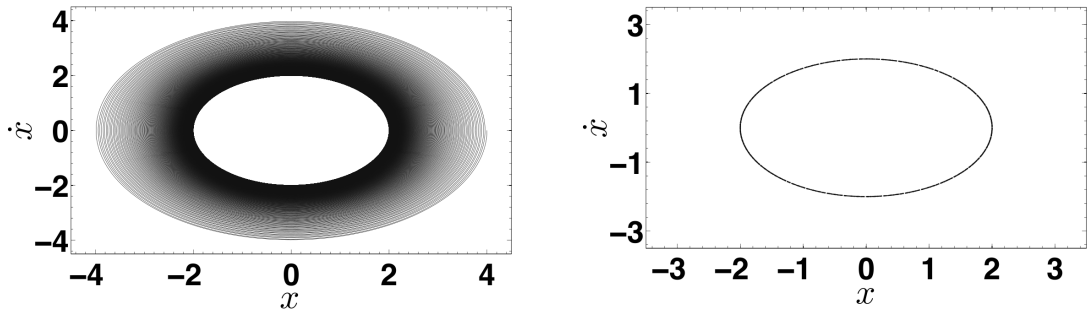

Figure 7.7: Quasiperiodicity in simplified PENVO ($\Omega = \sqrt{2}$): (*left*) A trajectory is seen to densely fill up a portion of $x - \dot{x}$ plane of phase space, and (*right*) an elliptical curve traced by the same trajectory in the relevant Poincaré section (cf. text).

7.6 Quasiperiodicity

In our numerical simulations of the PENVO we observed that for irrational values of the excitation frequency the oscillations exhibit *quasiperiodicity*. In order to further ensure the existence of quasiperiodicity, let us rewrite Eq. (7.7) as an autonomous set of three first order ordinary differential equations in the following manner:

$$\dot{x}_1 = x_2\,, \tag{7.27a}$$

$$\dot{x}_2 = -x_1 - \varepsilon(1+\cos x_3)(1-x_1^2)x_2\,, \tag{7.27b}$$

$$\dot{x}_3 = \Omega\,. \tag{7.27c}$$

The question that whether for irrational Ω, the oscillations are quasiperiodic or not, is best addressed by looking at the Poincaré section at some plane: x_3 =constant. Its the the third coordinate x_3 which is same as that of the parametric excitation. That's why the appropriate Poincaré section must be taken stroboscopically *i.e.* looking at the oscillations after intervals of $2\pi/\Omega$. In case of periodic oscillations, depending on what is the ratio between angular frequency of the oscillation and the excitation frequency Ω, the Poincaré section is expected to consist of finite number of discrete points. On the other hand, in case there exists quasiperiodic dynamics in a system it will generate infinite number of points on the Poincare section so that a dense curve will be seen. We have numerically determined the Poincaré section and the results are plotted in Fig. 7.7. As expected we see a trajectory densely filling up a portion of $x - \dot{x}$ plane of phase space when excitation frequency Ω is irrational in PENVO confirming the existence of quasiperiodicity.

7.7 A switch

The fact that when the excitation frequency, $\Omega = 2$, the limit cycle radius reduces as one increases γ from zero and that the radius saturates at $\sqrt{2}$ beyond $\gamma = 1$ creates the possibility of designing switching action using a pair of coupled PENVOs. Here we present an idea how a two-state-switch may be constructed by using this unique property of PENVO. Consider the set of coupled equations

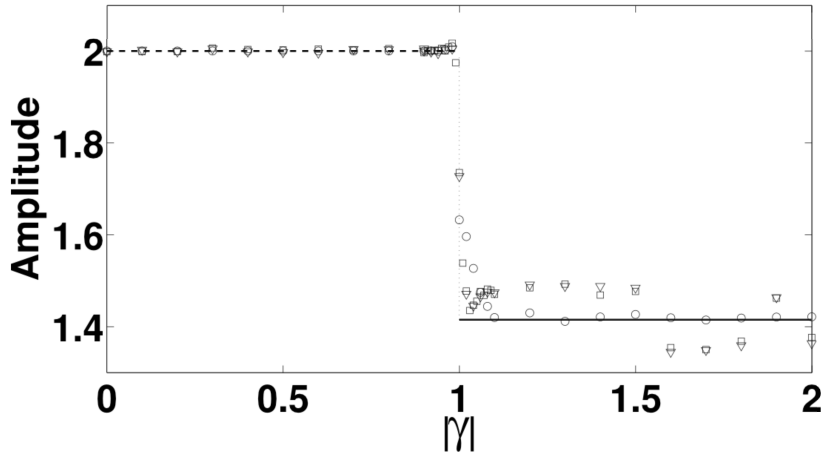

Figure 7.8: Two-state switch in coupled simplified PENVOs. The behavior of switch for $|\gamma| \geq 1$ is very sensitive to fluctuations in the value of '$2\sqrt{2}/A_{y_\infty}$'. To compensate this effect, A_{y_∞} has been calculated by taking average of 100 points over the limiting cycles of corresponding oscillations. The dashed line is for amplitude= 2, the solid line is for amplitude= $\sqrt{2}$ and; circles, squares and triangle are the numerical data points generated by system of equation (7.28) with no noise, uniform white noise and Gaussian white noise respectively. The noises have been added only in equation (7.28a) as a term $\nu\xi(t)$ in the R.H.S. ν which measures the strength of the noise is 0.01 in this figure. One can increase ν to at least 0.1 without any drastic change in the behavior of the switch.

given by

$$\ddot{x} + \varepsilon\left[1 + \gamma\cos\left(\frac{2\sqrt{2}}{\sqrt{y^2+\dot{y}^2}}t\right)\right](x^2-1)\dot{x} + x = 0\,, \tag{7.28a}$$

$$\ddot{y} + \varepsilon[1 + \gamma\cos(2t)](y^2-1)\dot{y} + y = 0\,. \tag{7.28b}$$

The above set of equations (7.28) has an one-way coupling. The amplitude of limit cycle of the second oscillator (Eq. (7.28b)) is used to modify the frequency of the parametric forcing in the first oscillator (Eq. (7.28a)). When the value of γ lies between 0 and 1, the second oscillator (Eq. (7.28b)) oscillates with an amplitude, $A_{y_\infty} \equiv (y_\infty^2 + \dot{y}_\infty^2)^{1/2} = (4 - 2|\gamma|)^{1/2}$ such that $A_{y_\infty} \in [2, \sqrt{2})$, where subscript A_{y_∞} is used to refer to the quantities computed in the asymptotic limit. This in turn means that the excitation frequency of the first oscillator lies between $\sqrt{2}$ to 2. Thus the amplitude of oscillations of the first oscillator will remain at the usual Van der Pol value of 2. So, for $\gamma \in [0,1)$, it is always $A_{x_\infty} = 2$.

However, the moment $|\gamma|$ becomes equal to or greater than unity, the second oscillator has limit-cycle oscillations of $A_{y_\infty} = \sqrt{2}$ and this makes the excitation frequency for the first oscillator equal to 2. No sooner does this happen, the amplitude of the oscillations of the first oscillator (Eq. (7.28a)) jumps to $\sqrt{2}$ for $|\gamma| \geq 1$. In short, the amplitude of the stable oscillations generated by Eq. (7.28a) *switches* from 2 (for $|\gamma| < 1$) to $\sqrt{2}$ (for $|\gamma| \geq 1$) at $|\gamma| = 1$ (cf. Fig. 7.8). We have also added Gaussian white noise to the system of coupled oscillators and have found that the switching action remains unaffected by it. It shows that the switch so constructed is quite robust in nature and can find applications in the real world.

7.8 Discussions and conclusion

Before we conclude this chapter lets return to the full PENVO given by the Eq. (7.5). We consider with $\omega = 1$ without any loss of generality. Let's first consider the case of $p(t)$ being an odd function so that the Fourier series (7.6) of $p(t)$ is an exclusively *sine* series. One can easily show using harmonic balance that for such a $p(t)$, there doesn't exist a single stable *node* or *focus* in the flow equations for $a,\ b$ in the Van der Pol plane. This means that the limit cycle in Van der Pol equation is destroyed and there is no *limiting* periodic oscillation. On the other hand if $p(t)$ is a even function, i.e. only *cosine* terms in the Fourier series of $p(t)$, then rescaling α and defining $\gamma_{ci} \equiv 2a_i/a_0$, we find that, to the leading order, only γ_{c2} and γ_{c4} can determine the conditions for stability of the periodic oscillations. Other γ_{ci}'s are irrelevant for this purpose: when $\gamma_{c2} = \gamma_{c4} = 0$, PENVO will always yield the limit cycle solution — $a^2 + b^2 = 4$ — irrespective of values that the other γ_{ci}'s take.

Thus the most general case which is enough for a complete description of PENVO effectively requires any arbitrary 2π periodic $p(t)$ to be represented by the following surprisingly small fraction of the full Fourier series (7.6):

$$\begin{aligned} \alpha p(t) = \alpha\,(1 &+ \gamma_{c2}\cos 2t + \gamma_{c4}\cos 4t + \gamma_{s2}\sin 2t + \gamma_{s4}\sin 4t \\ &+ \text{irrelevant terms})\,, \end{aligned} \tag{7.29}$$

where $a_0/2$ has been absorbed into α and $\gamma_{si} \equiv 2b_i/a_0$. When only '1+irrelevant terms' in Eq. (7.29) survive, the usual Van der Pol oscillator's limit cycle is recovered. Dealing with all four γ's simultaneously is cumbersome, but one can understand the relative importance of the sinusoidal terms in expression (7.29) by setting any two of the four γ's to zero at a time. Out of the six cases that can arise, only the following forms of $p(t)$ require independent study: (i) $1 + \gamma_{c2}\cos(2t) + \gamma_{c4}\cos(4t)$ and (ii) $1 + \gamma_{s2}\sin(2t) + \gamma_{s4}\sin(4t)$. The remaining four relevant combinations of *sine* and *cosine* can be re-

duced to these two forms by suitable rescaling of time t and γ's. However, no qualitatively new results other than the ones already obtained earlier, are found. We thus simply mention here that though the presence of four parameters in $p(t)$ makes the algebra tedious, one can in principle do the calculations in a quite straightforward manner using RG method which proves to be the most effective analytic tool for this system.

To conclude, we must emphasize that inclusion of parametric excitation in the nonlinear term definitely introduces novel and interesting effects. In light of this result there is a need to revisit other oscillators particularly ones capable of self-sustained oscillations and see what happens under such parametric excitation. It will also be interesting to look at conservative systems such as the Duffing oscillator with nonlinear parametric excitations. Its our belief that the results obtained here could be generic and, the resonance and the anti-resonance could be encountered in other such systems as well. Such resonance and anti-resonance can have direct implications on many nonlinear oscillators which are used to model various periodic phenomena across disciplines. Its quite a possibility that new applications may arise out of such studies.

Chapter 8

Postscript

We have presented in this thesis an unified tool to analyze a wide range of problems involving oscillators. Its clear from the various problems we have handled that a wide range of periodic responses can be probed using this method. We have handled planar systems for most part of this thesis but we have also shown via an example that this method is easily generalizable to systems with multiple-degrees-of-freedom. The criteria we have presented here can easily be used for $2N$-dimensional systems. But what if one has to deal with systems with odd number of dimensions? For instance, Lorenz system has three variables and straightforward application of our criteria is not possible. However, RG method can still be used for analyzing odd-dimensional systems. For example, the Lorenz system has been treated in [101] using a vectorial formulation of this technique. Its our suspicion that similar criteria as presented by us in this thesis can also be developed for such systems.

In our present work, we have no-where provided any kind of mathematical proof of the assertions we have made. Rather, we have relied on arguments based on physical considerations and intuitive understanding. Goldenfeld *et al* had initially presented the RG method as an unified global asymptotic analysis tool. Their work showed that results obtained by RG are equivalent and sometime improved than the results obtained by various singular perturbation techniques. In our investigations we have found that RG results are indeed equivalent to results obtained by various other techniques such as Lindstedt-Poincaré method, multiple scales method, Bogoliubov-Krylov method etc. It can safely be said that various singular perturbation methods can be understood in an unified manner in the spirit of renormalization group philosophy. Our work in exploring the RG method has led to development of a technique by which without explicitly solving the dynamics one can draw general conclusions about the asymptotic state of a given nonlinear dynamical system. However, a clear mathematical understanding of the renormalization group method and as to why it works so well is of course a very important question. In the last decade Kunihiro *et al* [100–102] have done some research which has shed some

light in this direction. They have showed that the renormalization group method can be understood in terms of the classical theory of envelopes. Their work identified RG equation as arising from an equation obtained from an envelope of a family of curves constructed by the naive expansion. The fact that RG method furnishes approximate answers which are globally improved has been attributed to the fact that an envelope of a family of curves has an improved global nature compared with the curves. Despite Kunihiro's work some basic issues about the RG method remained unsolved. In particular the question whether there is an invariant manifold involved and can its stability be detected remained a open one. Apart from this a basic definition of the RG equation was still lacking. In this regard the work by Ziane [392], Chiba [393–395] and DeVille [396] et al is worth mentioning. Ziane [392] and DeVille [396] *et al* have given a definition of the first order RG equation based on the averaging operator. Chiba [393] took this work further by giving a formula for the n-th order RG equation by calculating the envelope of the naive expansion solutions upto arbitrary order in the small parameter. Chiba has also presented a simplified form of the CGO-RG method [397]. CGO-RG method, as we have previously mentioned, takes advantage of the fact that along a given trajectory, every point can serve as an initial condition for a differential equation leading to the exactly same asymptotic state. Kunihiro took advantage of this fact and developed a short-cut prescription for the renormalization procedure without introducing an intermediate time τ but instead making use of an arbitrary initial time t_0 [103, 114]. In principle RG method can be formulated as a method dealing with the initial values at an arbitrary $t = t_0$, so that in terms of the initial values at $t = t_0$, the unperturbed solution (valid only locally at $t = t_0$) is continued smoothly to construct a globally valid solution at the asymptotic limit. Being perturbative RG can be applied recursively and owing to the works of Ziane and Chiba its now possible to formulate CGO-RG using "computer algebra systems" such as MATHEMATICA, MAPLE etc. This increases the effectiveness of our RG prescription manifold. For instance, in case of our method to determine conditions for isochrony we were limited due to the perturbative nature and the need to do higher order calculations but with help of a symbolic manipulation software its possible to do the tedious calculations with ease.2 With the present state of understanding of the CGO-RG method one can actually

In conclusion, a variety of $2D$-dynamical systems have been discussed in this thesis, most of which have been analytically handled by a renormalization group approach. We have presented a criterion, based on our observations from the numerous oscillator equations we have handled using RG in this thesis and using this criterion it is possible to discern between globally stable self-sustained oscillations (limit cycles) and neutrally stable oscillations (centres). For a given dynamical system, from the RG amplitude flow equation one can predict whether the system can have limit cycle solutions and if so its relative position and stability can also be determined in the weak nonlinearity limit. The only limitation of this method is the problem of finding a suitable state to perturb around. In case of limit

cycles coming into existence through a Hopf bifurcation (the most widely occurring way) one can always find such an unperturbed state. Besides this the RG flow equations also are useful to distinguish between *centre* and *focus* in problems where linear stability analysis fails to do so. The chief advantage of our method becomes clear in the fourth chapter where we have dealt with isochronous oscillators — without actually solving the dynamics we are able to find conditions under which a given planar system can have isochronous oscillations. We have also developed a way to artificially create a basic unperturbed state to do the perturbative RG around for a class of systems where such a state is missing to start with. In our investigations of various dynamical systems we have chanced upon systems with very interesting dynamical properties for instance the Riccati system, PENVO etc. Further studies into various aspects of the Van der Pol oscillator with nonlinear parametric excitation are sure to furnish more surprises. We had started out with an idea that exploring the potential of the RG method in dynamical systems can be a very interesting problem. To our delight the idea lived upto its promise and based on our systematic studies, we have presented here a method which we hope researchers will find useful in their study of oscillators in particular and dynamical systems.

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