A STUDY OF CERTAIN PROPERTIES OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

Thesis submitted for the degree of
Doctor Of Philosophy (Science)
of
WEST BENGAL STATE UNIVERSITY
Barasat, North 24 Parganas

By
BARUN KHANRA

JUNE, 2013

A STUDY OF CERTAIN PROPERTIES OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

Thesis submitted for the degree of
Doctor Of Philosophy (Science)

By
BARUN KHANRA

(Regn. No.: 10012416114000002 of 2011)
Department of Physics
WEST BENGAL STATE UNIVERSITY
Barasat, North 24 Parganas

Supervisors:

Dr. Partha Guha
Department of Theoretical Sciences,
S. N. Bose National Centre for
Basic Sciences,
Salt Lake, Kolkata, India.

Dr. Anindya Ghose Choudhury
Department of Physics,
Surendranath College
Kolkata, India.

Acknowledgements

I would like to thank Dr. Partha Guha, Associate Professor, S. N. Bose National Centre for Basic Sciences, Kolkata and Dr. Anindya Ghose Choudhury, Associate Professor, Department of Physics, Surendranath College, Kolkata, my supervisors, for their continuous guidance and constant support during my research work.

I wish to express my gratitude to the Director of S. N. Bose National Centre for Basic Sciences, Kolkata for giving me the opportunity for doing research work in the Institution as an external research scholar. I am also thankful to the Librarian and the office staff of S. N. Bose National Centre for Basic Sciences, for their support.

The cooperation and assistance rendered by the Secretary, the Headmaster and my colleagues of Sailendra Sircar Vidyalaya, Kolkata, is deeply acknowledged.

Last but not the least, I am grateful to my parents, parents in-law and wife for their patience and love. Without them this work would never have come into existence (literally).

List of publications:

Publications included in the Thesis

1. *Solution of some second order ODEs by the extended Prelle-Singer method and symmetries.*
A Ghose Choudhury, Partha Guha, **Barun Khanra**, J. Nonlin. Math. Phys.**15**(4),365-382(2008).

2. *On adjoint symmetry equation, integrating factors and solutions of nonlinear ODEs.*
Partha Guha, A Ghose Choudhury, **Barun Khanra**, J. Phys. A: Math. Theor. **42**(115206), 13 pp, (2009).

3. *On Jacobi last multiplier in integrating factors and the Lagrangian formulation of differential equation of the Painlevé- Gambier equation.*
A Ghose Choudhury, Partha Guha, **Barun Khanra**, J. Math. Anal. and applications. **360** (2), 651-664, (2009).

4. *Determination of elementary first integrals of a generalized RayChaudhuri equation.*
A Ghose Choudhury, Partha Guha, **Barun Khanra**, J. Math. Phys, **50**(102502),(2009).

5. *On Generalized Sundman Transformation Method, First Integrals and Solutions of Equations of Painlevé-Gambier Type.*
Partha Guha, **Barun Khanra**, A Ghose Choudhury, Nonlinear Analysis, **72**(2010), 3247-3257.

6. *First Integrals for the Time-Dependent higher-Order Riccati Equation by nonholonomic transformation.*
Partha Guha, A Ghose Choudhury, **Barun Khanra**, CNSNS, **16** (2011), 3062-3070.

7. *λ-symmetries, Isochronicity and Integrating factors of Nonlinear Ordinary Differential Equation.*
Partha Guha, A Ghose Choudhury, **Barun Khanra**, Journal of Eng. Math, Published online, February, 2013.

Publications not included in the Thesis

1. *Canonical Bäcklund transformation for the DST model under open boundary conditions.*
Barun Khanra, A Ghose Choudhury, Inverse Problems, **25** (2009), 085002 (11pp).

2. *On Bäcklund transformation of D_n type toda lattice.*
Barun Khanra, A Ghose Choudhury, Physics Letters A, **374** (2010), 4120-4127.

3. *On solution of third and fourth-order time dependent Riccati equation and the generalized Chazy system .*
Partha Guha, A Ghose Choudhury, **Barun Khanra**, CNSNS, **17**(2012), 4053-4063.

Contents

Chapter 1

Introduction

The study of ordinary and partial differential equations constitutes a fundamental aspect of mathematical analysis. Its importance is primarily because differential equations are by far the most preferred mathematical tool for modeling a wide variety of physical and economic phenomena, ranging from the colossal arena of astronomy to the microscopic universe of genetics, encompassing in between almost the entire gamut of human activity, including even stock markets.

Generally, when a system is modeled by an ordinary differential equation (ODE) it is done in such a way that the corresponding ODE describes the change with respect to time (the independent variable) of some dependent variable; and the solution of the equation represents the state of the system at that point in time. This enables us to predict its future behavior. Indeed this ability to predict the evolution of system is of fundamental importance and is the primary reason for their unique status.

Many of the mathematical models used for understanding physical, chemical, engineering, or biological processes are described by nonlinear ordinary differential equations and their widespread applicability to the sciences has generated in its trail a continuous stream of several new problems of both theoretical and practical interest. Consequently, it is a worthwhile endeavor to engage in an investigation of their properties and distinctive features.

Some of the earliest investigations of nonlinear phenomena arose in the analysis of water waves and in acoustics. Notable contributions were made in this context not only by celebrated continental mathematicians like Cauchy, Poisson, Euler and Lagrange, but also by several others such as Bossinesq, Cole, Stokes, Airy, Scott Russell, and Lord Rayleigh, to name just a few. These mathematicians took a more practical view of such phenomena as the propagation of shallow water waves in a canal or an artificial channel, or the propagation of plane progressive sound waves of finite amplitude in air. Aside from hydrodynamic or acoustic phenomena, it was realized quite early on that nonlinearity was an inherent feature of many problems with regard to the theory of elasticity. In fact, the problem of a flexible rod bent in one plane so that its two ends approach each other attracted the attention of many brilliant minds including the Bernoullis, Euler, and Lagrange. Extensive studies of viscous and compressible fluids by Goldstein, Reynolds, Prandtl, and Taylor also revealed the role of nonlinearity.

In the 1920's the Dutch electrical engineer Van der Pol first observed the phenomenon of relaxation oscillations in electrical circuits employing vacuum tubes. Today such oscillations are more commonly known as limit cycles. The Van der Pol oscillator is a non conservative oscillator with nonlinear damping. Besides electrical circuits the Van der Pol oscillator has also appeared in biology where Fitzhugh and Nagumo extended it as a model for action potentials of neurons. It is also useful for modeling heart beats in human physiology and surprisingly enough, has been applied even to model geological fault lines.

The use of modern computing devices has enabled us to make remarkable progress in solving several problems belonging to the domain of nonlinear mathematics which challenged many brilliant minds of the preceding centuries. In fact for over 250 years astronomers had struggled with the nonlinear system of equations describing the motions of planets. To this day the problem of stability of the solar system presents difficulties.

With the advent of high speed computers, most studies of nonlinear systems arising in the natural sciences and engineering are carried out numerically. This, however, should not be taken to mean that analytical methods are to be disregarded. Analytical methods of investigations are useful because that have the potential to provide exact results as opposed to graphical or numerical methods. Moreover the solution of a nonlinear equation often possesses singularities which can be discovered and described by analytical methods alone.

However, unlike linear differential equations, there are very few general methods for dealing with nonlinear ODEs. While planar nonlinear ODEs have been more extensively analysed the situation in case of non planar systems is generally acknowledged to be rather fuzzy.

It is therefore not surprising that the problem of linearization of nonlinear ODEs is an area of considerable importance from the practical standpoint owing to an abundance of tools and techniques for their solution. Closely related to this is the issue of symmetries of such equations. This follows from the general notion that a differential equation is solvable only if it displays some sort of inherent symmetry. The determination of its symmetries and their consequent applications for unearthing solutions or first integrals of nonlinear ODEs constitutes the very essence of Lie's seminal work. The search for first integrals of ODEs is a common feature of many of the methods employed for their analysis. A number of techniques have been devised for unearthing them, at times based on symmetries and in some cases shrewd insights. Their existence together with the Hamiltonian aspects of nonlinear ordinary and partial differential equations is also of profound interest from the geometrical point of view.

In view of the above we have focussed our attention primarily on the following four aspects of the study of nonlinear ordinary differential equations:
- Lie symmetries of nonlinear ODEs.
- Linearization of nonlinear ODEs.
- Determination of first integrals and Darboux integrability.
- Jacobi's Last multiplier and its applications.

1.1 Symmetries of Ordinary Differential Equation

The subject of symmetry analysis of ODEs was almost single handedly developed by Sophus Lie [41, 100, 82], in the third quarter of the nineteenth century. Such has been the impact of Lie's seminal work that by and large the methods discovered by him have remained more or less unchanged for over 100 years– a rare event indeed.

A symmetry transformation of an ODE is a transformation which maps the solution set of the equation to itself. This requires that the form of the ODE be invariant under the transformation. The essential concept being a preservation of form or structure.

In general to solve an ordinary differential equation one usually checks whether the ODE belongs to a known class such as the class of linear ODEs or if it can be transformed to a known class by a simple transformation of the dependent and independent variables. If the above techniques fail, one can try more elaborate transformations or even an *ad hoc* ansatz to find the solution. There is no guarantee in general that these efforts will succeed and it may happen that the problem remains unsolved. However, it is found that in the majority of cases when an exact solution of an ODE can be found, the underlying property is a symmetry of that equation. Therefore symmetry analysis of differential equations is of fundamental importance and its applications extend to several areas of physical and chemical sciences including crystallography as also within different branches of mathematics itself. This importance is further enhanced by the fact that according to the highly successful physical theories of the 20-th century all physical interactions (including gravity) act in accordance with an idea (the 'gauge connection') which depends crucially upon certain physical structures possessing an exact symmetry at the fundamental level of description.

Investigations of point symmetries of ordinary differential equations were principally motivated by physical problems, often involving linear equations [1]. In [61, 62] Mahomed and Leach showed that the number of point symmetries which a second-order equation can posses is exactly one of 0,1,2,3 or 8. If a nonlinear second-order equation has eight symmetries, which can only mean that its underlying algebra is the $sl(3, R)$ algebra, then it is linearizable via a point transformation. Consequently a second-order equation with eight

symmetries actually belongs to the class of a free particle equation [91].
For first-order ODEs, Lie [56] showed how to construct an integrating factor from each admitted point symmetry. Conversely, he also showed how each integrating factor could yield an admitted point symmetry. Furthermore if the system of ODEs is self-adjoint, then its integrating factors are necessarily solutions of its linearized system. Such solutions are the symmetries of the given system of ODEs. On the other hand if the system is not self-adjoint, then its integrating factors are necessarily solutions of the adjoint of its linearized system. Such solutions are known as *adjoint symmetries* of the given system of ODEs [36, 89, 90]. Anco and Bluman [2] have introduced an *adjoint-invariance condition* which is a necessary and sufficient condition for an admitted adjoint symmetry to be an integrating factor. They also present an explicit formula for the first integral corresponding to each integrating factor.

1.2 Linearization of Differential Equations

The linearization problem for a general second-order ordinary differential equation was first solved by Lie [58] who deduced the relevant criterion. Tresse [104] also worked on the same problem and deduced similar criterion for linearization in terms of relative invariants of the equivalence group of point transformation of the form

$$(t, x) \mapsto (T, X) \quad \text{where } T = G(t, x), X = F(t, x).$$

E. Cartan [7], on the other hand, studied the problem from a differential geometric point of view.

In recent times a generalization of the above methods involving nonlocal transformations has been achieved by means of which a given ODE can be linearized. Its basis can be traced to the pioneering techniques first employed by Sundman [103] in the context of celestial mechanics. The most general conditions under which a second-order ODE is transformable to the free particle equation (i.e. $X''(T) = 0$) by means of a generalized Sundman transformation of the form $X(T) = F(t, x)$ and $dT = G(t, x)dt$ were obtained by Duarté *et al* [25]. Here F and G are assumed to be smooth functions. Thereafter by using the fundamental invariants of this equation they obtained the first integrals of second-order ODEs, which could be linearized. The case of the general anharmonic oscillator was treated by Euler and Euler [27, 28, 29] who also investigated the Sundman symmetries of second-order and third-order nonlinear ODEs. It is important to note that these symmetries which are in general nonlocal can be calculated in a systematic manner and may be used to determine the first integrals of the equations as will be illustrated later on in this work.

Finally it must be mentioned that Meleshko *et al* [43, 64, 97] have also addressed the issue of linearization for coupled second and third-order ODEs by using a technique which gives more general linearization criteria then the usual linearization *via* point transformations. The process has also been extended to fourth-order ODEs in [42].

1.3 The Darboux Method of Integrability

It is well known that for a two dimensional system the existence of a first integral completely determines its phase portrait. It is well known that such systems, do not exhibit chaos because of the Poincaré-Bendixson theorem [40]. According to this theorem, for a two dimensional system of ordinary differential equations, which is real analytic and defined in a simply connected domain, any *compact* limit set of the system is either a fixed point, a cycle or a union of fixed points and connections, i.e. a polycycle. In three dimension this is no longer true. In the case of non-planar systems, the problem of determining first integrals is a non-trivial task in general, and various methods have been introduced for studying the existence of such first integrals. However, except for some special cases [22, 34, 35, 38, 39] there are few known satisfactory general methods for their determination. In 1878 Darboux [21] initiated the theory of planar polynomial differential systems and his work provided a link between algebraic geometry and the search of first integrals. He demonstrated how to construct first integrals of polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$. The extension of Darboux theory of integrability to polynomial systems in $\mathbb{R}^n$ and $\mathbb{C}^n$ (for $n \geq 3$) was given by Jouanolou [48]. This yielded the notion, of what is today known as Darboux integrability (cf. [13, 15, 16, 37]). Research in this area which lies at the crossroads of ODE theory with algebraic geometry and differential algebra, has deep implications for the problem of the *center*, as well as, for Hilbert's 16th problem on limit cycles. In an interesting survey, Schlomiuk [94] has described the early ideas of Darboux and related them to the influential paper of M. J. Prelle and M. F. Singer [87].

The existing literature on planar differential systems is indeed vast and has a rich history [14]. Moreover as the existence of first integrals leads to a reduction of the order of the ODEs under consideration it constitutes an important ingredient of their analysis.

A *first integral* of a system of ODEs is any non-constant globally differentiable function $I(t, x_1, \ldots, x_n)$ that retains a constant value on any integral curve of the system. This means its derivative with respect to the independent variable t must vanish on the solution curves. In many cases, the determination of a first integral is considerably simplified by the existence of what are known as *second integrals*. The polynomial second integrals for polynomial vector fields are called *Darboux polynomials* (monic irreducible polynomials) and their importance stems from the fact that the computation of a rational first integral $I = F/G$ for a polynomial vector field D is actually equivalent to the computation of its Darboux polynomials.

A polynomial system is said to be *Darboux integrable* if it possesses a first integral or an integrating factor given by Darboux polynomials. In particular, Darboux showed (see for example [13]) that a polynomial system of degree n with at least $n(n+1)/2+1$ invariant algebraic curves has a first integral which can be expressed by means of these algebraic curves. This theory of integrability also received contributions from the work of Poincaré, whose main interest was on rational first integrals.

Darboux showed that one can construct an integrating factor (and first integrals) of pla-

nar polynomial differential systems if there exists a sufficient number of invariant algebraic curves (real or complex).
Therefore, invariant algebraic curves are the key elements of the Darboux method. In 1979 Jouanolou [48] showed that if the number of invariant algebraic curves of a polynomial system of degree d is at least $[n(n+1)/2]+2$, then the vector field has a rational first integral and, in particular, all its solutions are algebraic curves.

The knowledge of algebraic curves can also be used to study the topological properties of the system, for example, the inverse of the integrating factor must be null over all the limit cycles which are isolated periodic orbits contained in the plane.

1.3.1 Planar differential systems

As one of our chief interests is in the integrability of a planar polynomial differential system we have made substantial use of the notion of Darboux polynomials and their determination by means of the Prelle-Singer semi algorithm to deduce first integrals of planar differential systems. A major step towards the construction of an algorithm for solving first order ordinary differential equations (ODEs) was put forward by Prelle and Singer [87]. This is a semi-algorithmic procedure for solving nonlinear first-order ordinary differential equations. The Prelle-Singer method [87] provides the form of the integrating factor when the solution of the associated system of differential equations is expressible in terms of elementary functions. Their paper has had a profound influence and has provided some of the fundamental algebraic results required for the automated solution of ODEs using computer algebra. Therefore, it has motivated many extension of the original idea [18, 63, 95]. An extension of their method provides the form of an integrating factor when the solution is expressible in terms of Liouvillian functions. In 1992, Singer [96] derived the form of the integrating factor when a polynomial system has a Liouvillian first integral. On the other hand, Christopher in [14] provided the Liouvillian first integrals of second-order polynomial differential equations. In an interesting paper Llibre and Pereira defined four different kinds of multiplicity of an invariant algebraic curve for a given polynomial vector field and investigated their relationships [59]. The introduction of the notion of multiplicity has led to an extension of the classical Darboux theory of integrabity. Duarte *et. al.* present a semi-decision procedure to tackle first-order ODEs with Liouvillion function in the solution [26].

In an interesting paper Man and MacCallum [63] have formulated a method to compute the elementary solutions of the second order differential equations, which shares certain commonalities with the Darboux and Prelle-Singer procedures and is quite useful in dealing with certain non-planar polynomial differential systems.

The Prelle-Singer method was extended by Duarte et al. [23, 24] to second-order ODEs. Recently the theory has been further generalized by Lakshmanan and his coworkers [8, 9, 10] to obtain rational first integrals and the general solution of a specific class of nonlinear second and third-order equations. Their work provides a systematic way for identifying integrable cases as well as constructing the integrating factors, integrals of motion and general solutions of second-order nonlinear ordinary differential equations.

1.3.2 Non-planar dynamical systems

Non planar dynamical systems are frequently encountered not only as theoretical idealizations of physical processes, such as rigid body dynamics but in concrete real life situations involving diverse phenomena ranging from atmospheric sciences (Lorenz system) to studies involving human physiology (Hindmarsh-Rose model) besides others. Compared to planar dynamical systems the situation in terms of available methods for their analysis is even more limited for non planar systems of ODEs.

The problem of integrating a set of ODEs describing a given non planar dynamical system is compounded by a number of hurdles such as the equations being in general nonlinear, usually involving several degrees of freedom and in addition are more likely to be coupled. In fact except for the Painlevé criteria there are no concrete tests for deciding whether a given dynamical systems is integrable or not. The Painlevé criteria requires that the solution of the differential equation posses no movable (i.e., initial condition dependent) singularities other than poles in the complex time plane [99]. Equations which satisfy this condition are said to possess the *Painlevé property* and the latter property has been used to identify new integrable Hamiltonian systems as well as integrable cases of non-Hamiltonian systems such as Lorentz equations, the Lotka-Voltera system etc. Generally dynamical systems described by coupled nonlinear ordinary differential equations are non Hamiltonian in character and describe the time evolution of physical processes which are mainly dissipative in nature. Consequently in course of their evolution the phase space volumes contract and the motion is often attracted by fixed points or even periodic orbits.

However, a majority of nonlinear systems which depend on one or more parameters often exhibit a range of values of their parameters for which the solution approaches a much more complicated type of attractor. These are subsets of the phase space with a cantor-like structure, called *strange attractors*, on which the motion is widely chaotic in the sense that it depends sensitively on the choice of initial conditions.

The self generated chaotic behavior of certain dynamical systems constitutes one of the most interesting areas in the study of dynamical systems. In fact most nonlinear systems are non-integrable and posses large classes of solutions with truly random properties. The identification of integrable dynamical systems and the determination of the size of the chaotic region of the corresponding phase space is therefore a major problem and often gives rise to exciting new results. The existence of first integrals in the study of non planar dynamical systems is extremely important since it greatly simplifies their analysis especially in the infinite time limit. Some of the existing methods for finding first integrals for non planar systems of ODEs are enumerated below.

1. The Frobenius integrability theorem provides a direct approach for finding first integrals for non-planar ordinary differential equations where the emphasis is on determining the values of the parameters for which the system admits a first integral. By employing this technique new constants of motion have found for the Lotka-Voltera and certain other systems [4, 101].

2. An alternative procedure is to assume a specific ansatz for the first integrals. The usual ansatz is to choose a polynomial of suitable degree in the phase space coordinates. This procedure was employed by Kus [54] for the determination of new constants of motion for the Lorentz model which does not fulfil the Painlevé criteria.

3. A variation of the above procedure due to Giacomini *et.al.* [33] also employs an ansatz for the first integral, but it is of a more general character. Here a polynomial in one of the coordinates of the system of suitable degree is first chosen whose coefficients are unknown functions of the remaining coordinates. These functions must satisfy an over determined set of partial differential equations and are consistent only for certain particular values of the parameters of the system.

1.4 The Jacobi Last Multiplier method

Carl Gustav Jacob Jacobi was a prolific mathematician who is often considered as the successor to the venerable Gauss. Although today Jacobi is known mostly for his contributions to the theory of elliptic functions, his work on differential equations and most notably on the theory of the last multiplier may only be ranked second in importance to his investigations in elliptic functions.

The earliest reference to Jacobi's last multiplier (JLM) can be traced to the summary of his lecture at the Congress of Italian Scientists in Lucca in 1843, which appeared under the English title "On the principle of the last multiplier and its use as a new broad principle of mechanics" in 1844 [45]. A more detailed account of the last multiplier along with its applications in Classical Mechanics appeared in his lectures on dynamics, published posthumously after his death in 1851 [46].

Although some studies on the last multiplier were carried out in various contexts, the most significant development thereafter was the discovery by Lie that there exists a close relationship between the JLM and Lie Symmetries. This promptly led to a keen interest in the last multiplier especially with regard to its relationships with first integrals of differential equations, its possible generalization, deductions of nonlocal Lie symmetries and also to its role in determining a Lagrangian for systems of second-order ODEs. Indeed the last feature seems to have been forgotten with the passage of time. Although a very clear and concise description of the JLM in the context of its role in reducing a systems of first-order ODEs to quadrature is given in Whittaker's book on analytical dynamics [105], the precise nature of the relationship between the JLM and the Lagrangian of a system of second-order ODEs is best dealt with in an article titled " On the reduction of dynamical equations to the Lagrangian form" by Madhava Rao published in 1940, in which he considers a Lagrangian which is a quadratic form in the generalized velocities. For a single second-order ODE $y'' = w(x, y, y')$ which admits a Lagrangian function $L(x, y, y')$ the Jacobi Last Multiplier, M, is given by $M = \frac{\partial^2 L}{\partial y'^2}$.

The inverse problem of the calculus of variations has always been of interest to physicists and mathematicians alike in their efforts to obtain a Lagrangian description of second and higher-order ODEs. In recent years there has also been a great deal of interest in dissipative dynamical systems, in studying the existence of limit cycles etc. The importance of the JLM stems from the fact that it plays an important part in all these problems besides providing a very simple yet powerful mechanism for finding a Lagrangian of any second-order differential equation. This together with its connection to Lie Symmetries [56, 57] makes it an ideal tool for probing various properties of ODEs as will be described later on in the work. In fact recently Leach and Nucci derived Lagrangians for many second-order differential equations using Jacobi's last multiplier [76, 77, 78, 79, 80] which provided in a sense the impetus for our investigations.

1.5 Outline of the Thesis

The present work consists of nine chapters including the Introduction. Chapter 2 begins with a brief survey of some of the elementary properties of ordinary differential equations before moving on to the basic principles of Darboux integrability and applications of the latter in determining first integrals for planar and nonplanar systems. This is followed by a short review of fixed point and phase space analysis of nonlinear ODEs.

As is well known symmetry analysis of ODEs is in itself a vast subject and even though any attempt to encapsulate its essential tools and techniques within the confines of a single chapter is bound to fail, we have nevertheless endeavoured to explain some of the key features of Lie symmetries which are relevant for understanding some of our subsequent results. This is followed by a discussion of the linearization problem of ODEs under point and nonpoint transformations. The chapter ends with an exposition of some of the fundamental properties of Jacobi's last multiplier which has made an appearance in the recent mathematical literature after a gap of almost a century. Much of the material presented in this chapter is in the nature of a review serving as a backdrop for the later chapters.

In Chapter 3 we address the problem of finding first integrals of the Painlevé-Gambier equations as well as of an equation of the Liénard type. The method employed for this purpose is based on an extension of the Prelle-Singer technique as formulated by Chandrasekhar et al. The systems studied also include two dimensional problems such as the Kepler problem in 2D. This is followed by an attempt to construct, using Darboux polynomials, first integrals of a non planar system of equations closely related to the Raychaudhuri equation in cosmology.

Chapter 4 introduces the notion of adjoint symmetries and investigates their role in the determination of first integrals of nonlinear ODEs. A concise review of the extended Prelle-Singer method is then furnished in order to unfold its relationship with the classical adjoint symmetry equation. The relative advantages of these two approaches are illustrated with a number of examples.

The concept of λ-symmetries has recently gained a lot of importance as they are deemed to be, in some sense, generalization of Lie symmetries. Chapter 5 investigates the relation between adjoint symmetries and λ-symmetries from the standpoint of finding integrating factors of ODEs. It then explains how λ-symmetries may be found for some second-order equations of the Painlevé-Gambier classification and ends by considering such symmetries for a few third-order ODEs.

Chapter 6 is devoted to a description of generalized Sundman transformations (GST) which are of a nonlocal character. After introducing the notion of a generalized Sundman transformation we define the associated Sundman symmetry and illustrate these concepts in the general case of the Jacobi equation . Explicit results are provided for several equations of the Painlevé-Gambier classification from the perspective of the GST and their associated Sundman symmetry including also their solution. New first integrals for some of these as well as other equations of the Painlevé-Gambier class.

An extension of the GST, in which both the independent and dependent variables transform in a completely nonlocal manner, is considered in Chapter 7. We illustrate the effectiveness of such nonlocal transformations for computation of first integrals of a generalized time-dependent Riccati equation after which a similar analysis is undertaken for certain third-order ODEs.

The first section of Chapter 8 contains a delineation of the relationship between the Jacobi Last Multiplier (JLM) and the Lagrangian of a second-order ODE. It includes a deduction of the Lagrangians for the six Painlevé equations and also their corresponding Hamiltonians. It also briefly outlines the procedure for other equations of the Painlevé-Gambier classification. This is followed by a similar analysis for second-order equations of the Liénard type in section 3 which also includes a specific instance of a generic equation of nonlinear oscillator type. Finally in section 4 we apply the technique to a number of systems of coupled second-order ODEs.

The work ends with a modest outlook in Chapter 9.

Finally we consider it pertinent to add a few remarks on the Painlevé-Gambier equations which have a rather interesting origin and frequently appear in many of the examples presented here. The fundamental problem which Painlevé, Gambier Fuchs addressed was a question first raised by Picard concerning second-order first-degree ODEs of the form $w'' = F(z, w, w')$ where F is rational in w', algebraic in w and locally analytic in z and having the property that all movable singularities of all solutions are poles. A differential equation is said have the Painlevé property if all solutions are single valued around all movable singularities. Within the Mobius transformation, Painlevé and his school found 50 such equations which are classified in Ince's book [44]. Among all these equations six of them are irreducible and define the classical transcendents $P_I, P_{II}, , P_{VI}$ while the remaining 44 equations are either solvable in terms of known functions or can be transformed into one of the six Painlevé equations. These equations may be regarded as the nonlinear counterparts of the classical special equations. For example P_{II} has solutions which has similar properties as Airy's functions [17].
Although the Painlevé equations were discovered from strictly mathematical considerations they have appeared in many physical problems including statistical mechanics, random matrices, plasma physics, nonlinear waves, quantum gravity, general relativity and nonlinear optics. Recently there has been considerable interest in the Painlevé equations as they arise as reductions of the soliton equations which are solvable by inverse scattering.

1.6 Summary of findings

1. Using the extended Prelle-Singer method we have derive a new first integral for the equation XXII of the Painlevé-Gambier class and have obtained a formula for deducing the known first integrals, of a particular form, belonging to this classification.

2. Existence of first integrals for a Generalized Raychaudhuri equation arising in modern string inspired cosmology, We described employing the notion of Darboux theory of integrability for polynomial ODEs.

3. The exact nature of the relationship between the so called extended Prelle-Singer method and the classical adjoint symmetry equation of the symmetry analysis has been unravelled. It is found that the extended Prelle-Singer method is actually a decomposition of the adjoint symmetry equation into a system first-order ODEs.

4. We have explored the λ-symmetries of some second-order equations of the Painlevé-Gambier type and have studied their relationship with the standard adjoint symmetry

equation used for determining the integrating factor of a second-order ODE. In Particular have also computed the λ-symmetries of the Painlevé-Gambier equation numbers III, VIII, XIX and XXX and have followed it up with a brief study of the λ-symmetries of certain special types of third-order ODEs.

5. By means of the Generalized Sundman transformation (GST) we have obtained the known and five new first integrals of Painlevé-Gambier equations bearing numbers XI, XVII, XXXVII, XLI and XLII. All the known integrals are time-independent and are identical to those given in Ince's book [44]. The time-dependent integrals appear to be new to the literature. We have also computed the Sundman symmetries of these equations as well as their solutions.

6. In order to generalize the notion of a GST considered a nonlocal transformations of the form

$$dX = A(x,t)dx + B(x,t)dt,$$
$$dT = C(x,t)dx + D(x,t)dt$$

and have shown that this transformation provide us with an effective tool for the determination of a first integral in several particular cases. Exploiting the above transformation we have deduce a time-dependent first integral for a generalized second-order nonautonomous Riccati differential equation. Finally, we have considered applications of the method to third-order time-dependent ODEs.

7. A fairly extensive account of the Jacobi Last multiplier is presented in this work. As an application of this concept to Lagrangian and Hamiltonian of the Six Painlevé transcendents using the Last multiplier. For the Liénard equation by means of a novel transformation we have shown how the JLM may be used to find the Lagrangian. Further application to coupled second-order system are also included.

Chapter 2

Ordinary Differential Equations

2.1 Introduction

An *ordinary differential equation* (ODE) is an equation of the form

$$F(x,\ y,\ y',\ldots,\ y^{(n)}) = 0. \tag{2.1.1}$$

which relates an independent variable x, the required function $y = y(x)$, and at least one of its derivatives $y', y'', \ldots, y^{(n)}$.
Here F is a specified function of its arguments and the prime denotes the usual derivative.

The *degree* of a differential equation is the power of the highest order derivative appearing in the equation.
The *order* of a differential equation is the order of the highest derivative appearing in the equation.

A *solution* of an n^{th} order ODE on the interval (a, b) is any function $y = \phi(x)$ which has derivatives up to order n inclusive on that interval such that the substitution of the function $y = \phi(x)$ and its derivatives into the differential equation turns the equation into an identity with respect to x on (a, b). The graph of the solution of a differential equation is called an *integral curve* of that equation.

2.1.1 The Cauchy problem

Assume that we have a *first-order differential equation* (FODE)

$$F(x,\ y,\ y') = 0.$$

We can solve this equation for y' when the inverse function of F exist and then the equation is of the form

$$\frac{dy}{dx} = f(x, y). \tag{2.1.2}$$

Here f is a given function of its arguments. To isolate a definite solution of equation (2.1.2), we must specify an initial condition which consist of preassigning for a certain value x_0 of the independent variable x a value y_0 of the required function $y(x)$, i.e.,

$$y|_{x=x_0} = y_0, \quad \text{or} \quad y(x_0) = y_0. \tag{2.1.3}$$

The problem of finding the solution $y(x)$ of equation (2.1.2) which satisfies the initial condition (2.1.3), is known as the *Cauchy problem* for equation (2.1.2).

2.1.2 Existence and Uniqueness of solution for given initial condition:

Given a first-order ordinary differential equation

$$\frac{dy}{dx} = f(x, y), \tag{2.1.4}$$

with $f(x, y)$ satisfying the following conditions:
(i) $f(x, y)$ is continuous in a given region A,
(ii) $|f(x, y)| \leq M$, a fixed real number in A,
(iii) $|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$, k being a fixed quantity for any two points (x, y_1), (x, y_2) in the region A, it can be shown that-
if (x_0, y_0) be any point in A such that the rectangle R given by $|x - x_0| \leq a, |y - y_0| \leq b$, where $b > aM$ is such that R lies wholly within A, then there exist one and only one continuous function $y = \phi(x)$ having continuous derivatives in $|x - x_0| \leq a$, which satisfies the differential equation (2.1.4) and takes up the value $\phi(x_0) = y_0$ when $x = x_0$.
The condition (iii) i.e., $|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$, k being a fixed quantity for any two points (x, y_1) and (x, y_2) in the region A, is known as the *Cauchy-Lipschitz condition*. If $f(x, y)$ admits continuous derivative and hence $|\frac{\partial}{\partial y} f(x, y)| < k$ then the Cauchy-Lipschitz condition is satisfied.
The proof may be found in [44].

2.1.3 Geometrical significance of an ordinary differential equation

Let us consider a differential equation

$$\frac{dy}{dx} = f(x, y) \tag{2.1.5}$$

of first-order and of first degree. The primitive of an ordinary differential of the first-order is a relation between the two variables x and y and a parameter c and the differential equation is said to represent a single parameter family of plane curves. Each curve of the family is said to be an *integral curve*.

In a Cartesian coordinate system the derivative dy/dx, of a curve gives the direction of the tangent to the curve at a given point. Let A be the domain in the (x, y) plane throughout which $f(x, y)$ is single valued and continuous. Let (x_0, y_0) be a point lying in the interior of the domain A. Let the value of dy/dx at the point (x_0, y_0) be m_0. Thus, as a point moves through (x_0, y_0), satisfying (2.1.5), with the direction of movement m_0 it defines a line element (x_0, y_0, m_0). The line element may be defined with sufficient accuracy as the line which joins the point x_0, y_0 and $(x_0 + \delta x, y_0 + \delta y)$ where δx and δy are small and $\frac{\delta y}{\delta x} = m_0)$. After that, if the point moves to the point (x_1, y_1) at an infinitesimal distance on this line element one can construct anther line-element (x_1, y_1, m_1). Continuing this process a broken line is obtained which may be regarded as an approximation to the integral curve which passes through (x_0, y_0).

Since it has been assumed that $f(x, y)$ is continuous and single valued at every point of A, it follows that through every point there will pass one and only one integral curve. Outside the region A there may be points at which $f(x, y)$ may be continuous and single valued. Such points are known as *singular points* and the behaviour of the integral curves may be exceptional.
Similarly, for a second-order differential equation the aggregate of integral curves will form a two parameter family.

2.1.4 Integrating Factors

Given an ODE in the form

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}, \tag{2.1.6}$$

the problem of finding its solution basically involves a process of separating the variables to rewrite the equation as $f(x)dx + g(y)dy = 0$. Once this is achieved the equation may be reduced to quadrature. One can achieve separation by multiplying the equation with a suitable function of x and y such that the equation becomes exact, as will be explained below.
From (2.1.6) we have

$$Pdx - Qdy = 0. \tag{2.1.7}$$

We assume (2.1.7) is not exact. Then the problem of integrating such an equation essentially requires finding a function $\mu(x, y)$ such that the expression

$$\mu(Pdx - Qdy) \equiv du$$

i.e., it is a total differential. It follows therefore that, $\partial u/\partial x = \mu P$ and $\partial u/\partial y = \mu Q$. The condition for equality of the mixed derivative then gives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{2.1.8}$$

The function $\mu(x, y)$ is called an *integrating factor* of the differential equation and is non unique.

Theorem 2.1.1 *The number of integrating factors of an equation which has a solution, is infinite.*

Proof:
Let $\mu(x, y)$ be an integrating factor of the equation (2.1.7), so that

$$\mu(Pdx - Qdy) = du.$$

Hence $u(x, y)$ is a solution of the equation. If $f(u)$ be any function of u, then

$$\mu\ f(u)(Pdx - Qdy) = f(u)du.$$

Now, the right hand expression is an exact differential since $f(u)du$ can easily be integrated to give $\phi(u)$. Thus the solution of the equation is

$$\phi(u) = c,$$

showing that $\mu f(u)$ is also an integrating factor of the equation (2.1.7). Since $f(u)$ is an arbitrary function of u the number of integrating factor is infinite.

2.1.5 Elementary rules for finding integrating factors

In many cases the integrating factors for an ODE can be found by inspection. When the integrating factor cannot be found by inspection, the following elementary methods are used to find it. Consider the differential equation $Pdx + Qdy = 0$ in which $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, so that the equation is not exact. The rules for finding integrating factors may be summarized as follows:

Rule I: If $Px + Qy \neq 0$ and the equation is homogeneous, then $1/Px + Qy$ is an integrating factor of the equation $Pdx + Qdy = 0$.

Rule II: If $Px - Qy \neq 0$ and the equation can be written as $\{f(x, y)\}ydx + \{g(x, y)\}xdy = 0$, then $\frac{1}{Px - Qy}$ is an integrating factor of the equation $Pdx + Qdy = 0$.

Rule III: If $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ be a function of x alone, say $f(x)$ then, $e^{\int f(x)dx}$ is an integrating factor of the equation $Pdx + Qdy = 0$.

Rule IV: If $\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ be a function of x alone, say $g(y)$ then, $e^{\int g(y)dy}$ is an integrating factor of the equation $Pdx + Qdy = 0$.

Rule V: If the equation be of the form

$$x^a y^b(mydx + nxdy) = 0,$$

with a, b, m and n being constants, then $x^{km-a-1}y^{kn-b-1}$, where k has any value, is an integrating factor of the equation.
However, there are many ODEs for which an integrating factor cannot be found by the above rules. In such cases other procedures have to be adopted. One of the main objectives of this section is to describe how one can deduce an integrating factor for a system of ODEs using the notion of Darboux polynomials which is now described.

2.2 The Method of Darboux Integrability

2.2.1 Planar differential equation

Consider a system of two first-order ODEs of the form

$$\frac{dx}{dt} = Q(t, x, y), \frac{dy}{dt} = P(t, x, y). \tag{2.2.1}$$

A solution of (2.2.1) namely $x = x(t), y = y(t)$, assuming the values $x(0), y(0)$ at $t = t_0$ say, defines in space a certain curve, which passes through the point $P_0(t_0, x(0), y(0))$ and is an *integral curve* of the system (2.2.1).

In geometrical terms the Cauchy problem amounts to finding the integral curve of (2.2.1) passing through the given point P_0. An alternative interpretation of the solution of (2.2.1), treats t as a parameter and $x = x(t), y = y(t)$ as the parametric equation of a curve in the $x - y$ plane called the *phase plane*. The projection of the integral curve on the phase plane, then gives the trajectory of the system. However, while from the integral curve one can define the phase trajectory uniquely, the converse is not true in general.

If the right hand side of (2.2.1) is not explicitly dependent on t then the system is said to be *autonomous* otherwise it is called a *non-autonomous* system.

2.2.2 Planar Polynomial Differential Equations:

For a planar system of ODEs

$$\frac{dx}{dt} = Q(x, y), \quad \frac{dy}{dt} = P(x, y). \tag{2.2.2}$$

(Q, P) represents the components of the tangent vector at the point (x, y). We define the vector field D at a point (x, y) by

$$D := Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} \tag{2.2.3}$$

where $\partial/\partial x$ and $\partial/\partial y$ represent the directional derivatives along the X and Y axes respectively. When P and Q are polynomials we say that D is a polynomial vector field of degree d on $\mathbb{C}[x, y]$ if the maximum degree of the polynomials $P(x, y)$ and $Q(x, y)$ is d.

2.2.3 Invariant Algebraic Curve:

Definition 2.2.1 ***(Invariant curve):*** *The curve $f(x, y) = 0$ is said to be an invariant curve if $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and (Q, P) are orthogonal over the curve $f(x, y) = 0$, i.e.,*

$$\frac{df}{dt} = (Q\frac{\partial f}{\partial x} + P\frac{\partial f}{\partial y})\Big|_{f=0} = 0.$$

Definition 2.2.2 ***(Algebraic curve):*** *An invariant curve $f(x, y) = 0$ is called an algebraic curve of degree m when $f(x, y)$ is a polynomial of degree m.*

Definition 2.2.3 ***(Invariant algebraic curve):*** *Let D be the vector field associated with a planar differential system. A curve $f(x, y) = 0$ is an invariant algebraic curve if $D[f]/f$ is a polynomial. The latter polynomial $\lambda_f = D[f]/f$ is usually called the cofactor of the invariant algebraic curve.*

2.2.4 First Integral

Next we introduce the notion of invariants for a system of differential equations. These invariants are usually called first integrals or constants of motion.

Definition 2.2.4 *A first integral of the system of ODE's*

$$\frac{dx_i}{dt} = X_i(t, x_1, \ldots, x_n), \qquad i = 1, \cdots, n, \tag{2.2.4}$$

is any non-constant globally differentiable function $\Phi(t, x_1, \ldots, x_n)$ that retains a constant value on any integral curve of the system.

This means its derivative with respect to t vanishes on the solution curves, i.e.,

$$\frac{d\Phi}{dt} = 0 \Rightarrow \sum_i \frac{\partial \Phi}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial \Phi}{\partial t} = 0 \Rightarrow \widetilde{D}[\Phi] = 0, \tag{2.2.5}$$

where $\widetilde{D} := \sum_i X_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}$. For an autonomous system this reduces to the following condition

$$\sum_i X_i \frac{\partial \Phi}{\partial x_i} = 0, \tag{2.2.6}$$

where $D = \sum_i X_i \frac{\partial}{\partial x_i}$ is just the vector field associated with the given autonomous system. In many cases, the determination of a first integral is considerably simplified, by the existence of what are known as *second integrals.*

Definition 2.2.5 *A second integral of a vector field D is a C^1 function, $f = f(x_1, \cdot, x_n) : \mathbb{K}^n \to \mathbb{K}$ such that $D[f] = \lambda f$ where $\lambda = \lambda(x_1, \ldots, x_n) : \mathbb{K}^n \to \mathbb{K}$.*

Here $\mathbb{K}$ be a field of characteristic zero[1] where $\mathbb{K}$ is real or complex, and for our purposes may either be $\mathbb{R}$ or $\mathbb{C}$. Let us introduce next the notion of *Darboux polynomials.*

Definition 2.2.6 *The polynomial second integrals for polynomial vector fields are called Darboux polynomials (monic irreducible polynomials).*

It is important to note in this context that the computation of a rational first integral $I = F/G$, for a polynomial vector field D is equivalent to the computation of its Darboux polynomials.

Proposition 2.2.1 *Let $F, G \in \mathbb{K}[x]$ be two relatively prime polynomials, then $I = F/G$ is a rational first integral of D_f iff there exist $g \in \mathbb{K}[x]$ such that*

$$D[F] = gF \quad and \quad D[G] = gG. \tag{2.2.7}$$

Proof: Let $I = F/G$ be a first integral of D, then $GD[F] = FD[G]$. Therefore, G divides $FD[G]$. Since F and G are relatively prime polynomials, $D[G] = gG$ and hence $D[F] = gF$.

Conversely, if $D[F] = gF$ and $D[G] = gG$, then $GD[F] - FD[G] = gGF - gFG = 0$ and F/G is a first integral.

Proposition 2.2.2 *Let P_1 and P_2 be Darboux polynomials then*
(i) P_1P_2 is a Darboux polynomials.
(ii) All irreducible factors of Darboux polynomials are also Darboux polynomials.

Proof: Let $D[P_1] = g_1P_1$ and $D[P_2] = g_2P_2$, where $g_1, g_2 \in \mathbb{K}[x]$. Then we have $D[P_1P_2] = (g_1 + g_2)P_1P_2$. Therefore P_1P_2 is Darboux polynomial.

Conversely, Let $P_1 = Q_1^rQ_2$ with Q_1 and Q_2 being relatively prime polynomials and Q_1 is irreducible. Then

$$rQ_1^{r-1}D[Q_1] - Q_1^rD[Q_2] = gQ_1^rQ_2.$$

Since $Q_1^rQ_2$ divides $rQ_1^{r-1}D[Q_1] - Q_1^rD[Q_2]$ and Q_1 is relatively prime with Q_2, Q_1 must divide $D[Q_1]$ and also Q_2 must divide $D[Q_2]$.
By induction on Q_2, all reducible factors of P_1 are Darboux polynomials.

[1] A field with multiplication $*$ is of characteristic zero if the only element a of the field such that $a * b$=0 for all b is a=0

2.2.5 Elementary First Integrals

By an elementary first integral we mean a first integral involving elementary functions only which for the present purpose may be roughly defined as follows.

Definition 2.2.7 *A function $F(x_1, \cdots, x_n), \in \mathbb{C}^n$ is said to be elementary if it belongs to the set S, which in turn is obtained from rational functions on $\mathbb{C}^k, k = 0, 1, ...,$ using a finite series of the following operations: (a) algebraic operations such as addition, subtraction, multiplication and division, (b) solution of algebraic equations, (c) derivations and (d) exponential and logarithmic operations.*

Note that if in addition we include the operation of integration, then S becomes the set of Liouvillian function.

2.2.6 An Algorithm for finding a Polynomial First Integral

Proposition 2.2.3 *An n-dimensional polynomial vector field of degree d may depend upon a certain number of parameters $(\lambda_1, \lambda_2, \ldots, \lambda_p)$. The problem is to determine the values of $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ such that the vector field admits a time independent polynomial first integral of a given degree N.*

Proof: Step 1: Let us start with degree N=1 for the first integral $I(x)$.
Step 2: The most general form of a first integral $I(x)$ of degree d is,

$$I(x) = \sum_{i=1}^{N} c_i x_i.$$

Step 3: We compute the time derivative of $I(x)$:

$$D[I] = \left(Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}\right)[I] = \left(Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}\right)\left(\sum_{i=1}^{N} c_i x_i\right) = \sum_{i=1}^{N+d-1} k_i x_i.$$

Step 4: Since I is a first integral, $D[I] = 0$, it implies $k_i = 0$. This system of equations is a linear system for the coefficients c_i of dimension at most $\begin{pmatrix} n+d+N-1 \\ n \end{pmatrix}$. So if there exist values of the parameters $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ and a set of constants c_i that are not all zero such that $k_i = 0$ for all i, then $I(x)$ is a first integral. Otherwise, we increase the value of N by 1 and return to the step 2.

Example 2.2.1 *Consider the system*

$$\dot{x} = 2z - 2x^2 \tag{2.2.8}$$

$$\dot{y} = -3xy \tag{2.2.9}$$

$$\dot{z} = 4xz - 2x(2x^2 - 9y^2). \tag{2.2.10}$$

A first integral for this system using the above algorithm is found to be given by

$$I = z - x^2 + 3y^2$$

2.2.7 The method of Darboux polynomials

Let us consider the planar polynomial differential system (2.2.2) where $P(x,y) = \sum_{i=0}^{m} P_i(x,y)$ and $Q(x,y) = \sum_{i=0}^{m} Q_i(x,y)$ are co-prime polynomials in $\mathbb{C}$ such that $\max\{deg\,P, deg\,Q\} = m$ and $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous components of degree i. The system may be described by the vector field (2.2.3) or the differential one form

$$\omega = Pdx - Qdy.$$

The corresponding phase-flow is given by the solutions of the first-order ordinary differential equation

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}. \tag{2.2.11}$$

Suppose the vector field admits d distinct invariant algebraic curves $f_i,\ i = 1, ..., d$.

(a) If there are $n_i \in \mathbb{C}$, not all zero, such that

$$\sum_{i=1}^{d} n_i \lambda_i = 0 \text{ then the function } \prod_{i=1}^{d} f_i^{n_i}$$

is a *first integral* of the vector field D.

(b) If there exists $n_i \in \mathbb{C}$ not all zero, such that $\sum_{i=1}^{d} n_i\lambda_i = -\mathrm{div}D$, then $\prod_{i=1}^{d} f_i^{n_i}$ is an *integrating factor* of D.
These results form the essential content of the *Darboux theory of integrability.*
Proof: We shall prove the second part of the proposition first. If $R(x,y)$ be an integrating factor of (2.2.11), then

$$RP\ dx - RQ\ dy = 0 \text{ and } (RP)_y = -(RQ)_x, \tag{2.2.12}$$

where the subscripts denote partial derivatives. The latter may be written as

$$D[R] := (Q\partial_x + P\partial_y)[R] = -\mathrm{div}(Q,P)R = -(Q_x + P_y)R. \tag{2.2.13}$$

This follows from the observation that the problem of determining the integrating factor R is essentially one of finding a solution of the linear partial differential equation (2.2.13). This being a first order PDE, we write the associated Lagrange one form:

$$\frac{dx}{Q} = \frac{dy}{P} = \frac{dR}{-(Q_x + P_y)R} \tag{2.2.14}$$

Let there exist a certain number of Darboux polynomials f_i satisfying

$$D[f_i] = \lambda_i f_i \tag{2.2.15}$$

where λ_i are suitable eigen polynomials. From (2.2.14) we have

$$\frac{dx}{Q} = \frac{dR}{-(Q_x + P_y)R} = \sum_i n_i \frac{f_{ix}}{f_i} dx = \sum_i n_i \frac{Q f_{ix} dR}{-(Q_x + P_y) f_i R} \tag{2.2.16}$$

where n_i's are some rational numbers. Similarly multiplying by f_{iy}/f_i we have

$$\frac{dy}{P} = \frac{dR}{-(Q_x + P_y)R} \Rightarrow \sum_i n_i \frac{f_{iy}}{f_i} dy = \sum_i n_i \frac{P f_{iy} dR}{-(Q_x + P_y) f_i R}. \tag{2.2.17}$$

Adding (2.2.16) and (2.2.17) we obtain

$$\sum_i n_i \frac{f_{ix} dx + f_{iy} dy}{f_i} = \sum_i n_i \frac{D[f_i]}{-f_i(Q_x + P_y)} \frac{dR}{R}. \tag{2.2.18}$$

But f_i's being Darboux polynomials we have

$$\sum_i n_i \frac{df_i}{f_i} = \frac{\sum_i n_i \lambda_i}{-(Q_x + P_y)} \frac{dR}{R}$$

Let us demand that the rational numbers n_i are such that

$$\sum_i n_i \lambda_i = -(Q_x + P_y), \tag{2.2.19}$$

whence it immediately follows

$$\sum_i n_i \frac{df_i}{f_i} = \frac{dR}{R} \Rightarrow \quad R = \prod_i f_i^{n_i} \tag{2.2.20}$$

Thus, whenever the ODE has an elementary first integral, its corresponding integrating factor may be expressed in the above form.

If we can identify a sufficient number of Darboux polynomials f_i satisfying

$$D[f_i] = \lambda_i f_i \tag{2.2.21}$$

where λ_i are suitable polynomials, then

$$\frac{D[R]}{R} = \sum_i n_i \frac{D[f_i]}{f_i} = -(Q_x + P_y). \tag{2.2.22}$$

Clearly Q_x, P_y are polynomials since Q, P are themselves polynomials; and therefore it is necessary that $f_i | D[f_i]$. Therefore,

$$\frac{D[R]}{R} = -\text{div} D. \tag{2.2.23}$$

Furthermore when $\text{div} D = 0$ then $D[R] = 0$ so that R is a first integral.

2.2.8 Extension of the Darboux method

In the following we describe an extension of Darboux's method which allows us to find an exponential integrating factor for an ODE. The integrating factor of some ODEs consists of an exponential factor which cannot be found by the method just described. However, the notion of Darboux polynomials continues to play a pivotal role in the determination of even exponential integrating factors as we shall now show.

Definition 2.2.8 *Let $e = exp\left(\frac{M}{N}\right)$ where M and N are co-prime polynomials in $\mathbb{R}^2$. If $D[e]/e = L_e$ then e is said to be an exponential factor of the vector field D of degree d when $D[e]/e$ is a polynomial of degree at most $(d-1)$. The polynomial L_e is called the cofactor of the exponential factor e.*

Proposition 2.2.4 *If the vector field D admits q exponential factors $e_1, \cdots, e_q$ and r algebraic inverse curves $f_j, (j = 1, \cdots, r)$ then $R = \prod_{i=1}^{q} e^{m_i} \prod_{j=1}^{r} f_j^{n_j}$ is an integrating factor of the differential equation $Pdx + Qdy = 0$ if*

$$\frac{D[R]}{R} = \sum_{i=1}^{q} m_i L_{e_i} + \sum_{i=1}^{q} n_j L_{f_j} = -(divD)$$

where L_{f_j} is the co-factor of f_j

Proof: Let $R = \prod_{i=1}^{q} e^{m_i} \prod_{j=1}^{r} f_j^{n_j}$ Therefore,

$$\frac{D[R]}{R} = D\left[\frac{M}{N}\right] + \sum_{i=1}^{r} n_i L_{f_i} = -(Q_x + P_y). \tag{2.2.24}$$

Since Q_x and P_y are polynomials of degree at most $(d-1)$ and L_{f_i} is also of maximum degree $(d-1)$. So,

$$D\left[\frac{M}{N}\right] = \Pi, \tag{2.2.25}$$

which is a polynomial of degree $\leq d-1$.
Consequently,

$$\frac{1}{N} D[M] - \frac{M}{N}\frac{D[N]}{N} = \Pi$$

implies,

$$D[M] - M\frac{D[N]}{N} = \Pi N. \tag{2.2.26}$$

Here $D[M]$ and ΠN are polynomials since M and N are polynomials. However, M and N being co-prime it follows that $\frac{D[N]}{N}$ is a polynomial. Therefore, as far as the structure of the unknown polynomial N goes, since $\frac{D[N]}{N}$ is a polynomial, we can express N in terms of

Darboux polynomials f_i.
Let

$$N = \prod_{j=1}^{q} f_j^{m_j}, \tag{2.2.27}$$

so that,

$$\frac{D[N]}{N} = \sum_{j=1}^{q} m_j L_{f_j}, \tag{2.2.28}$$

where m_j is a positive integer. Substituting (2.2.25) in (2.2.24), we get

$$\Pi + \sum_{i=1}^{r} n_j L_{f_i} = -(Q_x + P_y),$$

which implies

$$N\Pi = -N(Q_x + P_y + \sum_{i=1}^{r} n_j L_{f_i}). \tag{2.2.29}$$

Using (2.2.27), (2.2.28) and (2.2.29) in (2.2.26) we have,

$$D[M] - M\sum_{j=1}^{q} m_j L_{f_j} = -(\prod_{j=1}^{q} f_j^{m_j})(Q_x + P_y + \sum_{i=1}^{r} n_j L_{f_i}). \tag{2.2.30}$$

Thus we then have the following algorithm:
Step 1 : Calculate the Darboux polynomials for different order of polynomials.
Step 2 : Set bounds for the unknown polynomials M and N.
Step 3 : As the structure of N is known therefore with a preset bound of N there are only a finite number of choices for the positive integers m_j.
Step 4 : With a preset bound for M and knowledge of f_k's and possible finite number of choices of m_j, find the value of n_i's and coefficients of power of M from the above equation which involves solving a system of linear equations.

Example 2.2.2

Let us consider the differential equation

$$\frac{dy}{dx} = \frac{(x+1)y}{x - xy - y^2 + x^2}. \tag{2.2.31}$$

Here $P = (x+1)y$ and $Q = x - xy - y^2 + x^2$ so that $D = (x - xy - y^2 + x^2)\frac{\partial}{\partial x} + (x+1)y\frac{\partial}{\partial y}$
Step 1 : The Darboux polynomials are found to be $f_1 = y$ and $f_2 = x + y$ while the corresponding co-factors are $L_{f_1} = x + 1$ and $L_{f_2} = 1 + x - y$ respectively.
Step 2 : For the determination of N we assume at first N is of degree 1. Since $N = f_1^{m_1} f_2^{m_2}$ with f_1 and f_2 being of degree 1, we have just two possibilities $\{m_1 = 0, m_2 = 1\}$ or

$\{m_1 = 1, m_2 = 0\}$.
Step 3 : In this step we set a degree for M, starting with an M of degree 1 i.e., $M = a_0 + a_1 x + a_2 y$ where $a_i, i = 0, 1, 2$ are constants. Then $D[M] = a_1 Q + a_2 P$ and from (2.2.26) with $m_1 = 0, m_2 = 0$ we have $a_1(x - xy - y^2 + x^2) - (a_0 + a_1 x)(x + 1) = -y(2 - y + 3x + n_1(x+1)) + n_2(1 + x - y)$. Equating now the coefficients of powers of x, y etc. we get $a_0 = 0, a_1 = 1, n_1 = 0, n_2 = -2$ and a_2 is arbitrary. Setting $a_2 = 0$, we have $N = y, \quad M = x$ finally. Therefore the integrating factor is $R = e^{x/y} f_1^0 f_2^{-2} = e^{x/y}(1 + x - y)^{-2}$.

2.2.9 The Prelle-Singer (PS) method (1983)

The essence of the PS method is that whenever a vector field D associated with a first-order ODE has an *elementary first integral*, the latter can be computed using only the invariant algebraic curves by an almost algorithmic procedure [87]. Clearly, these first integrals may be found by using the Darboux approach. The method is attractive because if the given first-order ODE has a solution expressible in terms of elementary functions, then it guarantees that this solution can be found. The procedure depends upon the determination of Darboux polynomials of the elementary functions occuring in the ODE. The principal shortcoming of the PS method is that it does not specify the degree bound for the Darboux polynomials. Consequently one has to set a degree bound for the Darboux polynomial *a priori* and therefore it is a semi-algorithmic procedure. If it is possible to set a theoretical degree bound for the Darboux polynomial in the PS method, then the method will clearly become an algorithm.
If the first-order ODE, $y' = P/Q$, possesses an elementary general solution then the solution is of the form

$$I(x, y) = W_0 + \sum_{i>0} c_i \ln W_i$$

where W_0, W_i are algebraic functions of (x, y).
• The one form $\omega = Pdx - Qdy$ admits an integrating factor such that R^k is a rational function of (x, y) where

$$R = \prod_i f_i^{n_i}$$

Given an upper bound B on the Darboux polynomials one proceeds as follows:

- Step 1: Set current bound on the Darboux polynomials $N = 1$.
- Step 2: Search for irreducible polynomials f_i such that deg $f_i \leq N$ and $f_i | D(f_i)$
- Step 3: If there exists constants n_i not all zero, such that $\sum_{i=1}^m n_i \lambda_i = 0$ then $D[R]/R = 0$ and the ODE is exact. The solution is $I(x, y) = \prod_{i=1}^m f_i^{n_i} = c$. If no solutions for n_i exist then go the next step.

- Step 4: if there exists constants n_i not all zero, such that $\sum_{i=1}^{m} n_i\lambda_i = -(Q_x + P_y)$ (so that $R = \prod_i f_i^{n_i}$) then return the solution $I(x, y) = c$ with

$$I(x,y) = \int RPdx - \int (RQ + \partial_y \int RP)dy.$$

or

$$-\int RQdy + \int (RP + \partial_x \int RQ)dx$$

- Step 5: Set $N = N + 1$. If $N > B$ then exit with no result. Else go to Step 2

As stated earlier since no upper bound on the Darboux polynomials are known so the process is a semi decisive one.

Example 2.2.3

The following example serves to illustrate the procedure.

$$\frac{dy}{dx} = -\frac{x(1+y)}{y + x^2 + y^2}$$

Here the vector field is $D = (y + x^2 + y^2)\partial_x - x(1 + y)\partial_y$ and the Darboux polynomials along with their cofactors are $f_1 = (1 + y)$, $\lambda_1 = -x$ $f_2 = \left(x^2 + \frac{y^2}{2} + \frac{y}{3} - \frac{1}{6}\right)$, $\lambda_2 = 2x$. Consequently we have $n_1\lambda_1 + n_2\lambda_2 = -(Q_x + P_y) = -x$ so that $n_1 = n_2 = -1$. The integrating factor is therefore given by

$$R(x,y) = (1+y)^{-1}\left(x^2 + \frac{y^2}{2} + \frac{y}{3} - \frac{1}{6}\right)^{-1}$$

and the solution is finally found to be the following:

$$I(x,y) = -\frac{1}{2}\ln\left(x^2 + \frac{y^2}{2} + \frac{y}{3} - \frac{1}{6}\right) - \ln(1+y) = c$$

Our next example concerns the generalized Lotka-Volterra model.

Generalized Lotka-Volterra model

We focus on the following version of the Lotka-Volterra model, describing the population dynamics of two competing species represented by x and y.

Example 2.2.4

$$\dot{x} = x(a_1 + a_2x + a_3y) \quad \text{and} \quad \dot{y} = y(b_1 + b_2x + b_3y) \tag{2.2.32}$$

The corresponding phase-flow is given by the equation:

$$\frac{dy}{dx} = \frac{y(b_1 + b_2x + b_3y)}{x(a_1 + a_2x + a_3y)}. \tag{2.2.33}$$

Here $P(x,y) = y(b_1 + b_2x + b_3y)$ and $Q(x,y) = x(a_1 + a_2x + a_3y)$ while

$$D[R] = (x(a_1 + a_2x + a_3y)\partial_x + y(b_1 + b_2x + b_3y)\partial_y)[R] = -(Q_x + P_y)R.$$

Notice that

$$D[x] = x(a_1 + a_2x + a_3y) \;\text{ and }\; D[y] = y(b_1 + b_2x + b_3y)$$

which implies that the Darboux polynomials are:

$$(i)\;\; f_1 = x, \quad \lambda_1 = (a_1 + a_2x + a_3y)$$

$$(ii)\;\; f_2 = y, \quad \lambda_2 = (b_1 + b_2x + b_3y).$$

Thus

$$R = \prod_i f_i^{n_i} \Rightarrow \frac{D[R]}{R} = \sum_i n_i \frac{D[f_i]}{f_i} = -(Q_x + P_y) \tag{2.2.34}$$

where

$$Q_x = a_1 + 2a_2x + a_3y, \;\text{ and }\; P_y = b_1 + b_2x + 2b_3y.$$

We are therefore left with the problem of finding rational numbers n_i such that

$$n_1(a_1 + a_2x + a_3y) + n_2(b_1 + b_2x + b_3y) = -[(a_1 + b_1) + (2a_2 + b_2)x + (a_3 + 2b_3)y]. \tag{2.2.35}$$

Equating the coefficients of the various powers on either side, leads to the following set of equations:

$$n_1a_1 + n_2b_1 = -(a_1 + b_1)$$
$$n_1a_2 + n_2b_2 = -(2a_2 + b_2)$$
$$n_1a_3 + n_2b_3 = -(a_3 + 2b_3). \tag{2.2.36}$$

A consistent solution of this set of equations is obtained by imposing the following constraints on the parameters:

$$b_1 = a_1, b_2 = 3a_2, b_3 = -a_3 \tag{2.2.37}$$

which yield

$$n_1 = -\frac{1}{2} \;\text{ and }\; n_2 = -\frac{3}{2}.$$

With these constraints on the parameters the Lotka-Volterra equation (2.2.32) becomes:

$$\dot{x} = x(a_1 + a_2 x + a_3 y), \quad \text{and} \quad \dot{y} = y(a_1 + 3a_2 x - a_3 y) \tag{2.2.38}$$

and its integrating factor is given by

$$R = x^{-\frac{1}{2}} y^{-\frac{3}{2}}. \tag{2.2.39}$$

If $I(x, y)$ denotes a first integral of the equation, it follows that

$$I_x = (RP) = x^{-\frac{1}{2}} y^{-\frac{1}{2}} (a_1 + 3a_3 x - a_3 y)$$

$$I_y = -(RQ) = -x^{\frac{1}{2}} y^{-\frac{3}{2}} (a_1 + a_2 x + a_3 y)$$

It is now a matter of straight forward integration to see that the first integral is

$$I(x, y) = \frac{2x^{\frac{1}{2}}(a_1 + a_2 x - a_3 y)}{y^{\frac{1}{2}}}. \tag{2.2.40}$$

We should point out that while finding the first two Darboux polynomials is relatively obvious, one may assume a third polynomial $f_3 = (\alpha x + \beta y)$ and try to determine the values of α, β such that f_3 is a Darboux polynomial. The latter requires $f_3|D[f_3]$, and occurs when $\alpha = a_2$ and $\beta = -a_3$; with the associated eigen polynomial being $\lambda_3 = (a_1 + a_2 x - a_3 y)$. Notice that λ_3 is precisely the factor which occurs in our expression for the first integral (2.2.40). This is because having obtained three Darboux polynomials, we can verify that there exists a set of rational numbers ($s_1 = \frac{1}{2}, s_2 = -\frac{1}{2}, s_3 = 1$) such that $D[I] = 0$ where $I = f_1^{\frac{1}{2}} f_2^{-\frac{1}{2}} f_3$.

We will now consider the special case of homogeneous vector fields.

2.2.10 Homogeneous vector fields

Consider the planar ODE

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)} \tag{2.2.41}$$

when P and Q are homogeneous functions. The computation of Darboux polynomials for such cases is considerably simplified by the following lemma due to Collins (1996) [18].

Lemma 2.2.1 *Let $D = Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}$ be a homogeneous vector field. If $W = xP(x, y) - yQ(x, y)$ does not vanish identically, then $D[W] = \lambda W$, so that W is a Darboux polynomial for D and all irreducible homogeneous Darboux polynomials of D divide W.*

Proof: Suppose $P(x, y)$ and $Q(x, y)$ are homogeneous functions of degree n. Therefore by Euler's theorem

$$\begin{aligned} x\tfrac{\partial P}{\partial x} + y\tfrac{\partial P}{\partial y} &= nP \\ x\tfrac{\partial Q}{\partial x} + y\tfrac{\partial Q}{\partial y} &= nQ. \end{aligned} \tag{2.2.42}$$

Therefore

$$\begin{aligned} D[W] &= D[xP(x,y) - yQ(x,y)] = D[x]P + xD[P] - D[y]Q - yD[Q] \\ &= PQ + x\left(Q\frac{\partial P}{\partial x} + P\frac{\partial P}{\partial y}\right) - PQ - y\left(Q\frac{\partial Q}{\partial x} + P\frac{\partial Q}{\partial y}\right) \\ &= \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)(xP - yQ) - P\left(\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}\right) + Q\left(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}\right) \\ &= \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)(xP - yQ) - p(nQ) + Q(nP) \qquad \text{[Using (2.2.42)]} \\ &= \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)(xP - yQ) = \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)W. \end{aligned}$$

Therefore W is a Darboux polynomial and W divides $D[W]$.

It is interesting to note that when $P(x,y)$ and $Q(x,y)$ are homogenous functions of odd degree then a necessary and sufficient condition for (2.2.41) to exhibit a closed cycle about the origin is

$$\int_{-\infty}^{\infty} \frac{Q(1,u)}{P(1,u) - uQ(1,u)} du = 0. \tag{2.2.43}$$

This result is known as *Frommer's theorem* [20]. However, a closed trajectory can also arise in the case when $P(x,y)$ and $Q(x,y)$ are of even degree.

(a) Homogenous odd degree case:
We consider as our prototype example the following ODE:

$$\frac{dy}{dx} = \frac{x^3 + ax^2y + bxy^2 + cy^3}{ax^3 + bx^2y + cxy^2 - y^3} \tag{2.2.44}$$

By lemma 2.2.1 we find that

$$W = x^4 + y^4 = (x + e^{i\theta}y)(x + e^{-i\theta}y)(x + e^{3i\theta}y)(x + e^{-3i\theta}y) \tag{2.2.45}$$

where $\theta = \frac{\pi}{4}$. Clearly each factor is also a Darboux polynomial and these along with their cofactors are tabulated below:

$$\begin{aligned} &(i) \quad f_1 = x + e^{i\theta}y, \qquad \lambda_1 = (a + e^{i\theta})x^2 + (b - e^{2i\theta})xy + (c + e^{3i\theta})y^2, \\ &(ii) \quad f_2 = x + e^{-i\theta}y, \qquad \lambda_2 = (a + e^{-i\theta})x^2 + (b - e^{-2i\theta})xy + (c + e^{-3i\theta})y^2, \\ &(iii) \quad f_3 = x - e^{-i\theta}y, \qquad \lambda_3 = (a - e^{-i\theta})x^2 + (b - e^{-2i\theta})xy + (c - e^{-3i\theta})y^2, \\ &(iv) \quad f_4 = x - e^{i\theta}y, \qquad \lambda_4 = (a - e^{i\theta})x^2 + (b - e^{2i\theta})xy + (c - e^{3i\theta})y^2. \end{aligned}$$

The relation (2.2.22) then yields

$$\frac{D[R]}{R} = \sum_i n_i \frac{D[f_i]}{f_i} = \sum_i n_i \lambda_i = -(Q_x + P_y) = -4(ax^2 + bxy + cy^2), \tag{2.2.46}$$

from which it follows that $n_1 = n_2 = n_3 = n_4 = -1$ and hence a requisite integrating factor of the equation is

$$R = (x + e^{i\theta}y)^{-1}(x + e^{-i\theta}y)^{-1}(x - e^{-i\theta}y)^{-1}(x - e^{i\theta}y)^{-1}. \tag{2.2.47}$$

It may then be verified that

$$I(x,y) = \frac{1}{2}\left(1 + \frac{c-a}{\sqrt{2}}\right)\ln\sqrt{\left(x + \frac{y}{\sqrt{2}}\right)^2 + \frac{y^2}{2}} + \frac{1}{2}\left(1 - \frac{c-a}{\sqrt{2}}\right)\ln\sqrt{\left(x - \frac{y}{\sqrt{2}}\right)^2 + \frac{y^2}{2}}$$

$$+\frac{1}{2}\left(b - \frac{c+a}{\sqrt{2}}\right)\tan^{-1}\left(\frac{y}{\sqrt{2}x + y}\right) - \frac{1}{2}\left(b + \frac{c+a}{\sqrt{2}}\right)\tan^{-1}\left(\frac{y}{\sqrt{2}x - y}\right) = \text{constant} \tag{2.2.48}$$

is the solution of (2.2.44).

(b)Homogeneous even degree case:
We consider the following homogenous second degree system of equations:

$$\frac{dx}{dt} = -2xy, \quad \frac{dy}{dt} = x^2 - y^2, \tag{2.2.49}$$

which may alternatively be expressed as

$$\frac{dy}{dx} = \frac{x^2 - y^2}{-2xy}. \tag{2.2.50}$$

In a manner similar to that described in the previous example it is found that

$$R = \frac{1}{x(x+iy)(x-iy)}, \tag{2.2.51}$$

is an integrating factor of this equation. The corresponding logarithmic first integral is easily obtained as

$$\phi(x,y) = \log\left(\frac{x^2 + y^2}{x}\right). \tag{2.2.52}$$

2.3 Time dependent first integrals and non-planar dynamical systems:

In the previous section we have basically outlined the Prelle-Singer procedure for planar vector fields when the first integrals are not explicitly time dependent.
Here we consider a non-planar dynamical system given by

$$\frac{dx_i}{dt} = X_i(t, x_1, x_2, \ldots, x_n), \quad i = 1, \cdots, n. \tag{2.3.1}$$

The procedure for finding time dependent first integrals may be presented in the form of the following algorithm [63]:

1. **Step** 1 Set N=1.
2. **Step** 2 Find all monic irreducible polynomials g_α with degree $\leq N$ such that g_α divides $\widetilde{D}[g_\alpha]$ where $\widetilde{D} := \frac{\partial}{\partial t} + \sum_i X_i \frac{\partial}{\partial x_i}$.
3. **Step** 3 Let $\widetilde{D}[g_\alpha] = g_\alpha \lambda_\alpha$ and decide if there are constants n_α, not all zero such that $\sum_\alpha n_\alpha \lambda_\alpha = r$, where $r \in \mathbb{R}$. If such n_α's exist then $e^{-rt} \prod_\alpha g_\alpha^{n_\alpha}$ is a first integral.

The last conclusion follows from the following observation. If

$$R = e^{-rt} \prod_\alpha g_\alpha^{n_\alpha} \text{ then } \log R = -rt + \sum_\alpha n_\alpha \log g_\alpha \tag{2.3.2}$$

then

$$\frac{\widetilde{D}[R]}{R} = -r\widetilde{D}[t] + \sum_\alpha n_\alpha \frac{\widetilde{D}[g_\alpha]}{g_\alpha} = -r + \sum_\alpha n_\alpha \frac{D[g_\alpha]}{g_\alpha}, \tag{2.3.3}$$

since it is assumed that g_α is not explicitly time dependent. Next the assumption g_α's are Darboux polynomials ensures that $g_\alpha | D[g_\alpha] = \lambda_\alpha$ is a polynomial and hence

$$\frac{\widetilde{D}[R]}{R} = -r + \sum_\alpha n_\alpha \lambda_\alpha. \tag{2.3.4}$$

Thus if there exists rational numbers such that $\sum_\alpha n_\alpha \lambda_\alpha = r$ then $\frac{\widetilde{D}[R]}{R} = 0$ and hence the system is exact, so that the integrating factor R is itself a first integral.
As to the nature of the integrating factor when $\sum_\alpha n_\alpha \lambda_\alpha \neq r$, to the best of our knowledge is not clear what the next step of the above algorithm should be.
The following example which is a variant of the Hindmarsh-rose model [93] serves to illustrate the above algorithm.

Example 2.3.1

The Hindmarsh-Rose model consists of a system of three autonomous differential equations, with mild nonlinearities for modelling neurons that exhibit triggered firing. The usual form of the equations are:

$$\begin{aligned} \dot{x} &= y + \phi(x) - z - I \\ \dot{y} &= \psi(x) - y \\ \dot{z} &= r(s(x - x_R) - z) \end{aligned} \tag{2.3.5}$$

where $\phi(x) = ax^2 - x^3$ and $\psi(x) = 1 - bx^2$. Here I is a control parameter, while of the remaining five parameters s and x_R are usually fixed. Let us re-write them in the following form appending two extra parameters to the functions $\phi(x)$ and $\psi(x)$.

$$\begin{aligned} \dot{x} &= y - z - ax^3 + bx^2 + \alpha \\ \dot{y} &= \beta - dx^2 - y \\ \dot{z} &= px - rz - \gamma. \end{aligned} \tag{2.3.6}$$

We thus have a total of eight parameters given by $\alpha, \beta, \gamma, a, b, d, p$ and r.
A list of first integrals for specific parameter values of the system (2.3.6) is given below:
I. For $p = 0$, $I = (rz + \gamma)e^{rt}$.

II. $d = 0$ then $I = (y - \beta)e^t$.

III. For arbitrary values of d, β, γ and $a = 0, b = -d, p = -2, \alpha = \beta + \gamma$ and $r = 1$,

$$I = e^{2t}(x - y + z)$$

IV. For arbitrary values of α, γ, p and b and when $a = 0, d = 2b, r = -(p+1), \beta = 2(\frac{\gamma}{p} - \alpha)$ we find that

$$I = \frac{e^t}{2x + y + \frac{2z}{p}}.$$

Unfortunately we have not been able to find a first integral with $a \neq 0$ which is the dominant nonlinear term here. Notice may be taken of the presence of the time t in all the above first integrals.

2.4 Fixed point and Phase-Plane Analysis

Since a lot of information concerning a differential system can be called from an analysis of the system in the phase plane we consider an autonomous planar nonlinear system in the standard form

$$\dot{x} = P(x,y),\ \dot{y} = Q(x,y)\ , \tag{2.4.1}$$

where P and Q are nonlinear functions of x and y. It is interesting to note that, all ODEs arising from Newton's second law of motion, namely $\ddot{x} = F(x, \dot{x})$, where F is the forcing term, may be put in the standard form

$$\dot{x} = y \equiv P, \quad \dot{y} = F(x,y) \equiv Q. \tag{2.4.2}$$

Definition 2.4.1 *The points in the $x - y$ plane (i.e. the phase plane) where $\dot{x} = 0$ and $\dot{y} = 0$ are called fixed points.*

These points are also known as the stationary points, i.e., they represent the points of intersect of the graphs of

$$P(x,y) = 0, \quad Q(x,y) = 0. \tag{2.4.3}$$

and are found by solving the simultaneous nonlinear equations

$$P(x,y) = 0, \quad Q(x,y) = 0. \tag{2.4.4}$$

If $P(x,y)$ and $Q(x,y)$ are nonlinear functions then obviously there can exist more than one fixed point for a given system.

Example 2.4.1 *Consider the autonomous ODE*

$$\begin{array}{c} \dot{x} = y \\ \dot{y} = -(1+y)(x+x^2) \end{array} . \tag{2.4.5}$$

Here $P(x,y) = y$ and $Q(x,y) = -(1+y)(x+x^2)$. By solving (2.4.4) with these functions we find there are two fixed points at (0,0) and (-1,0) respectively.

The behavior of the solution curve or trajectory in the neighbourhood of the fixed points in the phase plane exhibits a number of features. Since P and Q do not explicitly depend upon the time t, it can be eliminated by dividing the two equations in (2.4.1) and we have

$$\frac{dy}{dx} = \frac{Q(x,y}{P(x,y)}, \tag{2.4.6}$$

which is the slope of the trajectory at an arbitrary point (x,y) in the phase plane. Since at the fixed point $P = Q = 0$, we have $dy/dx = 0/0$ and therefore the slope is indeterminate. At all other points the slope has a unique value in between 0 and ∞. Points other than fixed points are called *ordinary points*. As time advances the solution will advance along the trajectory determined by the initial values of x and y.

Tangent Field:

Graphically one can see all possible trajectories of the standard ODE by sketching a tangent field. Thus is done by forming a systematic grid in the phase plane, the ratio $Q(x, y)/P(x, y)$ is then calculated at each grid point. A small arrow with slope $\frac{dy}{dx} = \frac{Q(x,y}{P(x,y)}$ is then drawn at each grid point i.e., tangent to the trajectory at that grid point. The arrowhead should point in the direction of increasing t.

Vortex point:

If the tangent field arrows form a counter clockwise whirlpool or vortex around a fixed point then the fixed point are called *vortex point.*
The behavior of the tangent field arrows in the figure 2.1 imply that the only possible physical solution are *cyclical.*

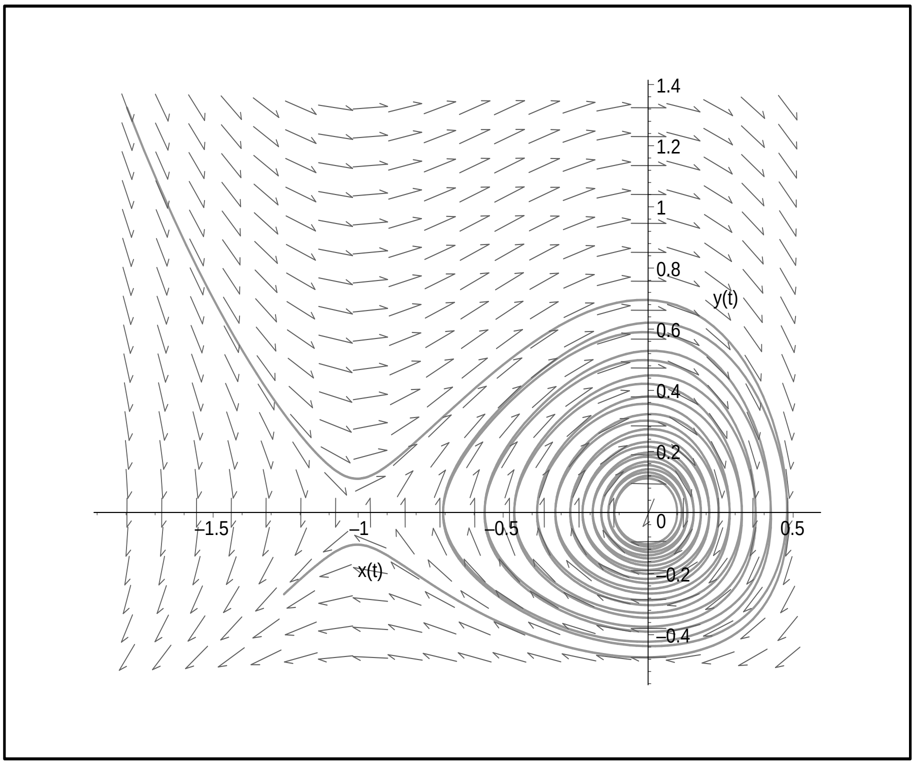


Figure 2.1: Phase portrait of the example 2.4.1

Saddle point:

If the tangent field arrows point away from the fixed point along the horizontal x-axis and point towards it along the vertical y-axis then this type of a fixed point is called a *saddle point.*

The name arising from analogy with a saddle point at the pass between two neighbouring mountain peaks with two valleys in the transverse direction. In the mountain situation, increasing height plays the role of increasing time. To represent the direction of increasing height, two arrows point away from the mountain away from the mountain saddle point to the peaks and two arrows point towards the saddle point as one ascends the pass from the valleys.

2.4.1 Linearization technique:

Let us now systematically identify the types of fixed points. Suppose (x_0, y_0) to be a fixed point. If the fixed point $(x_0, y_0) \neq (0,0)$, then by a translation $u = x - x_0$ and $v = y - y_0$, we have a new system with (0,0) as a fixed point. So, henceforth we shall assume (0,0) as a fixed point. So that $P(0,0) = Q(0,0) = 0$. Therefore. by Taylor's expansion about the fixed point we can express the functions P and Q by

$$P(x,y) = ax + by + R(x,y), \;\; Q(x,y) = cx + dy + S(x,y), \tag{2.4.7}$$

where $R(x,y) = O(r^2)$ and $S(x,y) = O(r^2)$ as $r = \sqrt{x^2+y^2} \to 0$, and

$$a = \frac{\partial P}{\partial x} \,|_{(0,0)}, \; b = \frac{\partial P}{\partial y} \,|_{(0,0)}, \; c = \frac{\partial Q}{\partial x} \,|_{(0,0)}, \; d = \frac{\partial Q}{\partial y} \,|_{(0,0)} \,. \tag{2.4.8}$$

The linear approximation in the neighbourhood of the origin is therefore the pair of linear ODEs.

$$\begin{array}{c} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{array} . \tag{2.4.9}$$

Assuming a nontrivial solution of (2.4.9) of the form $x = re^{\lambda t}$ and $y = se^{\lambda t}$, where r, s are real constants and λ is a either real or complex constant, we have upon substitution in (2.4.9)

$$\begin{array}{c} (a-\lambda)r + bs = 0 \\ cr + (d-\lambda)s = 0 \end{array} . \tag{2.4.10}$$

A non-trivial solution exists if and only if

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0, \tag{2.4.11}$$

which implies a quadratic equation in λ, namely

$$\lambda^2 + p\lambda + q = 0, \tag{2.4.12}$$

where $p = -(a+d)$ and $q = ad - bc$.
Therefore, we obtain two roots

$$\lambda_{\pm} = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}. \tag{2.4.13}$$

Note that, if $q = 0$, the roots $\lambda_+ = 0$ and $\lambda_- = -p$. In this case, higher-order terms in the Taylor expansion should be kept. For this reason, $q = 0$ corresponds to a higher-order fixed point. The fixed points which occurs for $q \neq 0$ are called *simple*. A detailed examination of the roots for this case shows that there are only four type of simple fixed points, saddle, focal or spiral, nodal and vortex point [47].
The range of q, p and $p^2 - 4q$ dictates the type of a fixed point and in case of the focal and nodal points, its stability. The following table summarizes the various possibilities:

Fixed point	$q = ad - bc$	$p = -(a+d)$	$p^2 - 4q$
Saddle	< 0	for all p	> 0
Higher-order	$= 0$	for all p	≥ 0
Stable focal	> 0	> 0	< 0
Stable nodal	> 0	> 0	≥ 0
Vortex or focal	> 0	$= 0$	< 0
Unstable focal	> 0	< 0	< 0
Unstable nodal	> 0	< 0	≥ 0

. (2.4.14)

While the above table serves to furnish a rough idea of the phase space behavior of the system, it should be noted that, for $q > 0, p = 0$ the fixed point is either a vortex or a focal point. The uncertainty arises because of the neglect of higher-order terms in the Taylor expansion which may turn a closed loop into a spiral (for the vortex). In this context the following theorem due to Poincaré's is useful.

Theorem 2.4.1 ***Poincaré's Theorem:***
If $P(x, -y) = -P(x, y)$ and $Q(x, -y) = Q(x, y)$ then the fixed point is a vortex and is not a focal point.

Example 2.4.2

Consider the differential equation

$$\ddot{x} + x - 2x^2 + x^3 = 0, \tag{2.4.15}$$

which may be written in the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + 2x^2 - x^3 \end{aligned} . \tag{2.4.16}$$

Here $P(x, y) = y$ and $Q(x, y) = -x(1-x)^2$. The fixed points are (0,0) and (1,0). For the fixed point at the origin, $a = 0,\ b = 1,\ c = -1,\ d = 0$, so that $p = -(a+d) = 0,\ q = ad - bc = 1 > 0$. Therefore, the origin is either a focal or a vortex point.
Applying Poincaré's theorem we find that

$$P(x, -y) = -y = -P(x, y)$$

$$Q(x,-y) = -x(1-x)^2 = Q(x,y).$$

Therefore it follows that the fixed point at the origin is a vortex point.
For the fixed point (1,0), we have $a = 0,\ b = 1,\ c = 0,\ d = 0$. Then $p = 0$ and $q = 0$. So the fixed point (1,0) is a higher-order fixed point.

In this section we have attempted to give a brief introduction to ODEs especially to the notion of first integrals and there evaluation and have highlighted some of there essential features as are relevant to the subsequent chapters. Now we will consider another important aspect of the analysis of ODEs namely that of Lie Symmetries.

2.5 Lie Symmetry

2.5.1 Symmetry of Nonlinear Differential Equations

Before trying to understand what we mean by a symmetry of an ODE, it is perhaps more appropriate to introduce the notion of symmetry for simple geometric objects, for which we have an intuitive sense.
Consider, for example, an equilateral triangle. If we rotate the triangle about its center by $2\pi/3$ in the anticlockwise sense, then after rotation the triangle is indistinguishable from the one we started with. This look alikeness is at the heart of symmetry. Further rotations by $4\pi/3$ and 2π also leave the original triangle unchanged. Again if we reflect the triangle about the medians then also it remains invariant. In fact anything which reflects invariance under a transformation indicates that the object with that property is actually simpler than it looks.

2.5.2 The Basic ideas

We say that a transformation is a symmetry if it satisfies the following conditions:
(a) The transformation is structure preserving.
(b) The transformation is a diffeomorphism, i.e., a smooth invertible mapping whose inverse is also smooth.
(c) The transformation maps the object to itself.
The last requirement is the one which we normally have in mind when discussing about symmetry.

Consider a unit circle, $x^2 + y^2 = 1$, made of any rigid material. Being of rigid nature, the distance between any two points on the circle is fixed. However, if the circle were to be made of any elastic material such as rubber, then we could have deformed it, in which case the distance between points on the circle, could vary. Next consider a transformations given by

$$\Gamma_\epsilon : (x, y) \mapsto (\hat{x}, \hat{y}) = (x \cos\epsilon - y \sin\epsilon, x \sin\epsilon + y \cos\epsilon)$$

for each $\epsilon \in (-\pi, \pi]$. In terms of polar coordinates this transformation is

$$\Gamma_\epsilon : (\cos\theta, \sin\theta) \mapsto (\cos(\theta + \epsilon), \sin(\theta + \epsilon)).$$

Such a transformation preserves the structure and is smooth and invertible. It is geometrically just a rotation by ϵ about the center of the circle. (The inverse of a rotation by ϵ is just a rotation by $-\epsilon$.)
Finally $\hat{x}^2 + \hat{y}^2 = x^2 + y^2 = 1$, in other words the transformation maps the unit circle to another unit circle.
Since Γ_ϵ is defined for all values of $\epsilon \in (-\pi, \pi]$, we have here an example of an infinite set of symmetries.

2.5.3 One parameter group of Transformation

Let us consider an appropriate change of variables from (x, y) to $(\tilde{x}, \tilde{y})$ given by

$$\tilde{x} = \tilde{x}(x, y), \quad \tilde{y} = \tilde{y}(x, y). \tag{2.5.1}$$

This will be called a *point transformation.*
In the context of symmetries we have to consider point transformations which depend upon at least one arbitrary parameter ϵ so that

$$\tilde{x} = \tilde{x}(x, y; \epsilon), \quad \tilde{y} = \tilde{y}(x, y; \epsilon), \tag{2.5.2}$$

subject to the condition

$$\tilde{x}(x, y; 0) = x, \quad \tilde{y}(x, y; 0) = y. \tag{2.5.3}$$

It is assumed that $\tilde{x}(x, y; \epsilon)$ and $\tilde{y}(x, y; \epsilon)$ are functionally independent i.e., their Jacobian

$$J := \begin{vmatrix} \tilde{x}_x & \tilde{x}_y \\ \tilde{y}_x & \tilde{y}_y \end{vmatrix} \neq 0. \tag{2.5.4}$$

In view of this condition the transformations are invertible and therefore repeated applications yield a transformation of the same family. The transformation (2.5.2) with these properties form a one-parameter group of point transformation.
A simple example of a one-parameter group is given by the rotation

$$\tilde{x} = x\text{cos}\epsilon - y\text{sin}\epsilon, \quad \tilde{y} = x\text{sin}\epsilon + y\text{cos}\epsilon. \tag{2.5.5}$$

However a reflection for which, $\tilde{x} = -x$ and $\tilde{y} = -y$, does not form a one parameter group even though it is a point transformation.

2.5.4 Group generator and the Lie equation

An infinitesimal transformation is an infinitesimal deformation from the identity and can be represented as

$$\tilde{x}(x, y; \epsilon) = x + \epsilon\xi(x, y) + \cdots = x + \epsilon X x + \cdots \tag{2.5.6}$$

$$\tilde{y}(x, y; \epsilon) = y + \epsilon\eta(x, y) + \cdots = y + \epsilon X y + \cdots , \tag{2.5.7}$$

For the case of two variables x and y the vector field X is defined by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.5.8}$$

where the functions ξ and η are defined by

$$\xi(x, y) = \frac{\partial \tilde{x}}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta(x, y) = \frac{\partial \tilde{y}}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.5.9}$$

Although we have restricted the number of variables to two for the purposes of clarity of presentation there can be any number of variables. The ϵ is the parameter of smallness and it is normal to ignore higher powers in this parameter.
From (2.5.8) it follows that the components of the tangents vectors are exactly ξ and η. The operator X is called the *infinitesimal generator* of the point transformation. The term *generator* indicates that the repeated application of the transformation will generate the finite transformation, which expresses the fact that the integral curves of the vector field X are the group orbits.
Given an *infinitesimal transformation* (2.5.6) and (2.5.7) or the generators of the transformation (2.5.8) are obtained by integrating the following system of ordinary differential equations called the *Lie equations*:

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x},\tilde{y}), \quad \tilde{x}|_{\epsilon=0} = x, \tag{2.5.10}$$

$$\frac{d\tilde{y}}{d\epsilon} = \eta(\tilde{x},\tilde{y}), \quad \tilde{y}|_{\epsilon=0} = y. \tag{2.5.11}$$

Example 2.5.1

For the rotation (2.5.5) in the $x-y$ plane, we have

$$\frac{\partial\tilde{x}}{\partial\epsilon}\bigg|_{\epsilon=0} = -y, \quad \frac{\partial\tilde{y}}{\partial\epsilon}\bigg|_{\epsilon=0} = x. \tag{2.5.12}$$

Therefore the corresponding generator is $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. One can easily verify that the Lie equations are satisfied since

$$\begin{aligned} \tfrac{d\tilde{x}}{d\epsilon} = -y, \quad \tilde{x}\Big|_{\epsilon=0} = x, \\ \tfrac{d\tilde{y}}{d\epsilon} = x, \quad \tilde{y}\Big|_{\epsilon=0} = y. \end{aligned} \tag{2.5.13}$$

Example 2.5.2

The inverse problem is to find the finite transformation when the generator is given. Suppose $X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$. The Lie equations (2.5.10) have the form

$$\begin{aligned} \tfrac{d\tilde{x}}{d\epsilon} = \tilde{x}^2, \quad \tilde{x}\Big|_{\epsilon=0} = x, \\ \tfrac{d\tilde{y}}{d\epsilon} = \tilde{x}\tilde{y}, \quad \tilde{y}\Big|_{\epsilon=0} = y. \end{aligned} \tag{2.5.14}$$

Solving the above system of differential equations we have, $\tilde{x} = -\frac{1}{\epsilon+C_1}, \quad \tilde{y} = \frac{C_2}{\epsilon+C_1}$. Applying the initial conditions we find that $C_1 = -1/x$ and $C_2 = -y/x$. Consequently we arrive at the following one parameter group:

$$\tilde{x} = \frac{x}{1-\epsilon x}, \quad \tilde{y} = \frac{y}{1-\epsilon x}. \tag{2.5.15}$$

Next we present certain basic definitions.

Definition 2.5.1 ***(Orbit of the group)***

The orbit of the group through (x, y) is the set of points (x, y) which can be mapped by a suitable choice of ϵ. The orbit through a typical point is a smooth curve.
The points each of which are mapped to itself by the Lie symmetries are called *invariant points*. An invariant point is a zero-dimensional orbit of the Lie group and are the fixed points of the flow.

Definition 2.5.2 ***(Invariants)***

A function $I(x, y)$ is called an *invariant of the group of transformations* (2.5.2) if the following condition holds:

$$I(\tilde{x}, \tilde{y}) = I(\tilde{x}(x, y, \epsilon), \tilde{y}(x, y, \epsilon)) = I(x, y). \tag{2.5.16}$$

Theorem 2.5.1

A necessary and sufficient condition for a function $I(x, y)$ to be an invariant is that it solves the following partial differential equation:

$$X(I) = \xi(x, y)\frac{\partial I}{\partial x} + \eta\frac{\partial I}{\partial y} = 0. \tag{2.5.17}$$

Proof: The necessary condition follows from the Taylor expansion of $I(x, y)$ with respect to ϵ:

$$I(\tilde{x}, \tilde{y}) \approx I(x+\epsilon\xi, y+\epsilon\eta) \approx I(x, y)+\epsilon\Big(\xi(x, y)\frac{\partial I}{\partial x}+\eta(x, y)\frac{\partial I}{\partial y}\Big) = I(x, y)+\epsilon X(I). \tag{2.5.18}$$

Since $I(x, y)$ is invariant, we have $I(\tilde{x}, \tilde{y}) = I(x, y)$. Using this relation we have from the above equation $\epsilon X(I) = 0$ and hence $X(I) = 0$. For sufficiency, if we substitute $X(I) = 0$ in (2.5.18) we have $I(\tilde{x}, \tilde{y}) = I(x, y)$. So, $I(x, y)$ is invariant.

Definition 2.5.3 ***(Characteristic)***

If $X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$ be a symmetry generator then we define $Q(x, y, y') = \eta(x, y) - y'\xi(x, y)$, as the *characteristic*. If C is the curve $y = y(x)$, the tangent to C in the direction $(1, y'(x))$, is parallel to $(\xi(x), \eta(x))$ if and only if

$$Q(x, y, y') = 0 \quad \text{on C.}$$

Every one-parameter group of point transformation in the plane has one independent invariant, which can be taken to be the left hand side of any first integral $\phi(x, y) = C$ of the characteristic equation:

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}. \tag{2.5.19}$$

Any other invariant has the form $I(x, y) = F(\phi(x, y))$.

Definition 2.5.4 ***(Canonical coordinates)***

Every one-parameter group of transformation (2.5.2) with the generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.5.20}$$

can be reduced to the group of translation $\tilde{r} = r,\ \tilde{s} = s + \epsilon$ with generator

$$\tilde{X} = \frac{\partial}{\partial s}, \tag{2.5.21}$$

by introducing new variables $r = r(x,y)$ and $s = s(x,y)$. We call (2.5.21) the *normal form* of the generator X. For the new coordinates, the tangent vector at the point (r,s) is $(0,1)$, i.e.,

$$\left.\frac{\partial \tilde{r}}{\partial \epsilon}\right|_{\epsilon=0} = 0, \quad \left.\frac{\partial \tilde{s}}{\partial \epsilon}\right|_{\epsilon=0} = 1. \tag{2.5.22}$$

Using the chain rule and (2.5.10) we obtain

$$\begin{aligned} \xi(x,y)r_x + \eta(x,y)r_y &= 0 \\ \xi(x,y)s_x + \eta(x,y)s_y &= 1. \end{aligned} \tag{2.5.23}$$

The change of coordinates should be invertible in some neighbourhood of (x,y). So, we impose the non-degeneracy condition

$$\begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix} \neq 0. \tag{2.5.24}$$

This condition ensures that if a curve of constant s and a curve of constant r meet at a point, they cross one another transversely. Any pair of functions $r(x,y)$ and $s(x,y)$ satisfying (2.5.22) and (2.5.24) is called a pair of *canonical coordinates.*

2.5.5 Extension of transformations and their generators:

Next we will describe the application of point transformations (2.5.2) on a differential equation

$$E(x,y,y',y'',\cdots,y^{(n)}) = 0, \quad y^{(k)} \equiv \frac{d^k y}{dx^k}, k = 1,\cdots,n \tag{2.5.25}$$

and also explain how to extend (or prolong) the point transformation to transform the derivatives $y^{(k)}, \quad (k = 1,\cdots,n)$ appearing in the equation.
This is accomplished by defining,

$$\tilde{y}' = \frac{d\tilde{y}}{d\tilde{x}} = \frac{d\tilde{y}(x,y;\epsilon)}{d\tilde{x}(x,y;\epsilon)} = \frac{y'(\frac{\partial \tilde{y}}{\partial y}) + (\frac{\partial \tilde{y}}{\partial x})}{y'(\frac{\partial \tilde{x}}{\partial y}) + (\frac{\partial \tilde{x}}{\partial x})} = \tilde{y}'(x,y,y';\epsilon), \quad \tilde{y}'' = \frac{d\tilde{y}'}{d\tilde{x}} = \tilde{y}''(x,y,y',y'';\epsilon), \text{ etc;} \tag{2.5.26}$$

Now, the extension of the infinitesimal generator X in (2.5.6) given by

$$\tilde{y}'(x,y;\epsilon) = y + \epsilon\eta^1(x,y,y') + O(\epsilon^2) = y' + \epsilon X y' + O(\epsilon^2), \tag{2.5.27}$$

$$\vdots$$

$$\tilde{y}^{(n)}(x,y;\epsilon) = y^{(n)} + \epsilon\eta^{(n)}(x,y,y',\cdots,y^{(n)}) + O(\epsilon^2) = y + \epsilon X y^{(n)} + O(\epsilon^2). \tag{2.5.28}$$

where $\eta^1, \eta^2, \ldots, \eta^{(n)}$ are defined by

$$\eta^1 = \frac{\partial \tilde{y}'}{\partial \epsilon}\bigg|_{\epsilon=0}, \ldots, \eta^{(n)} = \frac{\partial \tilde{y}^{(n)}}{\partial \epsilon}\bigg|_{\epsilon=0}. \tag{2.5.29}$$

Inserting the expressions (2.5.27) and (2.5.28) into (2.5.26), we obtain

$$\tilde{y}' = y' + \epsilon\eta^{(1)} + O(\epsilon^2) = \frac{d\tilde{y}}{d\tilde{x}} = \frac{dy + \epsilon d\eta + O(\epsilon^2)}{dx + \epsilon d\xi + O(\epsilon^2)} = \frac{y' + \epsilon\frac{d\eta}{dx} + O(\epsilon^2)}{1 + \epsilon\frac{d\xi}{dx} + O(\epsilon^2)} = y' + \epsilon\Big(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\Big) + O(\epsilon^2),$$

Similarly,

$$\tilde{y}^{(k)} = y^{(k)} + \epsilon\eta^{(k)} + O(\epsilon^2) = \frac{d\tilde{y}^{(k-1)}}{d\tilde{x}} = y^{(k)} + \epsilon\Big(\frac{d\eta^{(k-1)}}{dx} - y^{(k)}\frac{d\xi}{dx}\Big) + O(\epsilon^2), \quad k = 2, ..., n \tag{2.5.30}$$

From the above equations we can find

$$\eta^{(1)} = \frac{d\eta}{dx} - y'\frac{d\xi}{dx} = \frac{\partial\eta}{\partial x} + y'\Big(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\Big) - y'^2\frac{\partial\xi}{\partial y}, \tag{2.5.31}$$

$$\eta^{(k)} = \frac{d\eta^{(k-1)}}{dx} - y^{(k)}\frac{d\xi}{dx}, \quad k = 2, ..., n. \tag{2.5.32}$$

The last recursion formula can also be written as

$$\eta^{(k)} = \frac{d}{dx}\eta^{(k-1)} - y^{(k)}\frac{d\xi}{dx}, \qquad k = 1, 2, ... \tag{2.5.33}$$

The derivatives $\eta^{(k)}$ are called the kth-order prolongations of the vector field $X = \xi\partial_x + \eta\partial_y$ and satisfy the recursion relation (2.5.33) which may also be expressed in terms of the characteristics by the following formula

$$\eta^{(k)} = \frac{d^k Q}{dx^k} + y^{(k+1)}\xi, \qquad k = 0, 1, ... \tag{2.5.34}$$

With these preliminary concepts we are now ready to formally define the symmetry condition for a ODE.

Definition 2.5.1 *A differential equation $E(x, y, y', ..., y^{(n)}) = 0$ possesses a symmetry $X = \xi\partial_x + \eta\partial_y$, if $X^{(n)}E = 0$ when $E = 0$ has a nontrivial solution, where $X^{(n)}$ denotes the n-th prolongation of X. The latter being defined by*

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}. \tag{2.5.35}$$

Note that the symmetry condition $X^{(n)}E|_{E=0} = 0$, represents a linear partial differential equation which has to be solved for the coefficient functions ξ and η. Moreover we have considered the simplest case of one independent variable and one dependent variable and point transformations. The number of dependent variables can be increased and this requires a corresponding increase in the number of equations which should equal the number of dependent variables.

Definition 2.5.5 *(Symmetry group)*

The group of transformation (2.5.2) is called a *symmetry group* of an ordinary differential equation

$$\frac{dy}{dx} = \omega(x, y) \tag{2.5.36}$$

if the form of the differential equation remains invariant under the symmetry transformation (2.5.2) so that

$$\frac{d\tilde{y}}{d\tilde{x}} = w(\tilde{x}, \tilde{y}) \tag{2.5.37}$$

The definition being that same for higher-order equations. A symmetry group of a differential equation is also termed as *group admitted* by this equation. The generator X of a group admitted by a differential equation is also called an *admitted operator* of the equation.

2.5.6 ODEs and their equivalent first-order PDEs

Corresponding to an nth-order ordinary differential equation, there always exists an equivalent first-order linear partial differential equation in $n + 1$ variables. Suppose we are given a first order ODE $\frac{dy}{dx} = w(x, y)$. If $\phi(x, y) = c$ be its solution, we must have

$$d\phi = \left(\frac{\partial}{\partial x} + w(x, y)\frac{\partial}{\partial y}\right)\phi = 0.$$

In other words the given ODE is equivalent to the partial differential equation $Af = 0$ where

$$Af = \left(\frac{\partial}{\partial x} + w(x, y)\frac{\partial}{\partial y}\right) f = 0 \Leftrightarrow \frac{dy}{dx} = w(x, y).$$

To understand the nature of the equivalence in the general case, suppose our nth-order ODE is written as

$$y^{(n)} = w(x, y, y', ..., y^{(n-1)}) \tag{2.5.38}$$

and let us assign to it the following partial differential equation (PDE) in $n+1$ variables

$$Af = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots + w \frac{\partial}{\partial y^{(n-1)}} \right) f = 0. \tag{2.5.39}$$

Here the quantities $y', y'', ..., y^{(n-1)}$ are treated as independent variables and have the same status as x and y. If $I = I(x, y, y', ..., y^{(n-1)})$ is a first integral of the ODE then

$$\frac{dI}{dx} = \frac{\partial I}{\partial x} + y' \frac{\partial I}{\partial y} + y'' \frac{\partial I}{\partial y'} + \cdots + w \frac{\partial I}{\partial y^{(n-1)}} = 0, \tag{2.5.40}$$

where we have used the fact that on the solution curves of the given ODE, $y^{(n)} = w$. One can invert the relation $I(x, y, y', ..., y^{(n-1)}) = I_0$, a constant, and solve for $y^{(n-1)}$ to obtain

$$y^{(n-1)} = u(x, y, ..., y^{(n-2)}; I_0) \tag{2.5.41}$$

provided $I_{y^{(n-1)}} \neq 0$. This shows that the existence of a first integral causes a reduction in the order of the ODE by one. Comparison of (2.5.39) with the definition of the first integral (2.5.40) shows that every solution f^α of $Af = 0$ is a first integral I of the ODE $y^{(n)} = w$ and conversely.
Finally, every complete set of n functionally independent solutions ϕ^α of the PDE $Af = 0$ corresponds to the general solution $y = y(x, \phi^\alpha)$ of the ODE, obtained through elimination of all the derivatives of y from the system

$$\phi^\alpha(x, y, y', ..., y^{(n-1)}) = \phi_0^\alpha, \quad \alpha = 1, ..., n.$$

Thus ϕ_0^α essentially represent the constants of integration.

2.5.7 Determination of a Lie point symmetry

Given a general first-order ODE

$$\frac{dy}{dx} = w(x, y).$$

It may be recast as $E(x, y, y') = y' - w(x, y) = 0$. If $X = \xi \partial_x + \eta \partial_y$ denotes a symmetry generator of this equation then the prolonged generator is given by $X^{(1)} = \xi \partial_x + \eta \partial_y + \eta^{(1)} \partial_{y'}$. Alternatively, if we proceed in the same spirit as earlier then we find that

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{y' + \epsilon(\eta_x + y'\eta_y)}{1 + \epsilon(\xi_x + y'\xi_y)} = y' + \epsilon\{(\eta_x + y'\eta_y) - y'(\xi_x + y'\xi_y)\} + \cdots .$$

The first-order term in ϵ may be written in the following suggestive manner,

$$\eta^{(1)} := \frac{d\eta}{dx} - y' \frac{d\xi}{dx}.$$

In other words

$$\frac{d\tilde{y}}{d\tilde{x}} = \tilde{y}' = y' + \epsilon \eta^{(1)} + \cdots .$$

On the other hand for the function on the right hand side we have by a Taylor series expansion

$$w(\tilde{x}, \tilde{y}) = w(x, y) + \epsilon(\xi w_x + \eta w_y) + O(\epsilon^2).$$

However, as $y' = w(x, y)$ it follows that

$$\eta_x + w(\eta_y - \xi_x) - w^2\xi_y = \xi w_x + \eta w_y, \tag{2.5.42}$$

where we have used the expanded expression for $\eta^{(1)}$. In terms of the characteristic

$$Q = \eta - w\xi,$$

the expression (2.5.42) may be recast as

$$Q_x + wQ_y = w_y Q. \tag{2.5.43}$$

The same result is obtained by a direct application of $X^{(1)}$ to E since,

$$X^{(1)}E = \xi(-w_x) + \eta(-w_y) + \eta^{(1)} = 0,$$

implies upon using the expanded form of $\eta^{(1)}$ and the original ODE

$$\eta_x + w(\eta_y - \xi_x) - w^2\xi_y = \xi w_x + \eta w_y.$$

For the nth-order ODE

$$y^{(n)} = w(x, y, y', \cdots, y^{(n-1)}) \tag{2.5.44}$$

the linearized symmetry condition (LSC) is

$$\tilde{y}^{(n)} = w(\tilde{x}, \tilde{y}, \tilde{y}', \cdots, \tilde{y}^{(n-1)}) \text{ on } y^{(n)} = w. \tag{2.5.45}$$

First-order expansions give

$$y^{(n)} + \epsilon\eta^{(n)} = w(x + \epsilon\xi, y + \epsilon\eta, y' + \epsilon\eta^{(1)}, \cdots, y^{(n-1)} + \epsilon\eta^{(n-1)})$$

or

$$\eta^{(n)} = \xi w_x + \eta w_y + \eta^{(1)} w_{y'} + \cdots + \eta^{(n-1)} w_{y^{(n-1)}}. \tag{2.5.46}$$

In terms of the characteristics, $Q = \eta - y'\xi$, this condition may be written as

$$A^n Q - w_{y^{(n-1)}} A^{n-1} Q - \cdots - w_{y'} AQ - w_y Q = 0. \tag{2.5.47}$$

Example 2.5.1

Consider a second-order ODE
$$y'' = \frac{1}{y^3}.$$
Writing the ODE in the form
$$E = y^3 y'' - 1 = 0$$
we have upon applying the prolonged operator $X^{(2)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \eta^{(2)}\partial_{y''}$ to E the following condition
$$3y''\eta + y\eta^{(2)} = 0.$$
Substituting $y'' = y^{-3}$ and using the formula for $\eta^{(2)}$, the expanded form of the symmetry condition reads
$$\frac{3}{y^3}\eta + y\left[-y'^3\xi_{xx} + y'^2(\eta_{yy} - 2\xi_{xy}) + y'(2\eta_{xy} - \xi_{xx} - \frac{3}{y^3}\xi_y) + (\eta_{xx} + \frac{1}{y^3}\eta_y - \frac{2}{y^3}\xi_x)\right] = 0.$$
Equating the various powers of y' then gives the following system of equations:
$$\xi_{xx} = 0, \quad \eta_{yy} - 2\xi_{xy} = 0, \quad 2\eta_{xy} - \xi_{xx} - \frac{3}{y^3}\xi_y = 0, \quad \frac{3}{y}\eta + y\eta_{xx} + \frac{1}{y^2}\eta_y - \frac{2}{y^2}\xi_x = 0.$$
From the first of these we obtain
$$\xi(x, y) = a(x)y + b(x)$$
where $a(x)$ and $b(x)$ are to be determined. From the second it follows, upon using the above expression for ξ, that
$$\eta(x, y) = a'(x)y^2 + c(x)y + d(x).$$
Substituting these expressions for ξ and η into the third equation leads to the equation
$$3a''(x)y + 2c'(x) - b''(x) = 3a(x)\frac{1}{y^3}.$$
By equating coefficients of the different powers of y we conclude that
$$a(x) = 0 \quad c'(x) = \frac{1}{2}b''(x),$$
whence it follows that $c(x) = \frac{1}{2}b'(x) + \lambda$, where λ is a constant. Therefore the general forms of ξ and η are given by $\xi = b(x)$ and $\eta = c(x)y + d(x)$. Finally, using these expressions in the last equation leads to the conclusion that $d(x) = 0$ and $\lambda = 0$ together with $c''(x) = 0$ which implies $c(x) = \alpha x + \beta$ where α and β are arbitrary constants. On the other hand from this expression for $c(x)$ it is easy to deduce that $b(x) = \alpha x^2 + 2\beta x + \gamma$ where γ is also another arbitrary constant. Hence
$$\xi = \alpha x^2 + 2\beta x + \gamma \quad \text{and} \quad \eta = (\alpha x + \beta)y$$

so that

$$X = (\alpha x^2 + 2\beta x + \gamma)\partial_x + (\alpha x + \beta)y\partial_y = \alpha X_3 + \beta X_2 + \gamma X_1$$

where in the last step we have purposely written X as a linear combination of the basic symmetry generators X_i $(i = 1, 2, 3)$. Thus the given ODE admits a set of three symmetries which are given by

$$X_1 = \partial_x \quad X_2 = 2x\partial_x + y\partial_y \quad \text{and} \quad X_3 = x^2\partial_x + xy\partial_y.$$

2.5.8 An alternative formulation of the symmetry condition

We have already mentioned that an nth-order ODE, $y^{(n)} = w(x, y, y', ..., y^{(n-1)})$, is equivalent to the linear PDE

$$Af = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots + w\frac{\partial}{\partial y^{(n-1)}}\right) f = 0, \tag{2.5.48}$$

and furthermore any Lie point symmetry that may exists can be determined from a knowledge of the prolonged generator of the transformation,

$$X^{(n-1)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y'} + \cdots + \eta^{(n-1)}\frac{\partial}{\partial y^{(n-1)}}$$

where $\eta^{(k)}$'s denote the k-th prolongation of the infinitesimal generator. It is natural to inquire about the conditions under which the prolonged generator represents a symmetry of the PDE and, by equivalence, of the corresponding ODE. Consider a set of n independent solutions f^α of (2.5.48). Since a symmetry by definition has to map solutions into solutions; it follows that if f^α is a solution, then

$$X^{(n-1)}f^\alpha = G^\alpha(f^\beta), \qquad Af^\alpha = 0 = AG^\alpha, \tag{2.5.49}$$

the last part follows from the fact the any function G of the solutions is also a solution. One may now eliminate the functions G^α by constructing the commutator of $X^{(n-1)}$ and A,

$$\begin{aligned}\left[X^{(n-1)}, A\right] &= X^{(n-1)}A - AX^{(n-1)} \\ &= -(A\xi)\frac{\partial}{\partial x} + \left[(X^{(n-1)}y') - (A\eta)\right]\frac{\partial}{\partial y} + \cdots + \left[(X^{(n-1)}w) - (A\eta^{(n-1)})\right]\frac{\partial}{\partial y^{(n-1)}}\end{aligned} \tag{2.5.50}$$

and is a linear operator like $X^{(n-1)}$ and A. In view of (2.5.49) this commutator vanishes, since

$$\left[X^{(n-1)}, A\right] f^\alpha = X^{(n-1)}(Af^\alpha) - A(X^{(n-1)}f^\alpha) = 0.$$

Since this must hold for all functions f^α, the equation $\left[X^{(n-1)}, A\right] f = 0$ has the same set of solutions as the equation $Af = 0$, and therefore these operators can only differ by a non constant factor g:

$$\left[X^{(n-1)}, A\right] = g(x, y, ..., y^{(n-1)})A. \tag{2.5.51}$$

When the condition (2.5.51) is satisfied by the prolonged operator $X^{(n-1)}$, we say that it defines a symmetry of $Af = 0$.

Example 2.5.2

It will be recalled that for the ODE, $y'' = w = 1/y^3$, the symmetry generators were found to be

$$X_1 = \partial_x, \quad X_2 = 2x\partial_x + y\partial_y, \quad X_3 = x^2\partial_x + xy\partial_y.$$

Since this is a second-order ODE we need to construct only the first-order prolongations of these generators, i.e., $X_k^{(1)} = X + \eta^{(1)}\partial_{y'}$. Simple calculations show that

$$X_1^{(1)} = \partial_x, \quad X_2^{(1)} = 2x\partial_x + y\partial_y - y'\partial_{y'}, \quad X_3^{(1)} = x^2\partial_x + xy\partial_y + (y - xy')\partial_{y'},$$

respectively. On the other hand the ODE is equivalent to

$$Af = \left(\partial_x + y'\partial_y + \frac{1}{y^3}\partial_{y'}\right) f = 0.$$

In order to calculate the commutator $[X^{(1)}, A]$ we shall make use of (2.5.50). For instance in case of X_2 we have

$$\xi = 2x, \quad \eta = y \quad \eta^{(1)} = -y' \text{ so that } A\xi = 2, \quad A\eta = y' \text{ and } A\eta^{(1)} = -\frac{1}{y^3}.$$

On the other hand

$$X_2^{(1)}y' = -y', \quad X_2^{(1)}w = X_2^{(1)}\frac{1}{y^3} = -\frac{3}{y^3}.$$

Therefore we find that

$$\left[X_2^{(1)}, A\right] = -2\left[\partial_x + y'\partial_y + \frac{1}{y^3}\partial_{y'}\right] = -2A.$$

2.5.9 Algebra of Symmetry Generators:

A key feature of the symmetry generators is that they form an algebra under the operation of a Lie Bracket. Symmetries are differential operators and one can calculate their Lie brackets. For any two generators X_i and X_j, their Lie Bracket is defined by

$$[X_i, X_j] := X_iX_j - X_jX_i.$$

Given $X_1 = \xi_1(x,y)\partial_x + \eta_1(x,y)\partial_y$ and $X_2 = \xi_2(x,y)\partial_x + \eta_2(x,y)\partial_y$, the Lie bracket of X_1 and X_2 is

$$[X_1, X_2] = X_1X_2 - X_2X_1 = (\xi_1\partial_x + \eta_1\partial_y)(\xi_2\partial_x + \eta_2\partial_y) - (\xi_2\partial_x + \eta_2\partial_y)(\xi_1\partial_x + \eta_1\partial_y)$$

$$= (\xi_1\xi_{2x} + \eta_1\xi_{2y} - \xi_2\xi_{1x} - \eta_2\xi_{1y})\partial_x + (\xi_1\eta_{2x} + \eta_1\eta_{2y} - \xi_2\eta_{1x} - \eta_2\eta_{1y})\partial_y$$

The result of calculation of Lie Bracket is either zero or another differential operator. The resultant differential operation of symmetries is also a symmetry of the differential equation or differential function. The set of all symmetries form an algebra known as Lie algebra and this is the importance of the symmetries with its other uses.

One can easily verify that the symmetry generators satisfy the following $sl(2, R)$ Lie algebra, a representation of the special linear group in two dimension.

$$[X_1, X_2] = 2X_1 \quad [X_1, X_3] = X_2 \quad [X_2, X_3] = 2X_3.$$

As an example of a second-order ODE having a maximal number of symmetry generators, one can carry out a similar analysis for the equation

$$y'' = 0,$$

and verify that the list of symmetry generators are given by:

$$X_1 = y\partial_y$$

$$X_2 = 1\partial_y, \quad X_3 = x\partial_y$$

$$X_4 = \partial_x, \quad X_5, x\partial_x + \frac{1}{2}y\partial_y, \quad X_6 = x^2\partial_x + xy\partial_y$$

$$X_7 = y\partial_x, \quad X_8 = xy\partial_x + y^2\partial_y.$$

By explicitly calculating their Lie Brackets, it may further be verified that these eight generators form a representation of the $sl(3, R)$ Lie algebra. It is interesting however to understand the underlying implications of the symmetry operations associated with these generators. One will observe that whenever the ODE is a homogeneous one, we always have the first symmetry X_1. The symmetries X_2 and X_3 are called solution symmetries because their coefficient functions represent solutions of the differential equation. The next three symmetries X_4, X_5 and X_6 actually form a $sl(2, R)$ algebra amongst themselves, i.e., an $sl(2, R)$ subalgebra is formed by these generators. The last two symmetries are called non Cartan symmetries and the symmetry operations induced by them are distinct from the others.
There are a few algebras which occur which occur frequently and these are listed below.

1. Second order: The maximal number of the Lie point Symmetries is eight and the algebra is denoted by $Sl(2, R)$. Any scalar second-order equation which has eight Lie point symmetries can be transformed to $y'' = 0$ by means a point transformation.

2. Third order: There are maximum seven Lie point symmetries and is represented by the equation $y''' = 0$. there can also be four or five.

3. Fourth and Higher order: There are three possibilities and the number of symmetries can be $n + 4$, $n + 2$ or $n + 1$.

Algebra of Dimension Three:

There are eleven real Lie algebras of dimension three.

Algebra	Nonzero commutation relations
$3A_1$	
$A_1 \bigoplus A_2$	$[X_1, X_3] = X_1$
$A_{3,1}$ (Weyl)	$[X_2, X_3] = X_1$
$A_{3,2}$	$[X_1, X_3] = X_1, \quad [X_2, X_3] = X_1 + X_2$
$A_{3,3}$	$[X_1, X_3] = X_1, \quad [X_2, X_3] = X_2$
$A_{3,4(E(1,1))}$	$[X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2$
$A^a_{3,5}(0 < \|a\| < 1)$	$[X_1, X_3] = X_1, \quad [X_2, X_3] = aX_2$
$A_{3,6}E(2)$	$[X_1, X_3] = -X_2, \quad [X_2, X_3] = X_1$
$A^b_{3,7}(b > 0)$	$[X_1, X_3] = bX_1 - X_2, \quad [X_2, X_3] = X_1 + bX_2$
$A_{3,8}(Sl(2, R))$	$[X_1, X_2] = 2X_1,\ [X_1, X_3] = X_2,\ [X_2, X_3] = 2X_3$
$A_{3,9}(So(3))$	$[X_1, X_2] = X_3 \quad [X_1, X_3] = X_1 \quad [X_2, X_3] = X_2$

2.5.10 The Method of integrating factors for deducing a first integral

Let $I = x, y, y', \ldots, y^{(n-1)}$ denotes the first integral of

$$y^{(n)} = w(x, y, y', \ldots, y^{(n-1)}) \tag{2.5.52}$$

so that

$$\frac{dI}{dx} = 0 \quad \text{when equation (2.5.52) holds.} \tag{2.5.53}$$

As mentioned in section 2.5.6, the last equation can also be written as

$$AI = 0, \quad I_{y^{(n-1)}} \neq 0 \tag{2.5.54}$$

where

$$A = \partial_x + y'\partial_y + \cdots + y^{(n-2)}\partial_{y^{(n-3)}} + y^{(n-1)}\partial_{y^{(n-2)}} + w\partial_{y^{(n-1)}}. \tag{2.5.55}$$

A nonzero function $\Lambda(x, y)$ is an integrating factor of the ODE (2.5.52) if

$$\Big(y^{(n)} - w\Big)\Lambda = \frac{d\phi}{dx}. \tag{2.5.56}$$

for some function $\phi(x, y, y', \ldots, y^{(n-1)})$. Thus, ϕ is an first integral whenever (2.5.52) holds. Furthermore, every first integral ϕ satisfies

$$\frac{d\phi}{dx} = A\phi + (y^{(n)} - w)\phi_{y^{(n-1)}} = (y^{(n)} - w)\phi_{y^{(n-1)}}. \tag{2.5.57}$$

Comparing (2.5.56) and (2.5.57) we see that every integrating factor has to be of the form

$$\Lambda(x, y, y', \ldots, y^{(n-1)})) \equiv \phi_{y^{(n-1)}}, \tag{2.5.58}$$

for some first integral ϕ. Below we discuss how an integrating factor can be systematically determined.
For this purpose we note the following useful identities:

$$A\partial_{y^{(k)}} = \partial_{y^{(k-1)}}A - \partial_{y^{(k-1)}} - w_{y^{(k)}}\partial_{y^{(n-1)}}, \quad k = 1, \ldots, n-1. \tag{2.5.59}$$

In the above relation we assume the convention $y = y^{(0)}$ and $\partial_{y^{(-1)}} \equiv 0$. Using the identity (2.5.59) with $k = n-1$, and $A\phi = 0$, we obtain

$$A\phi_{y^{(n-1)}} = -\phi_{y^{(n-2)}} - w_{y^{(n-1)}}\phi_{y^{(n-1)}}.$$

This may also be written as

$$\phi_{y^{(n-2)}} = -(A\phi_{y^{(n-1)}} + w_{y^{(n-1)}}\phi_{y^{(n-1)}}). \tag{2.5.60}$$

Applying A to each $\phi_{y^{(k)}}$, we have

$$\phi_{y^{(k-1)}} = -(A\phi_{y^{(k)}} + w_{y^{(k)}}\phi_{y^{(n-1)}}) \quad k = 0, \ldots, n-1. \tag{2.5.61}$$

For $k = 0$ the equation becomes

$$0 = -(A\phi_y + w_y\phi_{y^{(n-1)}}), \tag{2.5.62}$$

so that by using (2.5.58) it is possible to write each partial derivative of ϕ in terms of Λ and its derivatives and as a result (2.5.62) can be expressed as follows

$$A^n\Lambda + A^{n-1}(w_{y^{(n-1)}}\Lambda) - A^{n-2}(w_{y^{(n-2)}}\Lambda) + \cdots + (-1)^{n-1}w_y\Lambda = 0. \tag{2.5.63}$$

Equation (2.5.63) is called the *adjoint* of the linearized symmetry condition (2.5.47) and its solutions are termed as *adjoint symmetries* of the ODE. We shall call any nonzero solution Λ of (2.5.63) a co-characteristic. Having found solutions Λ^i we may determines the ones that are integrating factors of the given ODE in the following manner. For that, we first calculate recursively the quantities:

$$\begin{gathered} P^i_{n-1} = \Lambda^i, \\ P^i_{k-1} = -AP^i_k - w_{y^{(k)}}\Lambda^i, \quad k = n-1, n-2, \dots, 1 \end{gathered} \tag{2.5.64}$$

From (2.5.58), (2.5.61), we see that Λ^i is an integrating factor if

$$\begin{gathered} P^i_k = \phi^i_{y^{(k)}}, \quad k = 0, \dots, n-1 \\ wP^i_{n-1} + \sum_{k=0}^{n-2} y^{(k+1)} P^i_k = -\phi^i_x. \end{gathered} \tag{2.5.65}$$

The integrability condition is satisfied if and only if

$$\frac{\partial P^i_{n-1}}{\partial y^{(j)}} = \frac{\partial P^i_j}{\partial y^{(n-1)}}, \quad 0 \le j \le n-2. \tag{2.5.66}$$

Thus, R is an integrating factor if and only if the integrability condition (2.5.66) is satisfied. The first integral ϕ^i is then lastly obtained as a line integral from

$$\phi^i = \int P^i_0(dy - y'dx) + P^i_1(dy' - y''dx) + \cdots + P^i_{n-1}(dy^{(n-1)} - wdx). \tag{2.5.67}$$

Another important aspect of the analysis of ODEs is the issue of their linearization to which we now turn our attention in the following section.

2.6 Linearization of Ordinary Differential Equation

As emphasized in the introduction the linearization problem of nonlinear ODEs is of fundamental importance in their analysis. This is because of the existence of a large number of techniques that have been accumulated, over almost two centuries, for the analysis of linear ODEs. Keeping this in mind we will now outline the process of linearization of an ODE by three specific classes of transformations which are explained below.

2.6.1 Point Transformation

Consider the general second-order ordinary differential equation

$$\ddot{x} + A_3(x,t)\dot{x}^3 + A_2(x,t)\dot{x}^2 + A_1(x,t)\dot{x} + A_0(x,t) = 0 \tag{2.6.1}$$

Under a point transformation $(t, x) \mapsto (T, X)$ of the form

$$T = G(t,x), X = F(t,x), \tag{2.6.2}$$

(2.6.1) may be mapped to the following equation of a free particle, viz.

$$\frac{d^2X}{dT^2} = 0 \tag{2.6.3}$$

provided the coefficients $A_i(x,t)$, $i = 0,\ldots,3$ are related to $F(x,t)$ and $G(x,t)$ by the following

$$A_3 =[G_xF_{xx} - G_{xx}F_x]/\Delta \tag{2.6.4}$$
$$A_2 =[G_tF_{xx} + 2G_xF_{tx} - 2F_xG_{xt} - F_tG_{xx}]/\Delta \tag{2.6.5}$$
$$A_1 =[G_xF_{tt} + 2G_tF_{tx} - 2F_tG_{tx} - F_xG_{tt}]/\Delta \tag{2.6.6}$$
$$A_0 =[G_tF_{tt} - G_{tt}F_t]/\Delta \tag{2.6.7}$$

where

$$\Delta = G_tF_x - G_xF_t \neq 0. \tag{2.6.8}$$

Given a second-order differential equation such that the implicit form of coefficients $A_i(x,t)$ are known, upon solving the above set of equations if one can deduce the functions $F(x,t)$ and $G(x,t)$ then the linearization transformation may be determined and equation (2.6.1) is said to be linearizable to (2.6.3).

For the linearized equation (2.6.3) a first integral is obviously given by

$$I_1 = \frac{dX}{dT} = \frac{F_x\dot{x} + F_t}{G_x\dot{x} + G_t}. \tag{2.6.9}$$

Hence from the knowledge of linearizing transformation (2.6.2) one can easily deduce a first integral for the original second-order equation (2.6.1).

Regarding the issue of compatibility we note that the compatibility conditions between the coefficients $A_i(x,t),\ i = 0,\ldots,3$ such that (2.6.1) is linearizable by means of a point transformation were first derived by Tresse *et. al.*[104] and are as follows:

$$A_{1xx} - 2A_{2xt} + 3A_{3tt} + 6A_3A_{0x} + 3A_0A_{3x} - 3A_3A_{1t} - 3A_1A_{3t} - A_2A_{1x} + A_2A_{2t} = 0, \quad (2.6.10)$$

$$2A_{1xt} - A_{2tt} - 3A_{0xx} + 6A_0A_{3t} + 3A_3A_{0t} - 3A_0A_{2x} - 3A_2A_{0x} - A_1A_{2t} + 2A_1A_{1x} = 0. \quad (2.6.11)$$

2.6.2 Non-Point Transformation

A non-point transformation may be regarded as a kind of nonlocal generalization of point transformation. The linearization problem for second-order ODEs under a nonlocal transformation of the form

$$X(T) = F(t,x), \quad dT = G(t,x)dt \quad (2.6.12)$$

was first studied by Duarté *et al* [25].
Here F and G are arbitrary smooth functions with the Jacobian $J \equiv \partial(T,X)/\partial(t,x) \neq 0$. Note that in comparison with the point transformation (2.6.2) only one half of the transformation (2.6.12) is of a nonlocal character, namely the transformation for the variable T. Of course if one knows the functional form of $x(t)$ then the latter part of (2.6.12) ceases to be nonlocal. But knowledge of $x(t)$ is what we are interested in the first place. Consequently (2.6.12) does indeed constitute a nonlocal transformation for the variable T. However, it has to be mentioned that as the term 'nonlocal' is of a very general nature and can mean different things depending on the context it is more appropriate to refer to the transformation (2.6.12) as a generalized Sundman transformation (GST) in view of its similarity to the transformation $dt = rd\tau$ introduced by Sundman to study the three-body problem in the context of celestial mechanics [103]. Here r stands for the dependent variable (radial component). In Chapter **6** we will demonstrate that a GST is an effective tool for deducing first integrals and parametric solutions of several nonlinear ODEs. We shall also introduce the notion of an associated symmetry corresponding to such nonlocal transformations.

In [25] the authors derived the most general condition under which a second-order ordinary differential equation is transformable to the linearized equation $X''(T) = 0$, (here $X' = \frac{dX}{dT}$) by means of a generalized Sundman transformation. By using the fundamental invariants of this equation they obtained the first integrals of several second-order ordinary differential equations, which could be linearized. The case of the general anharmonic oscillator was studied by Euler and Euler in [27]. Of late there have been a number of papers concerned with linearization of second and third-order ODEs by using the generalized Sundman transformations as also by other methods [11, 12, 28, 64, 71, 72, 73, 75].

Now second-order ordinary differential equation of the form

$$\ddot{x} + A_2(x,t)\dot{x}^2 + A_1(x,t)\dot{x} + A_0(x,t) = 0 \quad (2.6.13)$$

is equivalent to the free particle equation

$$\frac{d^2X}{dT^2} = 0 \tag{2.6.14}$$

under the transformation (2.6.12) when $A_i(x,t), i = 0, \ldots, 2$ are related with $F(x,t)$ and $G(x,t)$ by the following relations

$$A_2(x,t) = [GF_{xx} - F_xG_x]/\Delta' \tag{2.6.15}$$
$$A_1(x,t) = [2GF_{tx} - F_tG_x - F_xG_x]/\Delta' \tag{2.6.16}$$
$$A_0(x,t) = [GF_{tt} - F_tG_t]/\Delta' \tag{2.6.17}$$

with

$$\Delta' = GF_x. \tag{2.6.18}$$

An invariant I for the equation (2.6.13) can be found directly from the invariant $I = dX/dT$ of the equation (2.6.14):

$$I = \frac{dX}{dT} = \frac{F_x}{G}\dot{x} + \frac{F_t}{G}. \tag{2.6.19}$$

After some algebraic computation to eliminate the function F and G and their derivatives we find the compatibility conditions, which are analogous to the Tresse-Cartan conditions (2.6.10) and (2.6.11). These condition lead to the following possibilities:

(i) $S_1(x,t) := A_{1x} - 2A_{2t} = 0, \quad (2.6.20)$

$$S_2(x,t) := 2A_{0xx} - 2A_{1tx} + 2A_0A_{2x} - A_{1x}A_1 + 2A_{0x}A_2 + 2A_{2tt} = 0. \tag{2.6.21}$$

(ii) If $S_1(x,t) \neq 0$ and $S_2(x,t) \neq 0$ then

$$S_2^2 + 2S_{1t}S_2 - 2S_1^2A_{1t} + 4S_1^2A_{0x} + 4S_1^2A_0A_2 - 2S_1S_{2t} - S_1^2A_1^2 = 0, \tag{2.6.22}$$
$$S_{1x}S_2 + S_1^2A_{1x} - 2S_1^2A_{2t} - S_1S_{2x} = 0. \tag{2.6.23}$$

Example 2.6.1

The equation

$$\ddot{x} - \frac{2}{x}\dot{x}^2 + \frac{2x}{t^2} = 0. \tag{2.6.24}$$

This equation is linearizable by the non-point transformation

$$X = t^3x^{3/2}, \quad dT = tx^{5/2}dt, \tag{2.6.25}$$

as can be easily verified and a first integral obtained by using equation (2.6.19) and is given by

$$I = \frac{t}{x}\left(\frac{t}{2x}\dot{x} + 1\right). \tag{2.6.26}$$

Finally we end this section by noting that there exist a third possibility in which one can attempt to linearize a second-order ODEs by means of a transformation which is nonlocal in both the new variables X and T, i.e., a transformation of the form

$$dX = A(x,t)dx + B(x,t)dt$$

$$dT = C(x,t)dx + D(x,t)dt.$$

Such completely nonlocal transformation are considered in Chapter **7** and will be employed to find first integrals of a time dependent higher-order Riccati equation.

2.7 The Jacobi Last Multiplier

The Jacobi Last Multiplier is a useful tool originally introduced by Jacobi in the context of integrability of a system of first-order ODEs. It plays an important role in Lagrangian dynamics as will be explained in the Chapter 8, but prior to that we introduced here some of its basic features.

2.7.1 Jacobi's construction of the last multiplier

Let $M = M(x_1, \cdots, x_n)$ be a non-negative C^1 function non-identically vanishing on any open subset of $\mathbb{R}^n$. Consider a set of first order equations

$$\frac{dx_r}{dt} = W_r(x_1, \cdots, x_n), \quad r = 1, ..., n, \tag{2.7.1}$$

where the vector fields $(W_1, W_2, \cdots, W_n)$ are functions of $(x_1, \cdots, x_n, t)$. Let $(c_1, \cdots, c_k)$ be a set of constant of motions of these set of equations. The Jacobi Last Multiplier may be regarded as the density associated with the invariant measure $\int_\Omega M\, dx$, where Ω is any open subset of $\mathbb{R}^n$. Thus the invariance of flux implies

$$\int_\Omega M\delta x_1 \cdots \delta x_k = \int_{\phi_t(\Omega)} M \frac{\partial(x_1, \cdots, x_k)}{\partial(c_1, \cdots, c_k)} \delta c_1 \cdots \delta c_k, \tag{2.7.2}$$

where $\phi_t(.)$ is the flow generated by the solutions $x = x(t)$ of $\dot{x} = W(x)$. In other words, $\phi_t(\Omega)$ is the transformation of the domain Ω under the flow generated by the solution. This invariant condition yields

$$\frac{d}{dt}\Big\{M \frac{\partial(x_1, \cdots, x_k)}{\partial(c_1, \cdots, c_k)}\Big\} = 0 \tag{2.7.3}$$

$$\text{or} \quad \frac{dM}{dt}\frac{\partial(x_1, \cdots, x_k)}{\partial(c_1, \cdots, c_k)} + M\sum_{p=1}^{k} \frac{\partial(x_1, \cdots, x_{p-1}, W_p, \cdots, x_k)}{\partial(c_1, \cdots, c_k)} = 0$$

$$\text{so that} \quad \frac{dM}{dt}\frac{\partial(x_1, \cdots, x_k)}{\partial(c_1, \cdots, c_k)} + M\sum_{p=1}^{k} \frac{\partial W_p}{\partial x_p} \cdot \frac{\partial(x_1, \cdots, x_k)}{\partial(c_1, \cdots, c_k)} = 0,$$

and leads to the following equation:

$$\frac{dM}{dt} + M\sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0. \tag{2.7.4}$$

The equation serves as the defining equation for a Jacobi last multiplier. Note that the last multiplier is not unique and this has a useful consequence.

Proposition 2.7.1 *The ratio of two Jacobi Last Multiplier (JLM) is a first integral.*

Proof: The existence of more than one Jacobi multiplier leads to the following result. Suppose M_1 and M_2 are the Jacobi Last multipliers for system (2.7.1), then from (2.7.4) we have

$$\frac{dM_\alpha}{dt} + M_\alpha \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0,\ \alpha = 1, 2,$$

implying

$$\frac{d}{dt}\Big(\frac{M_1}{M_2}\Big) = 0,$$

i.e., the ratio of two JLM's is a first integral. Alternatively, M_1/M_2 satisfy the equivalent PDE of the differential equation i.e. $A(M_1/M_2 = 0)$.

Lemma 2.7.1 *If a system of differential equations*

$$\frac{dx_r}{dt} = W_r,\ (r = 1, \ldots, n) \tag{2.7.5}$$

is transformed by change of variables into another system

$$\frac{dy_r}{dt} = Y_r,\ (r = 1, \ldots, n) \tag{2.7.6}$$

then

$$\sum_{r=1}^{n} \frac{\partial W_r}{\partial x_r} = \frac{1}{D} \sum_{r=1}^{n} \frac{\partial Y_r}{\partial y_r} \tag{2.7.7}$$

where D denotes the Jacobian

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)}$$

Proof: Under the change of variables $x \to y$ we have

$$\sum_{r=1}^{n} \frac{\partial W_r}{\partial x_r} = \sum_{r=1}^{n} \frac{\partial}{\partial x_r}\Big(\sum_{k=1}^{n} Y_k \frac{\partial x_r}{\partial y_k}\Big) = \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial y_s}{\partial x_r}\frac{\partial}{\partial y_s}\Big(\sum_{k=1}^{n} Y_k \frac{\partial x_r}{\partial y_k}\Big)$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{k=1}^{n} \frac{\partial y_s}{\partial x_r}\Big(Y_k \frac{\partial^2 x_r}{\partial y_s \partial y_k} + \frac{\partial Y_k}{\partial y_s}\frac{\partial x_r}{\partial y_k}\Big).$$

In this expression the coefficient of $\frac{\partial Y_k}{\partial y_s}$ is $\sum_{r=1}^{n} \frac{\partial y_s}{\partial x_r}\frac{\partial x_r}{\partial y_k}$, which is zero or unity according as s is different from, or equal to k. Also $\frac{\partial y_s}{\partial x_r} = \frac{A_{rs}}{D}$, where A_{rs} denotes the minor of $\frac{\partial x_r}{\partial y_s}$ in the determinant D. Again the coefficient of Y_k in the above expression is

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial y_s}{\partial x_r}\frac{\partial^2 x_r}{\partial y_s \partial y_k},$$

and may be written as

$$\frac{1}{D}\sum_{r=1}^{n}\sum_{s=1}^{n}A_{rs}\frac{\partial^2 x_r}{\partial y_s \partial y_k} = \frac{1}{D}\sum_{r=1}^{n}\frac{\partial(x_1, x_2, \ldots, x_{r-1}, \partial x_r/\partial y_k, x_{r+1}, \ldots, x_r)}{\partial(y_1, y_2. \ldots, y_n)} = \frac{1}{D}\frac{\partial D}{\partial y_k}.$$

We have therefore

$$\sum_{r=1}^{n}\frac{\partial W_r}{\partial x_r} = \sum_{k=1}^{n}\frac{\partial Y_k}{\partial y_k} + \sum_{k=1}^{n} Y_k \frac{1}{D}\frac{\partial D}{\partial y_k} = \frac{1}{D}\sum_{k=1}^{n}\frac{\partial D Y_k}{\partial y_k}$$

which establishes the lemma.

The reason for calling it the 'last multiplier' is actually contained in Jacobi's original criterion for the integrability of the system of n first-order ODEs. In this formulation one requires almost complete knowledge of the system under consideration in the sense that if $n-2$ first integrals of the system are known then from the knowledge of a multiplier one can derive an additional i.e., the $(n-1)$-th first integral and thereby reduce the system to quadrature. We outline this process below [105].

Consider a system of n ODEs of the form (2.7.1). Let

$$f_r(x_1, \ldots, x_n) = c_r, \ (r = 1, \ldots, n-2) \tag{2.7.8}$$

denote $n-2$ known first integrals of the system and consider the change of variables

$$(x_1, \ldots, x_{n-1}, x_n) \to (c_1, \ldots, c_{n-1}, y_{n-1}, y_n), \tag{2.7.9}$$

with $y_{n-1} = x_{n-1}$ and $y_n = x_n$. If

$$\Delta = \frac{1}{D} = \frac{\partial(f_1, \ldots, f_{n-2}, x_{n-1}, x_n)}{\partial(x_1, \ldots, x_{n-2}, x_{n-1}, x_n)} = \frac{\partial(f_1, \ldots, f_{n-2})}{\partial(x_1, \ldots, x_{n-2}, x_{n-1})}, \tag{2.7.10}$$

then using (2.7.9) to eliminate $x_1, \ldots, x_{n-2}$ the system reduces to a planar one:

$$\frac{dx_{n-1}}{dt} = W'_{n-1}(x_{n-1}, x_n, \{c_r\}), \tag{2.7.11}$$

$$\frac{dx_n}{dt} = W'_n(x_{n-1}, x_n, \{c_r\}). \tag{2.7.12}$$

then by using lemma (2.7.1), we have

$$\sum_{i=1}^{n}\frac{\partial W_i}{\partial x_i} = \Delta\Big[\frac{\partial}{\partial x_{n-1}}\Big(\frac{W'_{n-1}}{\Delta'}\Big) + \frac{\partial}{\partial x_n}\Big(\frac{W'_n}{\Delta'}\Big)\Big], \tag{2.7.13}$$

where

$$\Delta' \equiv \Delta'(x_{n-1}, x_n, \{c_r\}).$$

Since the last multiplier is defined as a solution of

$$\frac{d\log M}{dt} + \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0, \tag{2.7.14}$$

it satisfies, in view of (2.7.13) the equation

$$\frac{d\log M}{dt} + \Delta\Big[\frac{\partial}{\partial x_{n-1}}\Big(\frac{W'_{n-1}}{\Delta'}\Big) + \frac{\partial}{\partial x_n}\Big(\frac{W'_n}{\Delta'}\Big)\Big] = 0,$$

or,

$$\frac{\partial}{\partial x_{n-1}}\Big(\frac{M'W'_{n-1}}{\Delta'}\Big) + \frac{\partial}{\partial x_n}\Big(\frac{M'W'_n}{\Delta'}\Big) = 0, \tag{2.7.15}$$

which yields the first integral

$$I(x_{n-1}, x_n) = \int \frac{M'}{\Delta'}(W'_{n-1}dx_n - W'_n dx_{n-1}). \tag{2.7.16}$$

Example 2.7.1 *(Whittaker's equation)*

$$\dot{x}_1 = x_3,\ \dot{x}_2 = x_4,\ \dot{x}_3 = x_1,\ \dot{x}_4 = x_3 \tag{2.7.17}$$

It is easy to verify that

$$f_1 = x_4 - x_1 = c_1$$

and

$$f_2 = x_2 - x_3 - t(x_4 - x_1) = c_2$$

are two first integrals of the above system of equations. From these two first integrals we obtain $x_1 = x_4 - c_1$ and $x_2 = x_3 + c_1 t + c_2$.

The resulting planar system has the form

$$\dot{x}_3 = x_4 - c_1,\ \ \dot{x}_4 = x_3.$$

On the other hand it follows from the first integral

$$\Delta = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = -1$$

and the simple calculation shows that the last multiplier is $M = 1$.
Hence by

$$I = \int (x_3 dx_3 - (x_4 - c_1)dx_4) = \frac{1}{2}(x_1^2 - x_3^2).$$

So that the third first integral may be taken as

$$f_3 = x_1^2 - x_3^2 = c_3.$$

2.7.2 Jacobi Last Multiplier for a planar vector field

Let us begin by considering a planar system

$$\dot{x} = f(x, y, \quad \dot{y} = g(x, y). \tag{2.7.18}$$

For such a system a knowledge of last multiplier enable us to find a first integral and hence the solution. Assuming $M = M(x, y)$ to be C^1 (non-zero) fuction such that

$$M(gdx - fdy) = dI. \tag{2.7.19}$$

Since $gdx - fdy = 0$, on all solutions we have $dI = 0$ and hence $I(x, y)$ is a first integral. Now,

$$dI = I_x dx + I_y dy. \tag{2.7.20}$$

Substituting (2.7.20) in (2.7.19), we obtain

$$I_x = Mg, \tag{2.7.21}$$

$$I_y = -Mf. \tag{2.7.22}$$

Therefore,

$$I = -\int^y Mfdy + J(x). \tag{2.7.23}$$

Where $J(x)$ is an arbitrary function. Next (2.7.21) implies

$$J'(x) = Mg + \frac{\partial}{\partial x}\int^y Mfdy. \tag{2.7.24}$$

From relation (2.7.21), equating the mixed derivatives $I_{xy} = I_{yx}$, we have

$$\frac{\partial(Mf)}{\partial x} + \frac{\partial(Mg)}{\partial y} = 0. \tag{2.7.25}$$

The function is also called the *density of the integral invariant* since on any bounded closed region D in the phase plane R^2 we have

$$\int_D \int M(x, y)dxdy = \int_{\phi(t)(D)} \int M(x, y)dxdy, \tag{2.7.26}$$

where $\phi(t)(D)$ is the transformation of the domain D under the flow generated by the solution. Therefore, for the determination of the first integral $I = I(x, y)$ of (2.7.18), we search for a non-trivial solution of (2.7.25) for the multiplier M, whose role here is similar to that of an integrating factor.

The formal definition of the Jacobi Last Multiplier for an n-th order ODE $y^{(n)} = w(x, y, y', \ldots, y^{n-1})$ is as follows.

Definition 2.7.1 *Given an nth order ODE or its equivalent linear PDE in* $(n+1)$ *variables*

$$Af = (\partial_x + y'\partial_y + y''\partial_{y'} + \cdots + w\partial_{y^{(n-1)}})f = 0, \tag{2.7.27}$$

the Jacobi last multiplier M *is defined by*

$$MAf := \frac{\partial(f, \phi^1, \phi^2 ... \phi^n)}{\partial(x, y, y' ... y^{(n-1)})} = \det \begin{pmatrix} f_x & f_y & \cdots & f_{y^{(n-1)}} \\ \phi^1_x & \phi^1_y & \cdots & \phi^1_{y^{(n-1)}} \\ \vdots & \vdots & \cdots & \vdots \\ \phi^{(n)}_x & \phi^{(n)}_y & \cdots & \phi^{(n)}_{y^{(n)}} \end{pmatrix} = 0. \tag{2.7.28}$$

From the above definition, it follows that the Jacobi last multiplier (JLM) can be varied by selecting a different set of $(n-1)$ independent solutions $\psi^1, \psi^2, ..., \psi^{n-1}$ of (2.7.27). If the corresponding JLM be $\tilde{M}$ then

$$\tilde{M}Af = \frac{\partial(f, \psi^1, \psi^2 ... \psi^{n-1})}{\partial(x, y, y' ... y^{(n-1)})} = \frac{\partial(f, \phi^1, \phi^2 ... \phi^n)}{\partial(x, y, y' ... y^{(n-1)})} \frac{\partial(\psi^1, \psi^2 ... \psi^{n-1})}{\partial(\phi^1, \phi^2 ... \phi^{n-1})} = M \frac{\partial(\psi^1, \psi^2 ... \psi^{n-1})}{\partial(\phi^1, \phi^2 ... \phi^{n-1})}$$

Indeed, each JLM as defined above turns out to be a solution of the following linear PDE

$$\frac{\partial M}{\partial x} + \sum_{k=1}^{n} \frac{\partial M y^{(k)}}{\partial y^{(k-1)}} = 0 \text{ on } y^{(n)} = w(x, y, y', ... y^{(n-1)}). \tag{2.7.29}$$

We verify this for the case of $n = 2$, i.e., for a second-order ODE, say $y'' = w(x, y, y')$ whose associated PDE is

$$(\partial_x + y'\partial_y + w\partial_{y'})f = 0.$$

With the above definition, we find that

$$(M\partial_x + My'\partial_y + Mw\partial_{y'})\, f = \det \begin{pmatrix} f_x & f_y & f_{y'} \\ \phi^1_x & \phi^1_y & \phi^1_{y'} \\ \phi^2_x & \phi^2_y & \phi^2_{y'} \end{pmatrix}.$$

Expanding the determinant on the right hand side and equating the coefficients of the partial derivatives of f, we get

$$\begin{aligned} M &= \phi^1_y \phi^2_{y'} - \phi^1_{y'} \phi^2_y, \\ My' &= \phi^1_{y'} \phi^2_x - \phi^1_x \phi^2_{y'}, \\ Mw(x, y, y') &= \phi^1_x \phi^2_y - \phi^1_y \phi^2_x. \end{aligned} \tag{2.7.30}$$

Using these expressions, it is easy to verify that

$$\frac{\partial M}{\partial x} + \frac{\partial (My')}{\partial y} + \frac{\partial (Mw)}{\partial y'} = 0, \quad \text{or} \quad \frac{d \log M}{dx} + \frac{\partial w}{\partial y'} = 0.$$

Example 2.7.2

$$\ddot{x} = x\frac{-a + \lambda\dot{x}^2}{(\lambda x^2 + 1)^2} \quad a, \lambda \in \mathbb{R}$$

Writing this equation as a system of first-order ODEs we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x\frac{-a + \lambda y^2}{(\lambda x^2 + 1)^2} \end{aligned} \tag{2.7.31}$$

It follows that

$$\frac{d}{dt}\log M + \frac{2\lambda x\dot{x}}{(\lambda x^2 + 1)^2} = 0,$$

whence

$$M = \frac{1}{\lambda x^2 + 1}.$$

2.7.3 First Integrals from Symmetries and JLM

It is evident that the classical definition of the JLM is overtly restrictive, requiring as it does almost complete knowledge of the system. However, being dependent on first integrals, it is natural to expect that it should be connected in some way to the symmetries of the equation under investigation. This connection was unravelled by Lie and its formulation in terms of the generators of the Lie symmetry algebra is outlined below.

In order to investigate the connection between the Jacobi last multiplier and infinitesimal transformations we consider an autonomous system of first-order ordinary differential equations of the form

$$\dot{x}_k = a_k(x_1, ..., x_n), \quad k = 1, ..., n. \tag{2.7.32}$$

As already explained in the previous section such a system of ODEs may be associated with the equivalent first-order partial differential equation

$$\tilde{A}f = \sum_{k=1}^{n} a_k(x_1, ..., x_n)\frac{\partial f}{\partial x_k} = 0 \tag{2.7.33}$$

Assume now that the system admits $n-1$ infinitesimal generators of symmetry given by

$$X_j = \sum_{k=1}^{n} \xi_{jk}(x_1, ..., x_n)\frac{\partial}{\partial x_k}, \quad j = 1, ..., n-1 \tag{2.7.34}$$

Define

$$\Delta = \det\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{n-1,1} & \xi_{n-1,2} & \cdots & \xi_{n-1,n} \end{pmatrix} \tag{2.7.35}$$

It may now be proved following Bianchi [2] [3] that Δ^{-1} is a multiplier. For this purpose let $w_1, ..., w_{n-1}$ be $(n-1)$ linearly independent solutions of the PDE $\tilde{A}f = 0$. Since the last multiplier is defined, in this case, by

$$M\tilde{A}f = \frac{\partial(f, w_1, ..., w_{n-1})}{\partial(x_1, x_2, ..., x_n)} = 0$$

it follows by setting $f = x_1$ that

$$Ma_1 = \frac{\partial(x_1, w_1, ..., w_{n-1})}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(w_1, ..., w_{n-1})}{\partial(x_2, ..., x_n)}. \tag{2.7.36}$$

Similarly setting $f = x_2$ we find

$$Ma_2 = (-1)\frac{\partial(w_1, w_2..., w_{n-1})}{\partial(x_1, x_3, ..., x_n)}, \tag{2.7.37}$$

so that finally with $f = x_n$ we obtain

$$Ma_n = (-1)^{n+1}\frac{\partial(w_1, w_2..., w_{n-1})}{\partial(x_1, x_2, ..., x_{n-1})}. \tag{2.7.38}$$

Let us assume the rhs of the last relation is non zero and consider the product

$$\Delta \times \frac{\partial(w_1, w_2..., w_{n-1})}{\partial(x_1, x_2, ..., x_{n-1})} = \det\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{n-1,1} & \xi_{n-1,2} & \cdots & \xi_{n-1,n} \end{pmatrix} \det\begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_2}{\partial x_1} & \cdots & \frac{\partial w_{n-1}}{\partial x_1} & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 \\ \frac{\partial w_1}{\partial x_{n-1}} & \frac{\partial w_2}{\partial x_{n-1}} & \cdots & \frac{\partial w_{n-1}}{\partial x_{n-1}} & 0 \\ \frac{\partial w_1}{\partial x_n} & \frac{\partial w_2}{\partial x_n} & \cdots & \frac{\partial w_{n-1}}{\partial x_n} & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} \tilde{A}w_1 & \tilde{A}w_2 & \cdots & \tilde{A}w_{n-1} & a_n \\ X_1w_1 & X_1w_2 & \cdots & X_1w_{n-1} & \xi_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{n-1}w_1 & X_{n-1}w_2 & \cdots & X_{n-1}w_{n-1} & \xi_{n-1,n} \end{pmatrix}$$

where use has been made of the properties of determinant multiplication. However as $w_k (k = 1, ..., n-1)$ are solutions of $\tilde{A}f = 0$ it follows that

$$\Delta \times \frac{\partial(w_1, ..., w_{n-1})}{\partial(x_1, x_2, ..., x_{n-1})} = (-1)^{n+1}a_n \det\begin{pmatrix} X_1w_1 & X_1w_2 & \cdots & X_1w_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ X_{n-1}w_1 & X_{n-1}w_2 & \cdots & X_{n-1}w_{n-1} \end{pmatrix} \tag{2.7.39}$$

Since we have assumed that X_j are symmetry generators it follows from definition that

$$[X_j, \tilde{A}]f = \lambda\tilde{A}f.$$

[2] I would like to thank Prof. M.C. Nucci for drawing my attention to [3]

Hence as w_k are solutions of $\tilde{A}f = 0$ we have $\tilde{A}(X_j w_k) = 0$ implying thereby that $X_j w_k$ is also a solution for each $j, k = 1, ..., n-1$. Therefore,

$$\det \begin{pmatrix} X_1 w_1 & X_1 w_2 & \cdots & X_1 w_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ X_{n-1} w_1 & X_{n-1} w_2 & \cdots & X_{n-1} w_{n-1} \end{pmatrix}$$

is a solution of $\tilde{A}f = 0$. Let

$$\Omega = \det \begin{pmatrix} X_1 w_1 & X_1 w_2 & \cdots & X_1 w_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ X_{n-1} w_1 & X_{n-1} w_2 & \cdots & X_{n-1} w_{n-1} \end{pmatrix}.$$

Therefore (2.7.39) can be written as

$$\Delta M = \Omega, \tag{2.7.40}$$

where M is given by (2.7.38). Which can be written as

$$\frac{M}{\Delta^{-1}} = \Omega. \tag{2.7.41}$$

Clearly

$$\frac{d}{dt}\left(\frac{M}{\Delta^{-1}}\right) = \frac{d\Omega}{dt} = 0. \tag{2.7.42}$$

Therefore from Proposition 2.7.1 it follows that Δ^{-1} is a multiplier.

Alternatively for the n-th order ODE the above result is as follows:

Given an n-th order ODE $y^{(n)} = w(x, y, y', ..., y^{(n-1)})$ or its equivalent linear PDE

$$f_x + y' f_y + y'' f_{y'} + \cdots + w f_{y^{(n-1)}} = 0$$

if we know $n-1$ symmetries of the ODE/PDE say

$$X_i = \xi_i \partial_x + \eta_i \partial_y \qquad i = 1, ...n-1$$

with prolongations

$$X_i^{(n-1)} = \xi_i \partial_x + \eta_i \partial_y + \eta_i^{(1)} \partial_{y'} + \cdots + \eta_i^{(n-1)} \partial_{y^{(n-1)}} \qquad i = 1, ...n-1$$

then Jacobi's last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$ [3], where

$$\Delta = \det \begin{pmatrix} 1 & y' & y'' & \cdots & w \\ \xi_1 & \eta_1 & \eta_1^{(1)} & \cdots & \eta_1^{(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \xi_{n-1} & \eta_{n-1} & \eta_{n-1}^{(1)} & \cdots & \eta_{n-1}^{(n-1)} \end{pmatrix}. \tag{2.7.43}$$

This provides us with a very straightforward tool to search for the Last Multiplier. In addition since the ratio of two Last Multipliers is a first integral, hence the search for first integrals is greatly simplified and involves simple algebra. The following example provides an illustration.

Example 2.7.3

$$y'' = \frac{3y'^2}{y} + \frac{y'}{x}$$

This equation is found to admit an eight dimensional Lie symmetry algebra generated by the following operators

$$X_i = \xi_i \partial_x + \eta_i \partial_y \qquad (i = 1, ...8)$$

such that

$$X_1 = y\partial_y, \quad X_2 = y^3\partial_y, \quad X_3 = x\partial_x, \quad X_4 = \frac{1}{x}\partial_x$$

$$X_5 = x^2y^3\partial_y, \quad X_6 = \frac{1}{xy^2}\partial_x, \quad X_7 = \frac{x}{y^2}\partial_x - \frac{1}{y}\partial_y, \;\; X_8 = x^3\partial_x - x^2y\partial_y.$$

One can calculate all the 14 possible determinants of the form Δ_{ij} using prolongations of the symmetry generators X_i and X_j up to the first-order. The ratio of these determinants actually give rise to the first integrals. Note that while several expressions for such ratios can be written, one has to check whether these expressions are functionally independent. Only the functionally independent expressions arising from the ratio of the determinants (or equivalently from the ratios of the Last Multipliers) determine the first integrals. For instance in the present case

$$\frac{M_{12}}{M_{13}} = \frac{\Delta_{13}}{\Delta_{12}} = -\frac{xy' + y}{y^3} = I_1.$$

In this chapter we have attempted to give a brief overview of some of the techniques relevant to our work. After a few preliminary definitions concerning ordinary differential equations we have focused on four principle features namely the issue of integrability of an ODE, techniques for finding their Lie symmetries, the linearization problem for ODEs and have finally ended with a description of Jacobi's last multiplier. Since the issue of integrability is closely associated with the existence of first integrals we have described in some detail the essential features of Darboux's classical method and the Prelle-Singer semi algorithm. The subject of Lie symmetries of ordinary and partial differential equations is itself a vast topic and we have purposely limited our description here to the barest minimum touching upon only those points which are relevant to our subsequent work.

In the following chapters we shall give a more detailed account of some of our work with the hope that they will illustrate more clearly the tools and techniques outlined in this chapter.

Chapter 3

The Extended Prelle-Singer method

3.1 Introduction

The problem of finding integrating factors and first integrals is fundamental to any analysis of ordinary differential equations. In this chapter we address these features in case of two classes of ODEs, namely the equation

$$\ddot{x}+\frac{1}{2}\psi_x\dot{x}^2+\psi_t\dot{x}+B(t,x)=0$$

which is sometimes known as the Jacobi equation and the more well known Liénard equation which has the generic form

$$\ddot{x}+f(x)\dot{x}+g(x)=0. \tag{3.1.1}$$

The former is interesting because it includes many equations of Painlevé-Gambier classification as given in Ince's book [44]. The Liénard equation is famous in the other hand because it includes a number of important physical systems such as those listed below.

1. $f(x)=k,\quad g(x)=w_0^2x,\ \Rightarrow\ \ddot{x}+k\dot{x}+w_0^2x=0$ *damped harmonic oscillator.*
2. $f(x)=\alpha x,\quad g(x)=\beta x^3,\ \Rightarrow\ \ddot{x}+\alpha x\dot{x}+\beta x^3=0$ *Modified Emden equation.*
3. $f(x)=\alpha+\beta x^2,\quad g(x)=-\gamma x+x^3,\ \Rightarrow\ \ddot{x}+(\alpha+\beta x^2)\dot{x}-(\gamma x+x^3)=0$ *Duffing Van der Pol oscillator.*
4. $f(x)=(k_1x^q+k_2),\ g(x)=k_3x^{2q+1}+k_4x^{q+1}+\lambda_1x$, where $q\in\mathbb{R}$

The last case includes many systems like the anharmonic oscillator, force free Helmholtz and Duffing oscillator as special cases. In [10], the authors have studied this system for $q=$ arbitrary and deduced a number of new completely integrable cases.

In subsection 2.2.9 we described the Prelle-Singer method for deducing integrating factor for a first-order ODEs when it has an elementary first integral. Here we describe an extension of their procedure applicable to second-order ODEs [8, 9, 10].

It is clear that a second-order ODE is written in the form:

$$\ddot{x} = \phi(x, \dot{x}), \tag{3.1.2}$$

may be written either as a system of first-order equations

$$\dot{x} = y, \quad \dot{y} = \phi(x, y) \tag{3.1.3}$$

or as a pair of differential one forms $\theta_i, \ i = 1, 2$:

$$\theta_1 = dx - ydt = 0, \quad \theta_2 = dy - \phi dt = 0. \tag{3.1.4}$$

Multiplying θ_1 with an unknown function $S(x, y)$ and adding it to to θ_2 we have

$$(Sy + \phi)dt = Sdx + dy.$$

Assuming R to be an integrating factor of this equation we have upon multiplication

$$R(Sy + \phi)dt - RSdx - Rdy = 0, \tag{3.1.5}$$

which implies that if $I(t, x, y)$ be the corresponding first integral such that

$$I_t dt + I_x dx + I_y dy = 0$$

we must have

$$I_t = R(Sy + \phi), \quad I_x = -RS, \quad I_y = -R. \tag{3.1.6}$$

The compatibility of these equations requires

$$I_{xy} = I_{yx}, I_{tx} = I_{xt} \ \text{ and } \ I_{ty} = I_{yt}. \tag{3.1.7}$$

From these conditions it is straightforward to derive the following equations

$$D[R] = -((RS) + \phi_y R), \tag{3.1.8}$$

$$D[RS] = -\phi_x R, \tag{3.1.9}$$

where

$$D = y\partial_x + \phi\partial_y.$$

Two subcases may be distinguished.
A: When $I_t = 0$, that is when the system is conservative and
B: when $I_t \neq 0$ for a non conservative system.

In case of the former, it is easy to see that $S = -\phi/y$. Therefore one needs to determine only the unknown function R, which is the required integrating factor. We shall analyze case A

first, since it is somewhat simpler, and postpone a discussion of the latter.
For case A, (3.1.8) simplifies to

$$D[R] = (\frac{\phi}{y} - \phi_y)R, \tag{3.1.10}$$

Substituting the ansatz

$$R = \frac{y}{T(x,y)}, \tag{3.1.11}$$

causes (3.1.10) to simplify further and it reduces to

$$D[T] = yT_x + \phi T_y = \phi_y T. \tag{3.1.12}$$

Let us consider an example to illustrate the method developed thus far.

Example 3.1.1

Consider the equation

$$\ddot{x} + \frac{1}{2}\psi_x \dot{x}^2 + \psi_t \dot{x} + B(t,x) = 0.$$

This is equivalent to the system of equations

$$\dot{x} = y$$
$$\dot{y} = \phi(t,x,y) = -\left[\frac{1}{2}\psi_x \dot{x}^2 + \psi_t \dot{x} + B(t,x)\right]$$

so that

$$\phi_y = -(\psi_x y + \psi_t) = -D[\psi].$$

Thus (3.1.12) becomes

$$D[\log T + \psi] = 0$$

which implies $T = K\exp(-\psi)$. Hence

$$R = \frac{y}{K}\exp(\psi) = -I_y$$

implies

$$I = -\frac{e^{\psi}}{K}\frac{y^2}{2} + \frac{J(x)}{K},$$

where K is a numerical constant. On the other hand $I_x = -RS$ implies

$$J'(x) = e^{\psi}(-\psi_t y - B(t,x))$$

. Clearly one must have $\psi_t = 0$ and $B(t,x) = B(x)$ for a time independent first integral. In that case we obtain

$$I(x,y) = -\frac{1}{K}\left[e^{\psi}\frac{y^2}{2} + \int^x e^{\psi} B dx\right]. \tag{3.1.13}$$

Such a first integral occurs, therefore for all equations having the generic form

$$\ddot{x} + \frac{1}{2}\psi_x \dot{x}^2 + B(x) = 0, \tag{3.1.14}$$

and may be a treated as a formula for deriving an time independent first integral for them.

3.1.1 First integrals of Painlevé-Gambier equations

It will be evident that the above method may be applied, in principle to a number of equations of the Painlevé-Gambier classification. We shall introduce a slight change of notation wherein x will denote the independent variable and illustrate this below.

Painlevé-Gambier equation XII

Let us consider the Painlevé-Gambier XII equation

$$y'' = \frac{1}{y}y'^2 + \alpha y^3 + \beta y^2 + \gamma + \frac{\delta}{y}$$

Comparison with (3.1.14) above indicates that $\frac{1}{2}\psi_y = -1/y$ and hence $e^{\psi} = y^{-2}$, while $B(x,y) = -\left[\alpha y^3 + \beta y^2 + \gamma + \frac{\delta}{y}\right]$. Then (3.1.13) yields the following first integral

$$y'^2 = \alpha y^4 + 2\beta y^3 - 2\gamma y - \delta + K_1 y^2,$$

where we have set $K_1 = -2KI(y, y')$. We have checked that *all* the known x- independent first integrals of the Painlevé-Gambier classification [44], can be obtained from (3.1.13).

Clearly it is of interest to know whether there exists other first integrals depending perhaps on the independent variable x, for equations having a first integral given by the above formula. This brings us actually to a discussion of case B, i.e. ($I_t \neq 0$) of the previous section.

Painlevé-Gambier equation XXII

We illustrate next, the existence of an x dependent first integral for equation XXII of the Painlevé-Gambier classification,

$$\frac{d^2y}{dx^2} = \frac{3y'^2}{4y} - 1. \tag{3.1.15}$$

A known first integral of this equation is

$$K = \left(\frac{y'^2 - 4y}{4y^{3/2}}\right), \tag{3.1.16}$$

which may be obtained from (3.1.13).
From (3.1.8) and (3.1.9), we have

$$D[R] = -(S + \phi_{y'})R \tag{3.1.17}$$

$$D[RS] = -\phi_y R, \tag{3.1.18}$$

as a result of our change in notation. Here $D = \partial_x + y'\partial_y + \phi\partial_{y'}$ with

$$\phi(x,y,y') = \frac{3y'^2}{4y} - 1 = \phi_0(y)y'^2 - 1, \quad \phi_0(y) = \frac{3}{4y}. \tag{3.1.19}$$

Closer inspection of equations (3.1.17) and (3.1.18) reveals that they are a pair of coupled first-order equations in the variables R and RS respectively. In [8, 9, 10] the authors have deduced first integrals of oscillator type systems under very general conditions using the above formulation. Regarding the issue of finding solutions of R and S they have assumed certain specific rational forms depends on y' and have refered to their procedure as an extension of the Prelle-Singer method to second-order ODEs. Following their prescription we assume the unknown quantities R and S admit rational solutions of the form

$$R = \frac{f}{g} \text{ and } RS = \frac{h}{g} \ \Rightarrow\ S = \frac{h}{f}, \tag{3.1.20}$$

so that (3.1.17) and (3.1.18) have the following appearance

$$gD[f] - fD[g] = -(h + \phi_{y'}f)g \tag{3.1.21}$$

$$gD[h] - hD[g] = -\phi_y fg. \tag{3.1.22}$$

However, as we will show in the next chapter the so called extended Prelle-Singer method of the authors of [8, 9, 10] is nothing but a decomposition of the existing adjoint symmetry equation into a first-order system and therefore falls within the domain of Lie's symmetry analysis.
For the time being however, from a leading order analysis of the above equations, assuming $f{\sim}y'^{\alpha}$, $h \sim y'^{\gamma}$ and $g \sim y'^{\beta}$ and with ϕ as in (3.1.19), it follows that $\gamma = \alpha + 1$ with β being arbitrary. This suggests the following ansatz for the y' dependence of the functions f, g and h namely:

$$\begin{aligned} f(y,y') &= f_0 + f_1y' + f_2y'^2, \\ h(y,y') &= h_0 + h_1y' + h_2y'^2 + h_3y'^3, \\ g(y,y') &= g_0 + g_1y' + g_2y'^2 + g_3y'^3 + g_4y'^4. \end{aligned} \tag{3.1.23}$$

Substituting these into (3.1.21) and equating different powers of y' leads to the following set of equations:

$$-g_0f_1 + f_0g_1 = -h_0g_0, \tag{3.1.24}$$

$$g_0F_1 - f_0G_1 = -\{(h_1 + 2\phi_0 f_0)g_0 + h_0g_1\}, \tag{3.1.25}$$

$$-g_2f_1 + g_1F_1 + g_0F_2 + f_2g_1 - f_1G_1 - f_0G_2 = -\{(h_2 + 2\phi_0f_1)g_0 + (h_1 + 2\phi_0f_0)g_1 + h_0g_2\}, \tag{3.1.26}$$

$$\begin{aligned} &-g_3f_1 + g_2F_1 + g_1F_2 + g_0F_3 - f_2G_1 - f_1G_2 - f_0G_3 \\ &= -\{(h_3 + 2\phi_0f_2)g_0 + (h_2 + 2\phi_0f_1)g_1 + (h_1 + 2\phi_0f_0)g_2 + h_0g_3\}, \end{aligned} \tag{3.1.27}$$

$$-g_4f_1 + g_3F_1 + g_2F_2 + g_1F_3 - f_2G_2 - f_1G_3 - f_0G_4 = -\{(h_3 + 2\phi_0f_2)g_1 +$$

$$(h_2 + 2\phi_0 f_1)g_2 + (h_1 + 2\phi_0 f_0)g_3 + h_0 g_4\}, \quad (3.1.28)$$

$$g_4F_1+g_3F_2+g_2F_3-f_2G_3-f_1G_4-f_0G_5 = -\{(h_3+2\phi_0 f_2)g_2+(h_2+2\phi_0 f_1)g_3+(h_1+2\phi_0 f_0)g_4\}, \quad (3.1.29)$$

$$g_4F_2 + g_3F_3 - f_2G_4 - f_1G_5 = -\{(h_3 + 2\phi_0 f_2)g_3 + (h_2 + 2\phi_0 f_1)g_4\}, \quad (3.1.30)$$

$$g_4 f_{2y} - f_2 g_{4y} = -h_3 g_4. \quad (3.1.31)$$

Here we have defined

$$F_1 = f_{0y} - 2f_2,\ F_2 = f_{1y} + \phi_0 f_1,\ F_3 = f_{2y} + \phi_0 f_2 \quad (3.1.32)$$

and

$$G_1 = g_{0y} - 2g_2,\ G_2 = g_{1y} + \phi_0 g_1 - 3g_3, \quad (3.1.33)$$

$$G_3 = g_{2y} + 2\phi_0 g_2 - 4g_4,\ G_4 = g_{3y} + 3\phi_0 g_3,\ G_5 = g_{4y} + 4\phi_0 g_4. \quad (3.1.34)$$

On the other hand from (3.1.22) we obtain the following equations:

$$-h_0 g_1 + g_0 h_1 = 0, \quad (3.1.35)$$

$$h_0 G_1 - g_0 H_1 = 0, \quad (3.1.36)$$

$$-h_2 g_1 + h_1 G_1 + h_0 G_2 + g_2 h_1 - g_1 H_1 - g_0 H_2 = \phi_{0y} f_0 g_0 \quad (3.1.37)$$

$$-h_3 g_1 + h_2 G_1 + h_1 G_2 + h_0 G_3 + g_3 h_1 - g_2 H_1 - g_1 H_2 - g_0 H_3 = \phi_{0y}(f_1 g_0 + f_0 g_1), \quad (3.1.38)$$

$$h_3 G_1 + h_2 G_2 + h_1 G_3 + h_0 G_4 + g_4 h_1 - g_3 H_1 - g_2 H_2 - g_1 H_3 - g_0 H_4 = \phi_{0y}(f_2 g_0 + f_1 g_1 + f_0 g_2), \quad (3.1.39)$$

$$h_3 G_2 + h_2 G_3 + h_1 G_4 + h_0 G_5 - g_4 H_1 - g_3 H_2 - g_2 H_3 - g_1 H_4 = \phi_{0y}(f_2 g_1 + f_1 g_2 + f_0 g_3), \quad (3.1.40)$$

$$h_3 G_3 + h_2 G_4 + h_1 G_5 - g_4 H_2 - g_3 H_3 - g_2 H_4 = \phi_{0y}(f_2 g_2 + f_1 g_3 + f_0 g_4), \quad (3.1.41)$$

$$h_3 G_4 + h_2 G_5 - g_4 H_3 - g_3 H_4 = \phi_{0y}(f_2 g_3 + f_1 g_4), \quad (3.1.42)$$

$$h_3 G_5 - g_4 H_4 = \phi_{0y} f_2 g_4, \quad (3.1.43)$$

where

$$\begin{aligned} H_1 &= h_{0y} - 2h_2, \\ H_2 &= h_{1y} + \phi_0 h_1 - 3h_3, \\ H_3 &= h_{2y} + 2\phi_0 h_2 \\ H_4 &= h_{3y} + 3\phi_0 h_3. \end{aligned} \quad (3.1.44)$$

To solve the system of first order coupled PDEs given by (3.1.24)-(3.1.31) and (3.1.35)-(3.1.43) we observe that, one can satisfy one half of each set identically, by making a second ansatz, namely

$$f_{odd} = g_{odd} = h_{even} = 0. \quad (3.1.45)$$

It then follows that

$$H_1 = H_3 = G_2 = G_4 = 0,$$

and from (3.1.35) we find

$$h_1 = 0. \tag{3.1.46}$$

Taking this in to account we are now left with the following equations from the set (3.1.24)-(3.1.31):

$$g_0(f_{0y} - 2f_2) - f_0(g_{0y} - 2g_2) = -2\phi_0 f_0 g_0, \tag{3.1.47}$$

$$g_2(f_{0y}-2f_2)+g_0(f_{2y}+2\phi_0 f_2)-f_2(g_{0y}-2g_2)-f_0(g_{2y}+2\phi_0 g_2-4g_4) = \{(h_3+2\phi_0 f_2)g_0+2\phi_0 f_0 g_2\} \tag{3.1.48}$$

$$g_4(f_{0y}-2f_2)+g_2(f_{2y}+2\phi_0 f_2)-f_2(g_{2y}+2\phi_0 g_2-4g_4)-f_0(g_{4y}+4\phi_0 g_4) = \{(h_3+2\phi_0 f_2)g_2+2\phi_0 f_0 g_4\} \tag{3.1.49}$$

$$g_4 f_{2y} - f_2 g_{4y} = h_3 g_4. \tag{3.1.50}$$

On the other hand from the set of equations (3.1.35)-(3.1.43), with $h_1 = 0$ we obtain the following four equations:

$$3h_3 = \phi_{0y} f_0, \tag{3.1.51}$$

$$h_3(g_{0y} + g_2 - 3\phi_0 g_0) - g_0 h_{3y} = \phi_{0y}(f_2 g_0 + f_0 g_2), \tag{3.1.52}$$

$$h_3(g_{2y} - g_4 - \phi_0 g_2) - g_2 h_{3y} = \phi_{0y}(f_2 g_2 + f_0 g_4), \tag{3.1.53}$$

$$h_3(g_{4y} + \phi_0 g_4) - g_4 h_{3y} = \phi_{0y} f_2 g_4. \tag{3.1.54}$$

Since $\phi_0 = \frac{3}{4y}$ it follows $\phi_{0y} = -\frac{\phi_0}{y}$ and upon rearranging (3.1.54), we have

$$h_3 g_{4y} - g_4 h_{3y} = -\phi_0 g_4 (h_3 + \frac{f_2}{y}).$$

Making the assumption that the coefficients of the highest powers of y' in the expressions for g, h are constants, say $g_4 = \mu$ and $h_3 = \nu$ so that $g_{4y} = h_{3y} = 0$, one obtains the following relation determining the coefficient of the y'^2 in f:

$$h_3 + \frac{f_2}{y} = 0 \Rightarrow f_2 = -\nu y. \tag{3.1.55}$$

While from (3.1.51) we obtain

$$f_0 = -4\nu y^2. \tag{3.1.56}$$

The remaining equations (3.1.52) and (3.1.53) determine the coefficients g_0 and g_2, from solutions of the following coupled linear equations:

$$g_{0y} - \frac{3}{y} g_0 = 2g_2, \tag{3.1.57}$$

$$g_{2y} - \frac{3}{2y} g_2 = 4\mu. \tag{3.1.58}$$

These conditions are consistent with the set of equations (3.1.47)-(3.1.50), as may be verified. Furthermore, the solutions of (3.1.57) and (3.1.58) are easy to construct and are given by

$$g_2 = -8\mu y \;\text{ and }\; g_0 = 16\mu y^2.$$

Hence we finally get

$$R = \frac{f}{g} = \frac{-\nu y(4y + y'^2)}{\mu(y'^2 - 4y)^2} \quad \text{and} \quad S = \frac{h}{f} = \frac{\nu y'^3}{-\nu y(4y + y'^2)}. \tag{3.1.59}$$

It is now straightforward to obtain the corresponding first integral as

$$I(x, y, y') = \frac{\nu}{4\mu}\left(x - \frac{4yy'}{y'^2 - 4y}\right). \tag{3.1.60}$$

This is a new first integral not listed in Ince's book [44].

3.1.2 Equations of the Liénard type

The Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3.1.61}$$

and frequently occurs in physical applications. Instead of writing this as a system of first-order equations in the usual form

$$\dot{x} = y, \quad \dot{y} = \phi(x, y),$$

where $\phi = -(f(x)y + g(x))$, let us write it as

$$\dot{x} = v - r\frac{g(x)}{f(x)}, \tag{3.1.62}$$

$$\dot{v} = -\frac{1}{r}f(x)v, \tag{3.1.63}$$

subject to the condition

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \frac{1}{r}\left(1 - \frac{1}{r}\right)f(x), \quad r \neq 0, 1. \tag{3.1.64}$$

Here r is a parameter. In order to determine a first integral for the system (3.1.62) and (3.1.63), we follow the same formulation as outlined in section (3.1) and demand that the one form

$$R[S(v - r\frac{g}{f}) - \frac{1}{r}fv]dt - RSdx - Rdv = 0, \tag{3.1.65}$$

be exact. This means there exists a function, $I(t, x, v)$, such that

$$I_t = R[S(v - r\frac{g}{f}) - \frac{1}{r}fv],$$

$$I_x = -RS \quad \text{and} \quad I_v = -R. \tag{3.1.66}$$

If we are interested in a time independent first integral, so that $I_t = 0$, we immediately obtain

$$S = \frac{fv}{r\left(v - r\frac{g}{f}\right)}. \tag{3.1.67}$$

From the compatibility of (3.1.66), using the above expression for S, we have the following equation for determining the integrating factor R, *viz*

$$R_x + \frac{fv/r}{\left(r\frac{g}{f} - v\right)} R_v = -\frac{g}{\left(r\frac{g}{f} - v\right)^2} R. \tag{3.1.68}$$

Let us now make the ansatz

$$R = \frac{\left(r\frac{g}{f} - v\right)}{T(x,v)}. \tag{3.1.69}$$

Inserting this into (3.1.68), we obtain the following equation for determining $T(x,v)$,

$$D[T] := \left(r\frac{g}{f} - v\right)\frac{\partial T}{\partial x} + \frac{fv}{r}\frac{\partial T}{\partial v} = fT. \tag{3.1.70}$$

As Chandrasekar *et al* have shown, it is not necessary to obtain the general solution of (3.1.70). Any particular solution of it is sufficient to determine a first integral, when it exists. In principle this leads to a considerable simplification, which cannot be underestimated.
For the problem of determining a particular solution of T, we shall use the technique of Darboux polynomials. Notice that if $f(x)$ be a polynomial, then in view of (3.1.64), we conclude that g/f must also be a polynomial. For the vector field D as defined by (3.1.70) we find that

$$D[h_1] = D[v] = \frac{f}{r} h_1 \tag{3.1.71}$$

and

$$D[h_2] = D\left[\frac{g}{f} - \frac{(r-1)}{r(r-2)}v\right] = \frac{(r-1)}{r} f h_2. \tag{3.1.72}$$

In other words,

$$h_1 = v \text{and} h_2 = \frac{g}{f} - \frac{(r-1)}{r(r-2)}v$$

are Darboux polynomials of the vector field D with cofactors

$$\lambda_1 = \frac{f}{r} \text{and} \lambda_2 = \frac{(r-1)}{r} f$$

respectively. Consequently, for $T(x,v) = h_1^{n_1} h_2^{n_2}$, we can find rational numbers such that $D[T] = fT$ namely $n_1 = n_2 = 1$. Thus we have the following particular solution of (3.1.70):

$$T(x,v) = v\left(\frac{g}{f} - \frac{(r-1)}{r(r-2)}v\right). \tag{3.1.73}$$

This completes the determination of the integrating factor R as

$$I_v = -R = -\frac{rg/f - v}{v/r(rg/f - (r-1)v/(r-2))} \quad \text{and} \quad I_x = -RS = \frac{f}{rg/f - (r-1)v/(r-2)}. \tag{3.1.74}$$

The corresponding first integral is now given by

$$I(x, v) = \log\left[\frac{\left(r\frac{g}{f} - \frac{r-1}{r-2}v\right)}{v^{r-1}}\right]^{\frac{r}{r-1}}, \quad r \neq 0, 1, 2, \tag{3.1.75}$$

which essentially means that

$$C(x, v) = \left[\frac{\left(r\frac{g}{f} - \frac{r-1}{r-2}v\right)}{v^{r-1}}\right], \quad r \neq 0, 1, 2 \tag{3.1.76}$$

is a constant of motion.

Of course one could have obtained this first integral in a much more simpler way, by observing that (3.1.70) admits a solution $T = v^r$. This in turn gives $R = (rg/f - v)/v^r$ and $RS = -fv^{1-r}/r$ from which one gets the first integral (3.1.76).

A Liénard type nonlinear oscillator – the second order Riccati equation

We illustrate the above method with a well known example of the modified Emden equation

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0. \tag{3.1.77}$$

Here $f(x) = \alpha x$ and $g(x) = \beta x^3$. The condition (3.1.64) gives a quadratic equation for the parameter r, with solution

$$\frac{1}{r} = \frac{1}{2}\left[1 \pm \sqrt{1 - 8\beta/\alpha^2}\right].$$

If we choose a particular value of r, then these solutions determines a relation between the parameters α and β of the equation; conversely, given the parameters it fixes the value of r. For example, the choice $r = 3$ yields $\beta = \frac{\alpha^2}{9}$. Thus setting $\alpha = 3k$ we have $\beta = k^2$ and the equation becomes

$$\ddot{x} + 3kx\dot{x} + k^2x^3 = 0. \tag{3.1.78}$$

This particular form is often called the second Riccati equation (and is also the Painlevé-Gambier equation VI with $q(Z) = 0$ of [44]). Its first integral from (3.1.76) is therefore

$$C_1(x, v) = \frac{kx^2 - 2v}{v^2}. \tag{3.1.79}$$

The phase flow for the equation, under these circumstances, as determined from (3.1.62) and (3.1.63) is

$$\frac{dv}{dx} = \frac{2kxv}{kx^2 - 2v},$$

which may be separated by using the above expression for C_1 *viz*

$$\frac{dv}{dx} = \frac{2kx}{C_1 v} \quad \Rightarrow \quad \frac{1}{2}C_1 v^2 - kx^2 = K_2$$

where K_2 is an integration constant.
If we desire to express $C_1(x, v)$ in terms of x and the actual velocity $\dot{x}$, then we simply eliminate v using (3.1.62) to get

$$C_1(x, \dot{x}) = -\frac{2\dot{x} + kx^2}{(\dot{x} + kx^2)^2},$$

which coincides with the results in [5]. In fact it has been shown by Cariñena *et al* [5] that this first integral plays the role of the Hamiltonian for (3.1.78).

3.1.3 A generalized 2D- Kepler system

In [8], the authors considered a system of second-order ODE's of the generic form

$$\ddot{x} = \frac{P_1}{Q_1} = \phi_1 \text{ and } \ddot{y} = \frac{P_2}{Q_2} = \phi_2$$

where it is assumed that $\phi_i (i = 1, 2)$ depend on $t, x, \dot{x}, y, \dot{y}$ in general. They illustrate the general procedure and finish off with the following example of the two-dimensional Kepler problem.

$$\ddot{x} = -\frac{x}{(x^2 + y^2)^{\frac{3}{2}}}, \ \ddot{y} = -\frac{y}{(x^2 + y^2)^{\frac{3}{2}}}. \tag{3.1.80}$$

Their analysis yielded the following first integrals:

$$I_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{\sqrt{x^2 + y^2}}$$

$$I_2 = y\dot{x} - x\dot{y}$$

$$I_3 = \dot{x}(y\dot{x} - x\dot{y}) - \frac{y}{\sqrt{x^2 + y^2}}. \tag{3.1.81}$$

corresponding to the Hamiltonian, the angular momentum and the Runge Lenz vector respectively.
We shall consider a system which is similar to this, but of the form:

$$\ddot{x} = -\frac{x(x^2 + b)}{(x^2 + y^2)^{\frac{3}{2}}} = -\frac{xg_1(x, y)}{(x^2 + y^2)^{\frac{3}{2}}},$$

$$\ddot{y} = -\frac{y(3x^2 + 2y^2 + b)}{(x^2 + y^2)^{\frac{3}{2}}} = -\frac{yg_2(x, y)}{(x^2 + y^2)^{\frac{3}{2}}}, \tag{3.1.82}$$

where

$$g_1(x,y) = (x^2+b), \text{ and } g_2(x,y) = (3x^2+2y^2+b). \qquad (3.1.83)$$

Assuming I to be a first integral of the coupled system such that

$$dI = I_t dt + I_x dx + I_y dy + I_{\dot{x}} d\dot{x} + I_{\dot{y}} d\dot{y} = 0,$$

let us write the coupled system of equations as:

$$(\phi_1 + S_1\dot{x})\,dt - S_1 dx - d\dot{x} = 0, \qquad (3.1.84)$$

$$(\phi_2 + S_2\dot{y})\,dt - S_2 dy - d\dot{y} = 0, \qquad (3.1.85)$$

then we must have

$$\begin{aligned} I_t &= R_1(\phi_1 + S_1\dot{x}) + R_2(\phi_2 + S_2\dot{y}), \\ I_x &= -R_1 S_1, \\ I_y &= -R_2 S_2, \\ I_{\dot{x}} &= -R_1, \\ I_{\dot{y}} &= -R_2. \end{aligned} \qquad (3.1.86)$$

Here R_1 and R_2 represent the respective integrating factors of the system of equations (3.1.84)-(3.1.85). Compatibility of the set of equations (3.1.86) then yields the following:

$$\begin{aligned} D[S_1] &= -\phi_{1x} - \frac{R_2}{R_1}\phi_{2x} + \frac{R_2}{R_1}S_1\phi_{2\dot{x}} + S_1\phi_{1\dot{x}} + S_1^2, \\ D[S_2] &= -\phi_{2y} - \frac{R_1}{R_2}\phi_{1y} + \frac{R_1}{R_2}S_2\phi_{1\dot{y}} + S_2\phi_{2\dot{y}} + S_2^2, \\ D[R_1] &= (R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}} + R_1S_1), \\ D[R_2] &= -(R_2\phi_{2\dot{y}} + R_1\phi_{1\dot{y}} + R_2S_2), \\ S_1R_{1y} &= R_1S_{1y} + S_2R_{2x} + R_2S_{2x}, \\ R_{1x} &= \frac{\partial}{\partial\dot{x}}(R_1S_1),\ R_{2y} = \frac{\partial}{\partial\dot{y}}(R_2S_2), \\ R_{1y} &= \frac{\partial}{\partial\dot{x}}(R_2S_2),\ R_{2x} = \frac{\partial}{\partial\dot{y}}(R_1S_1),\ R_{1\dot{y}} = R_{2\dot{x}}, \end{aligned}$$

Where D represents the vector field

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \phi_1\frac{\partial}{\partial\dot{x}} + \phi_2\frac{\partial}{\partial\dot{y}}.$$

The problem is therefore reduced to finding solutions (particular) satisfying these equations. For the system (3.1.82) we find a particular solution to be the following:

$$R_1 = \dot{x}, \quad R_2 = \dot{y}, \quad S_1 = \frac{x(x^2+b)}{\dot{x}(x^2+y^2)^{\frac{3}{2}}}, \quad S_2 = \frac{y(3x^2+2y^2+b)}{\dot{y}(x^2+y^2)^{\frac{3}{2}}}.$$

With these values of R_i, S_i $(i = 1, 2)$ we obtain the following first integral:

$$I(x, y, \dot{x}, \dot{y}) = -\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{x^2+2y^2-b}{\sqrt{x^2+y^2}}\right]. \tag{3.1.87}$$

In fact it is easy to verify that this integral is actually the Hamiltonian. However, we have not been able to deduce the analogs of the angular momentum or the Lenz vector for this case.

We shall next consider a nonplanar system of ODEs corresponding to a generalized version of the Raychaudhuri equation to give further illustration of the method outlined above.

3.2 The Generalized Raychaudhuri equation in a two dimensional deformable media and its first integrals

The Raychaudhuri equations constitute a coupled first-order system of nonlinear ODEs which were originally introduced in the context of cosmology in 1955 [54]. Basically these equations represent largely geometric/mathematical statements about the nature of geodesics in a Riemannian/pseudo-Riemannian geometry. They describe the kinematics of flows (i.e., the integral curves) generated by a vector field. The flows may be geodesic or non-geodesic in nature. The kinematic nature of these equations means that one does not ask how the flow is generated but is only interested in the characteristics of such flows. In order to understand which quantities may characterize a flow suppose λ denotes the parameter labeling points on the curves in the flow, then, the gradient of the velocity field, which is a second rank tensor, can be split up into a symmetric traceless part, an anti symmetric part and lastly the trace. These three quantities define the shear, rotation and the expansion of the flow and are the variables of interest.

In [19, 49], the authors considered geodesic flows on the surface of a deformable media and deduced how the expansion, shear and rotation of such flows evolve with time. The deformations of the media (at least locally) may be characterized in terms of time evolution of a deformation vector (θ, σ, w), where θ, σ and w represent the expansion(E), shear(S) and rotation(R) respectively. The kinematics can be quantified in terms of these (ESR) variables; and leads to the Raychaudhuri equation for a two dimensional curved surface of constant curvature. When the exact solutions of the geodesic equations are used in them, one is led to the following system, after suitable relabelling of the variables involved (see eq.(2.20)-(2.23) of [19]):

$$\dot{x} + \frac{1}{2}x^2 + \alpha x + 2(y^2 + z^2 - t^2) + 2\beta = 0 \tag{3.2.1}$$

$$\dot{y} + (\alpha + x)y + \gamma = 0 \tag{3.2.2}$$

$$\dot{z} + (\alpha + x)z + \delta = 0 \tag{3.2.3}$$

$$\dot{t} + (\alpha + x)t = 0 \tag{3.2.4}$$

One must not interpret t here as the time, it is simply at par with variables x, y, z. However $\dot{x}, \dot{y}$... etc., stand for the derivative of these variables with respect to the appropriate 'temporal variable' relevant to the model. Thus from a mathematical point of view the above equations form a non-planar dynamical system. Note that α, β, γ and δ are suitable parameters of the model.

The vector field D is in this case given by

$$-D = (\frac{1}{2}x^2 + \alpha x + 2(y^2 + z^2 - t^2) + 2\beta)\frac{\partial}{\partial x} + ((\alpha + x)y + \gamma)\frac{\partial}{\partial y} + ((\alpha + x)z + \delta)\frac{\partial}{\partial z} + ((\alpha + x)t)\frac{\partial}{\partial t}$$

It can be easily verified that with $f_1 = -\frac{\delta}{\gamma}y + z$ we have,

$$D[f_1] = D[-\frac{\delta}{\gamma}y + z] = -(\alpha + x)f_1, \text{ so that } \lambda_1 = -(\alpha + x) \tag{3.2.5}$$

Similarly we find, $f_2 = t$, to be another Darboux polynomial whose associated eigenpolynomial is again $\lambda_2 = -(\alpha + x) = \lambda_1$. Consequently the exactness condition, $D[R] = 0$, which implies, $\sum_i^2 n_i\lambda_i = 0$, leads to $(\alpha+x)(n_1+n_2) = 0$ or $n_2 = -n_1$. Making the choice $n_1 = -1$ we obtain the first integral given by

$$I_1(x, y, z, t) = \frac{t}{\left(-\frac{\delta}{\gamma}y + z\right)}. \tag{3.2.6}$$

3.2.1 Additional new first integrals

On the other hand for the specific choice of the parameters, $\gamma = \delta = 0$, one finds the following Darboux polynomials:

$$D[g_i] = -(\alpha + x)g_i \ \ (i = 1, 2, 3) \ \text{ with } \ g_1 = y, g_2 = z, g_3 = t, \tag{3.2.7}$$

and

$$D[g_4] := D[z^2 + t^2 + zt] = -2(\alpha + x)(z^2 + t^2 + zt). \tag{3.2.8}$$

Hence, the exactness condition $D[R] = 0$ implies

$$\sum_i n_i\lambda_i = 0 \Rightarrow \ (n_1 + n_2 + n_3 + 2n_4)(\alpha + x) = 0. \tag{3.2.9}$$

Choosing $n_1 = n_2 = 1$ and $n_3 = -1, n_4 = -\frac{1}{2}$ we get another first integral of the form

$$I = \frac{yz}{t(z^2 + t^2 + zt)^{\frac{1}{2}}}. \tag{3.2.10}$$

It will be noticed that all the above first integrals are independent of the variable x. To get a first integral explicitly dependent on x, y, z, t we notice that when all the parameters $\alpha = \beta = \gamma = \delta = 0$ then the following Darboux polynomial depending on x is obtainable

$$-D[g_1] := -D[y^2 + z^2 - t^2 - \frac{1}{4}x^2] = x(y^2 + z^2 - t^2 - \frac{1}{4}x^2) \tag{3.2.11}$$

with associated eigenpolynomial given by $\lambda_1 = -x$. In addition the following are Darboux polynomials of degree two:

$$-D[g_2] := -D[zt] = 2xg_2 \ \Rightarrow \ \lambda_2 = -2x,$$

$$-D[g_3] := -D[yz] = 2xg_3 \ \Rightarrow \ \lambda_3 = -2x,$$

$$-D[g_4] := -D[yt] = 2xg_4 \;\Rightarrow\; \lambda_4 = -2x. \tag{3.2.12}$$

The exactness condition

$$\sum_i n_i\lambda_i = 0 \;\text{ implies }\; -x[n_1 + 2(n_2 + n_3 + n_4)] = 0,$$

which may then be satisfied by the following choice $n_2 = n_3 = n_4 = -\frac{1}{2}$ and $n_1 = 3$, leading to the rational first integral

$$I(x, y, z, t) = \frac{(y^2 + z^2 - t^2 - \frac{1}{4}x^2)^3}{yzt}. \tag{3.2.13}$$

In this chapter the method devised by Prelle and Singer for first order ODEs using Darboux polynomials has been extended and adopted to two generic classes of second-order equations namely the Jacobi equation and the Liénard equation. The former includes many of the Painlevé-Gambier (PG) equations listed in the Ince's book while the latter includes many physical problems of interest. We have derived a general formula for a time independent first integral corresponding to the reduction, $\psi_t = B_t = 0$, of the Jacobi equation. In particular we consider two specific equations PG-XII and PG-XXII and for the latter we have obtained a new time dependent rational first integral. For the Liénard class of equations we have illustrated how the extended Prelle-Singer technique allows us to obtain the integrating factor and associated first integral when the function f and g are polynomials. Explicit results are presented for the second Riccati equation. Further generalization to a coupled second-order system describing a 2D Kepler system are also presented. Finally after a brief introduction to the Raychaudhuri equation we have deduced first integrals for specific parameter values of generalized system of this equation using Darboux theory.

In the following chapter we shall show how the extended Prelle-Singer technique may be encapsulated within the general theory of Lie symmetries thereby illustrating the power and depth of Lie's seminal work.

Chapter 4

Adjoint Symmetry Equations, Integrating Factors and Solutions

4.1 Introduction

In this chapter we consider the role of the adjoint symmetry equation in determining explicit integrating factors of nonlinear ODEs. We also review briefly the so called extended Prelle-Singer method developed by the authors [8, 9, 10] and show that it is actually a reformulation of the adjoint symmetry equation as a first-order system.

As is well known symmetries play a crucial role in the solutions of differential equations. In fact much of the existing literature on symmetries of ODEs is devoted to what are known as Lie point symmetries. We begin by considering an n-th order ODE or its equivalent PDE in $(n+1)$ variables as given by (2.5.38)/(2.5.39). Such an ODE/PDE is said to admit a Lie point symmetry with generator

$$\mathbf{X} = \xi(x,y)\partial_x + \eta(x,y)\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(k)}\partial_{y^{(k)}}, \text{ where } \eta^{(i)} = \frac{d\eta^{(i-1)}}{dx} - y^{(i)}\frac{d\xi}{dx},$$

if

$$[\mathbf{X}, A] = gA \tag{4.1.1}$$

holds. Here $g = g(x, y, y', \ldots, y^{(n-1)})$ is some function and $\eta^{(i)}$'s denote the prolongations of the vector field (infinitesimal generators) $\mathbf{X}^{(0)} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$. For an n-th order ODE (2.5.38) the infinitesimal symmetry generators, when they exist, are determined from the linearized symmetry condition (2.5.46)

$$\eta^{(n)} = \xi w_x + \eta w_y + \eta^{(1)} w_{y'} + \cdots + \eta^{(n-1)} w_{y^{(n-1)}}$$

when (2.5.38) holds [41]. In terms of the characteristic, $Q := \eta - y'\xi$, this condition is given by (2.5.47), namely

$$A^n Q - w_{y^{(n-1)}} A^{(n-1)} Q - \cdots - w_{y'} AQ - w_y Q = 0.$$

For example when $y'' = w(x, y, y')$, the linearized symmetry condition is a second order linear PDE

$$A^2Q - w_{y'}AQ - w_yQ = 0, \tag{4.1.2}$$

with the vector field given by

$$A = \partial_x + y'\partial_y + w(x, y, y')\partial_{y'}.$$

4.2 Adjoint symmetries and Integrating factors

The following equation is known as the adjoint of the linearized symmetry condition (4.1.2), and its solutions are called the adjoint symmetries

$$A^n\Lambda + A^{n-1}(w_{y^{(n-1)}}\Lambda) - A^{n-2}(w_{y^{(n-2)}}\Lambda) + \cdots + (-1)^{n-1}w_y\Lambda = 0. \tag{4.2.1}$$

It must be stressed however,that these solutions are neither symmetries nor generators of symmetries, and it is more appropriate to call a solution as a *cocharacteristic*[41]. Normally a systematic procedure for finding the solutions of (4.2.1) is by making an ansatz for Λ; for example, to assume that they are independent of $y^{(n-1)}$ or to even assume a suitable rational structure. In order to illustrate the connection between the methods used by authors of [8, 9, 10] and the adjoint symmetry equation consider the equation

$$y^{(n)} = w(x, y, y', \ldots, y^{(n-1)}), \tag{4.2.2}$$

together with the base one-forms $dx, (dy - y'dx), \ldots, (dy^{(n-1)} - wdx)$. The null form obtained by multiplying, all but the first one-form by functions $S_i(x, y, y', \ldots, y^{(n-1)})$ where $i = 0, \ldots, n-1$ and demanding that after addition the resultant one-form be exact is

$$-(S_0y' + S_1y'' + \cdots + S_{n-2}y^{(n-1)} + S_{n-1}w)dx + (S_0dy + S_1dy' + \cdots + S_{n-2}dy^{(n-2)} + S_{n-1}dy^{(n-1)})$$
$$= dI(x, y, y', \ldots, y^{(n-1)}) = 0. \tag{4.2.3}$$

This implies

$$I_x = -(S_0y' + S_1y'' + \cdots + S_{n-2}y^{(n-2)} + wS_{n-1}) \tag{4.2.4}$$
$$I_y = S_0,\ I_{y'} = S_1, \ldots, I_{y^{(n-1)}} = S_{n-1}. \tag{4.2.5}$$

Clearly I is a first integral of the equation (4.2.2), provided it satisfies the integrability criteria

$$I_{xy^{(j)}} = I_{y^{(j)}x}, \quad j = 0, \ldots, n-1, \tag{4.2.6}$$
$$I_{y^{(j)}y^{(k)}} = I_{y^{(k)}y^{(j)}}, \quad 0 \le j < k \le n-1. \tag{4.2.7}$$

The vector field associated with (4.2.2) is

$$A = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \cdots + w\frac{\partial}{\partial y^{(n-1)}}, \tag{4.2.8}$$

in terms of which the integrability conditions (4.2.6) may be expressed as follows:

$$-A[S_{n-1}] = (w_{y^{(n-1)}}S_{n-1} + S_{n-2}), \tag{4.2.9}$$

$$-A[S_{n-2}] = (w_{y^{(n-2)}}S_{n-1} + S_{n-3}), \tag{4.2.10}$$

$$\vdots$$

$$-A[S_1] = (w_{y'}S_{n-1} + S_0), \tag{4.2.11}$$

$$-A[S_0] = w_y S_{n-1}. \tag{4.2.12}$$

The remaining integrability conditions (4.2.7) are all satisfied if

$$\frac{\partial S_{n-1}}{\partial y^{(j)}} = \frac{\partial S_j}{\partial y^{(n-1)}}, \quad 0 \leq j \leq n-2. \tag{4.2.13}$$

Our primary interest is to know S_{n-1}, since the remaining ones can be determined algebraically from eqns. (4.2.9)-(4.2.12) in a recursive manner. Eliminating the S_i's by successively applying the vector field A to eqn.(4.2.9) and using the remaining ones, we obtain finally

$$A^n[S_{n-1}] + A^{n-1}[w_{y^{(n-1)}}S_{n-1}] - A^{n-2}[w_{y^{(n-2)}}S_{n-1}] + \cdots + (-1)^{n-1}w_y S_{n-1} = 0. \tag{4.2.14}$$

But this is precisely the adjoint equation corresponding to the linearized symmetry equation (4.2.1). Thus the integrating factors of eqn. (3.1.5) are just the solutions of (4.2.14), which fulfil the integrability criteria stated in eqn.(4.2.13). Consequently, determination of the integrating factor S_{n-1} of (4.2.2) is basically equivalent to finding a solution of this equation. (The connection to the notation used in [10] is given by the following substitutions: $S_j \longrightarrow RS_{j+1}, \forall \quad j = 0, \ldots, n-3$ and $S_{n-1} \longrightarrow R$). The usual procedure to tackle such PDEs is to make a ansatz for S_{n-1}, for example assuming it to be a polynomial in $y^{(n-1)}$ of some suitable degree, and then obtaining its coefficients in a recursive manner. In their work, Chandrasekhar *et al* have assumed a rational form for S_{n-1}. As a consequence, instead of solving the adjoint equation directly, they solved the set (4.2.9)-(4.2.12), of first order equations by making appropriate ansätze for the S_i's. Suppose Λ^i be the solution(s) of the adjoint equation. Setting, $S_{n-1} = \Lambda^i$, one can calculate the remaining S_j's, in a recursive manner and check if (4.2.13) holds. In the event such a integrating factor exists and satisfies the integrability condition, its associated first integral may be obtained from the relation

$$I^i = \int S_0^i(dy - y'dx) + S_1^i(dy' - y''dx) + \cdots S_{n-1}^i(dy^{(n-1)} - wdx). \tag{4.2.15}$$

Essentially therefore, one can choose to either solve the adjoint equation directly and obtain S_{n-1} through some suitable ansatz, or make suitable ansätze for the S_k's and solve a set of n first order PDEs. In general the former involves solving a single higher-order equation, while the latter involves solving a system of first-order linear PDEs.

4.2.1 Some illustrative examples

Example 4.2.1 $y'' = w(x, y, y') = \frac{3y'^2}{y} + \frac{y'}{x}$

Here the system of coupled first-order PDEs, for the unknown functions S_0, S_1 are:

$$A[S_1] = -(w_{y'}S_1 + S_0) \tag{4.2.16}$$

$$A[S_0] = -w_y S_1 \tag{4.2.17}$$

where $A = \partial_x + y'\partial_y + w\partial_{y'}$; the integrability condition is simply

$$S_{1y} = S_{0y'}. \tag{4.2.18}$$

and the adjoint equation is

$$A^2[S_1] + A[w_{y'}S_1] - w_y S_1 = 0. \tag{4.2.19}$$

Assuming $\Lambda = S_1$ to be a solution of (4.2.19) independent of y', we have upon equating the coefficients of different powers of y' the following set of equations:

$$15\Lambda + 9y\Lambda_y + y^2\Lambda_{yy} = 0$$

$$3\Lambda + 3x\Lambda_x + y\Lambda_y + xy\Lambda_{xy} = 0$$

$$-\Lambda + x\Lambda_x + x^2\Lambda_{xx} = 0.$$

Their structure suggests an ansatz of the form $\Lambda = x^\alpha y^\beta$. One can verify that this leads to four solutions, namely:

$$\Lambda^1(x, y) = \frac{x}{y^3},\ \ \Lambda^2(x, y) = \frac{1}{xy^3},\ \ \Lambda^3(x, y) = \frac{1}{xy^5},\ \text{and}\ \Lambda^4(x, y) = \frac{x}{y^5}.$$

However, only Λ^1, Λ^2 are acceptable, as the others do not satisfy the integrability criterion (4.2.13). The results are summarized below, along with the respective first integrals:

$$(i)\quad \Lambda^1 = S_1^1 = \frac{x}{y^3},\quad S_0^1 = -\frac{x}{y^3}(\frac{2}{x} + \frac{3y'}{y}),\ \text{with}\ I^1(x, y, y') = \frac{xy' + y}{y^3}.$$

$$(ii)\quad \Lambda^2 = S_1^2 = \frac{1}{xy^3},\quad S_0^2 = -\frac{3y'}{xy^4},\ \text{with}\ I^2(x, y, y') = \frac{y'}{xy^3}.$$

The first integral I^2 was obtained by Duarte et al in [23]. However, I^1, was not obtained by them.

Example 4.2.2

$$y'' = w(x, y, y') = -(kyy' + \lambda y)$$

Here k and λ are constants and the equation represents a damped harmonic oscillator. As before one has to solve the adjoint symmetry equation (4.2.1), for $n = 2$, namely,

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + w_{y'}\Lambda_x + (w + y'w_{y'})\Lambda_y + (w_x + 2ww_{y'} + y'w_y)\Lambda_{y'} + \Lambda_{xx}$$

$$+2y'\Lambda_{xy} + y'^2\Lambda_{yy} + 2w\Lambda_{xy'} + 2wy'\Lambda_{yy'} + w^2\Lambda_{y'y'} = 0$$

Solving this PDE is a rather daunting task even when $w(x, y, y')$ ia a fairly simple. It is therefore natural to make certain simplifying assumptions regarding the functional dependence of Λ. For instance one can begin by assuming Λ to be independent of a particular variable, say x, and see that if that leads to a more manageable form of the adjoint equation. Alternatively, one may at the very outset assume that Λ depends on only one of the three variables x, y or y'. The choice of procedure to be adopted is one of sheer convenience. We illustrate this by first making the following assumption, $\Lambda_x = 0$, which leads to

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + (w + y'w_{y'})\Lambda_y + (w_x + 2ww_{y'} + y'w_y)\Lambda_{y'}$$

$$+y'^2\Lambda_{yy} + 2wy'\Lambda_{yy'} + w^2\Lambda_{y'y'} = 0$$

This is a linear parabolic PDE. Since $w = -(kyy' + \lambda y)$ we have

$$w_x = w_{y'y'} = 0, \; w_{y'} = -ky, \;\; w_y = -(ky' + \lambda), \;\text{ and }\; w_{yy'} = -k.$$

As solving this PDE is still rather formidable, let us further assume $\Lambda_y = 0$. In other words Λ is just a function of y' and our equation simplifies further to

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + (w_x + 2ww_{y'} + y'w_{y'})\Lambda_{y'} + w^2\Lambda_{y'y'} = 0.$$

Plugging in the expressions for partial derivatives of w and equating the coefficients of different powers of y, then leads to the following set of equations:

$$(ky' + \lambda)y'\Lambda_{y'} = \lambda\Lambda$$

$$2k\Lambda_{y'} + (ky' + \lambda)\Lambda_{y'y'} = 0.$$

These equations admit the particular solution $\Lambda^1(y') = \frac{y'}{(ky'+\lambda)}$ and one finds with $S_1^1 = \Lambda^1 = y'/(ky' + \lambda)$ that $S_0^1 = y$. The integrability condition $S_{1y}^1 = S_{0y'}^1$ is trivially satisfied and the corresponding first integral is

$$I^1(x, y, y') = y' + \frac{1}{2}ky^2 - \frac{\lambda}{k}\log(ky' + \lambda).$$

Note that this first integral is independent of x by construction. For such first integrals, the method devised by Chandrasekhar *et al*[10] allows us to determine the form of S_0 *a priori*. We dwell on this aspect in the following subsection.

4.2.2 First integrals independent of a particular coordinate

An interesting feature occurs when the first integral, I, is independent of a particular variable, say x, i.e., $I_x = 0$. Then in general, (4.2.4) implies

$$S_0 = -\frac{1}{y'}\left(y''S_1 + \cdots + S_{n-2}y^{(n-1)} + S_{n-1}w\right),$$

which enables us to eliminate S_0, and causes a reduction in the order of the equations for determining the integrating factor. For instance in case of a second-order ODE, we have $S_0y' + wS_1 = 0$, leading to $S_0 = -\frac{w}{y'}S_1$. As a result, one is left with a first-order PDE for determining S_1 namely

$$A[S_1] = -(w_{y'} - \frac{w}{y})S_1. \tag{4.2.20}$$

On the other hand for a third-order equation, we have

$$S_0 = -\frac{y''S_1 + wS_2}{y'}.$$

Elimination of S_0, from the system of equations (4.2.9)-(4.2.12) with $n = 3$, then requires us to solve for S_1 and S_2 from the coupled system:

$$A[S_2] = -(w_{y''}S_2 + S_1)$$

$$A[S_1] = -\left((w_{y'} - \frac{w}{y'})S_2 - \frac{y''}{y'}S_1\right),$$

which are equivalent to the following second-order equation for the integrating factor S_2:

$$A^2S_2 + A(w_{y''}S_2) - \frac{y''}{y'}AS_2 - \{(w_{y'} - \frac{w}{y'}) + \frac{y''}{y'}w_{y''}\}S_2 = 0. \tag{4.2.21}$$

Thus the absence of one 'coordinate' in a first integral causes only marginal simplification, namely a reduction, by one, in the order of the equation to be solved for the integrating factor. Nevertheless this is extremely useful for second-order equations, $y'' = w(x, y, y')$, since one is then required to solve a *single* first-order linear PDE for the integrating factor S_1. This fact was exploited to the hilt in [8, 9]. Although in general for $n \geq 3$, the existence of an x independent first integral, may not always lead to substantial reduction of computational labor, nevertheless it is instructive to look into the method of the authors of [8, 9, 10] more carefully, as it has proved to be quite successful in determining first integrals of many highly nonlinear oscillator type systems. Generally, for equations of the generic form $y'' = -f_1(y)y' - f_0(y)$, eqn (4.2.20) reduces to

$$A[S_1] = -\frac{f_0(y)}{y'}S_1.$$

The solution S_1^1 of example 4.2.2, suggests the ansatz $S_1 = \frac{y'}{h(y,y')}$, with the consequence that

$$A[S_1] = \frac{A(y')}{h} - \frac{y'}{h}Ah = -\frac{f_0(y)}{h}.$$

The problem now therefore reduces to a determination of the function $h(y, y')$ from the following relation (since $A(y') = w$), viz,

$$y'A[h] = (w + f_0)h = -f_1(y)y'h,$$

$$A[h] = -f_1(y)h. \tag{4.2.22}$$

The resulting PDE for h is explicitly given by

$$y'h_y + (-f_1y' - f_0)h_{y'} = -f_1y'h.$$

For $f_1 = ky$ and $f_0 = \lambda y$, assuming furthermore that h is independent of y we obtain $h(y') = C(ky' + \lambda)$. Thus once again we get the solutions (after setting constant $C = 1$),

$$S_1 = \frac{y'}{(ky' + \lambda)} \quad \text{and} \quad S_0 = y,$$

which satisfy the integrability criterion.
As pointed out in [10], it is often more convenient to modify the ansatz for S_1 to, $S_1 = \frac{y'}{h(y,y')^r}$, to handle more complicated situations.

For generic equations of the form (Liénard type)

$$y'' = -f_1(y)y' - f_0(y)$$

with this ansatz for S_1, (4.2.22) is modified to

$$rA[h] = -f_1(y)h. \tag{4.2.23}$$

Assuming, $h(y, y') = A'(y) + B(y)y' + C(y)y'^2$, substitution into (4.2.23) leads to the following set of equations for determining the unknown functions A, B and C upon equating coefficients of different powers of y', namely

$$C_y = 0, \;\; rB_y = (2rf_0 - f_1)C, \;\; rA'_y = (rf_0 - f_1)B - 2rCf_1 \;\text{ and }\; rf_0B = f_1A'. \tag{4.2.24}$$

Suppose now

$$f_0(y) = \lambda y^\xi \quad \text{and} \quad f_1(y) = \mu y^\eta,$$

where λ and μ are parameters and ξ and η are constants. We obtain then the following solutions for C, B and A:

$$C(y) = \gamma, \quad rB(y) = \mu\gamma\frac{(2r-1)}{\eta+1}y^{\eta+1} + \beta$$

$$rA'(y) = \frac{2\lambda r\gamma}{\xi+1}y^{\xi+1} + \mu(r-1)\left[\frac{(2r-1)\mu\gamma}{2r(\eta+1)^2}y^{2(\eta+1)} + \frac{\beta}{r(\eta+1)}y^{\eta+1}\right] + \alpha,$$

where α, β and γ are constants of integration. From the last condition in eqn. (4.2.24), i.e., $rf_0B = f_1A'$, it follows, assuming $\xi \neq \eta$, that $\alpha = \beta = 0$ and leads to the following relation,

$$\lambda r\left[\frac{(2r-1)}{(\eta+1)} - \frac{2}{(\xi+1)}\right]y^{\xi+\eta+1} = \frac{\mu^2(r-1)(2r-1)}{2r(\eta+1)^2}y^{3\eta+2}. \tag{4.2.25}$$

One can then identify two possible cases.
(a) When, $r = 1$, we have $\xi = 2\eta+1$, $A'(y) = \frac{\lambda\gamma}{(\eta+1)}y^{2(\eta+1)}$ and $B(y) = \frac{\mu\gamma}{(\eta+1)}y^{(\eta+1)}$. The corresponding integrating factor is given by

$$S_1^a = \frac{y'}{[\frac{\lambda\gamma}{(\eta+1)}y^{2(\eta+1)} + \frac{\mu\gamma}{(\eta+1)}y^{(\eta+1)}y' + \gamma y'^2]} \quad \text{and} \quad S_0^a = \frac{\mu y^\eta y' + \lambda y^{2\eta+1}}{y'}S_1.$$

(b) For, $r \neq 1$, assuming the exponents of y in (4.2.25) to be equal we find once again $\xi = 2\eta+1$. Upon equating their coefficients we obtain a quadratic equation for the exponent r occurring in the denominator of the integrating factor with solution

$$r = \frac{\mu^2}{4\lambda(\eta+1)}\left[1 \pm \sqrt{1 - \frac{4\lambda}{\mu^2}(\eta+1)}\right].$$

Therefore, in this case $S_1^b = \frac{y'}{h^r}$ where

$$h(y,y') = \frac{\gamma}{(\eta+1)}\left[\lambda + \mu^2\frac{(r-1)(2r-1)}{2r^2(\eta+1)}\right]y^{2(\eta+1)} + \frac{\gamma\mu(2r-1)}{r(\eta+1)}y^{\eta+1}y' + \gamma y'^2.$$

4.3 Coupled second-order equations

In this section we consider an application of the adjoint equation to a coupled system of equations of the form

$$\ddot{x} = \phi_1(x,y) \text{ and } \ddot{y} = \phi_2(x,y). \tag{4.3.1}$$

As before, consider the following base one forms $(dx-\dot{x}dt)$, $(dy-\dot{y}dt)$, $(d\dot{x}-\phi_1 dt)$, $(d\dot{y}-\phi_2 dt)$ Let S_1, S_2 and R_1, R_2 be functions such that

$$S_1(dx-\dot{x}dt) + S_2(dy-\dot{y}dt) + R_1(d\dot{x}-\phi_1 dt) + R_2(d\dot{y}-\phi_2 dt) = dI(t,x,y,\dot{x},\dot{y}) = 0. \tag{4.3.2}$$

Hence

$$I_t = -(S_1\dot{x} + S_2\dot{y} + R_1\phi_1 + R_2\phi_2), \tag{4.3.3}$$

$$I_x = S_1, \quad I_y = S_2, \quad I_{\dot{x}} = R_1, \quad I_{\dot{y}} = R_2. \tag{4.3.4}$$

The functions R_1 and R_2 are the integrating factors. Compatibility of the set of equations. (4.3.3) and (4.3.4) namely:

$$I_{tx} = I_{xt}, \quad I_{ty} = I_{yt}, \quad I_{t\dot{x}} = I_{\dot{x}t}, \quad I_{t\dot{y}} = I_{\dot{y}t},$$

$$I_{xy} = I_{yx}, \quad I_{x\dot{x}} = I_{\dot{x}x}, \quad I_{x\dot{y}} = I_{\dot{y}x}, \quad I_{y\dot{x}} = I_{\dot{x}y}, \quad I_{y\dot{y}} = I_{\dot{y}y}, \tag{4.3.5}$$

requires that the following hold:

$$D[R_1] = -(S_1 + R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}}), \tag{4.3.6}$$

$$D[R_2] = -(S_2 + R_1\phi_{1\dot{y}} + R_2\phi_{2\dot{y}}), \tag{4.3.7}$$

$$D[S_1] = -(R_1\phi_{1x} + R_2\phi_{2x}), \tag{4.3.8}$$

$$D[S_2] = -(R_1\phi_{1y} + R_2\phi_{2y}), \tag{4.3.9}$$

where $D = \partial_t + \dot{x}\partial_x + \dot{y}\partial_y + \phi_1\partial_{\dot{x}} + \phi_2\partial_{\dot{y}}$. It is evident that once R_1 and R_2 are known the remaining S_1 and S_2 can be determined algebraically from (4.3.6) and (4.3.7). Since our basic aim is to determine the integrating factors, we can eliminate, say S_1, by differentiating (4.3.6) and using (4.3.8) to get

$$D^2[R_1] + D[R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}}] - (R_1\phi_{1x} + R_2\phi_{2x}) = 0 \tag{4.3.10}$$

Similarly eliminating S_2 yields

$$D^2[R_2] + D[R_1\phi_{1\dot{y}} + R_2\phi_{2\dot{y}}] - (R_1\phi_{1y} + R_2\phi_{2y}) = 0. \tag{4.3.11}$$

Equations (4.3.10)-(4.3.11) constitute the coupled version of the adjoint equation (4.2.1) when $n = 2$.

One needs to check, of course, that the solutions of the coupled adjoint equations indeed satisfy the compatibility conditions (4.3.5). In general one employs an ansatz for R_1 and R_2 in order to solve the system of PDEs (4.3.10)-(4.3.11). From a knowledge of R_1, R_2 and S_1, S_2 it is straightforward to obtain the first integral from

$$I = \int S_1(dx - \dot{x}dt) + S_2(dy - \dot{y}dt) + R_1(d\dot{x} - \phi_1 dt) + R_2(d\dot{y} - \phi_2 dt). \tag{4.3.12}$$

Example 4.3.1

Consider the following system of second-order equations:

$$\begin{array}{c} \ddot{x} + \frac{\alpha}{x^2} g(u) - \frac{\lambda}{x^3} = 0 \\ \ddot{y} + \frac{\beta}{x^2} f(u) - \frac{\mu}{y^3} = 0, \quad u = \frac{y}{x}. \end{array} \tag{4.3.13}$$

Here α, β, λ and μ are parameters and f and g are arbitrary functions. Writing these equations in the form $\ddot{x} = \phi_1(x, y)$ and $\ddot{y} = \phi_2(x, y)$, we identify

$$\phi_1(x, y) = -\frac{\alpha}{x^2} g(u) + \frac{\lambda}{x^3} \quad \text{and} \quad \phi_2(x, y) = -\frac{\beta}{x^2} f(u) + \frac{\mu}{y^3}$$

Note here that ϕ_1 and ϕ_2 are velocity independent and for a time independent first integral, $I_t = 0$, we may take $D = \dot{x}\partial_x + \dot{y}\partial_y + \phi_1\partial_{\dot{x}} + \phi_2\partial_{\dot{y}}$. In that event with the following ansatz for R_1 and R_2 namely

$$R_1 = a_1(x, y)\dot{x} + a_2(x, y)\dot{y} \text{ and } R_2 = b_1(x, y)\dot{x} + b_2(x, y)\dot{y}, \tag{4.3.14}$$

(4.3.10) and (4.3.11) yield the following equations:

$$\dot{x}^3 a_{1xx} + \dot{x}^2\dot{y}(a_{2xx} + 2a_{1xy}) + \dot{x}\dot{y}^2(2a_{2xy} + a_{1yy}) + a_{2yy}\dot{y}^3 + \dot{x}\{(\phi_1 a_1 + \phi_2 a_2)_x + 2a_{1x}\phi_1 + (a_{2x} + a_{1y})\phi_2\}$$

$$+\dot{y}\{(\phi_1 a_1 + \phi_2 a_2)_y + 2a_{2y}\phi_2 + (a_{2x} + a_{1y})\phi_1\} = \dot{x}(\phi_{1x}a_1 + \phi_{2x}b_1) + \dot{y}(\phi_{1x}a_2 + \phi_{2x}b_2), \tag{4.3.15}$$

$$\dot{x}^3 b_{1xx} + \dot{x}^2\dot{y}(b_{2xx} + 2b_{1xy}) + \dot{x}\dot{y}^2(2b_{2xy} + b_{1yy}) + b_{2yy}\dot{y}^3 + \dot{x}\{(\phi_1 b_1 + \phi_2 b_2)_x + 2b_{1x}\phi_1 + (b_{2x} + b_{1y})\phi_2\}$$

$$+\dot{y}\{(\phi_1 b_1 + \phi_2 b_2)_y + 2b_{2y}\phi_2 + (b_{2x} + b_{1y})\phi_1\} = \dot{x}(\phi_{1y}a_1 + \phi_{2y}b_1) + \dot{y}(\phi_{1y}a_2 + \phi_{2y}b_2). \tag{4.3.16}$$

Equating coefficients of different powers of the velocities we get the following system of equations:

$$a_{1xx} = 0, \quad a_{2xx} + 2a_{1xy} = 0, \quad a_{1yy} + 2a_{2xy} = 0, \quad a_{2yy} = 0, \tag{4.3.17}$$

$$(\phi_1 a_1 + \phi_2 a_2)_x + 2a_{1x}\phi_1 + (a_{2x} + a_{1y})\phi_2 = (\phi_{1x}a_1 + \phi_{2x}b_1), \tag{4.3.18}$$

$$(\phi_1 a_1 + \phi_2 a_2)_y + 2a_{2y}\phi_2 + (a_{2x} + a_{1y})\phi_1 = (\phi_{1x}a_2 + \phi_{2x}b_2) \tag{4.3.19}$$

$$b_{1xx} = 0, \quad b_{2xx} + 2b_{1xy} = 0, \quad b_{1yy} + 2b_{2xy} = 0, \quad a_{2yy} = 0, \tag{4.3.20}$$

$$(\phi_1 b_1 + \phi_2 b_2)_x + 2b_{1x}\phi_1 + (b_{2x} + b_{1y})\phi_2 = (\phi_{1y}a_1 + \phi_{2y}b_1), \tag{4.3.21}$$

$$(\phi_1 b_1 + \phi_2 b_2)_y + 2b_{2y}\phi_2 + (b_{2x} + b_{1y})\phi_1 = (\phi_{1y}a_2 + \phi_{2y}b_2). \tag{4.3.22}$$

Observes that the choice a_k = constant and b_k = constant $k = 1, 2$ satisfies (4.3.17) and (4.3.20), while there remaining equations then simplify to

$$\phi_{2x}(b_1 - a_2) = 0, \quad \phi_{1y}(a_2 - b_1) = 0,$$

$$\phi_{1x} - \phi_{2y}a_2 - \phi_{1y}a_1 + \phi_{2x}b_2 = 0,$$

$$\phi_{1y}a_1 + (\phi_{2y} - \phi_{1x})b_1 - \phi_{2x}b_2 = 0.$$

The first two equations imply, $a_2 = b_1$, which renders the second and the third equations identical, namely

$$(\phi_{1x} - \phi_{2y})a_2 - \phi_{1y}a_1 + \phi_{2x}b_2 = 0. \tag{4.3.23}$$

Clearly if the system (4.3.13) is derivable from a potential then it is necessary that $\phi_{1y} = \phi_{2x}$. With this in mind (4.3.23) can be satisfied by making the choice $a_2 = b_1 = 0$ whilst a_1 and b_2 are arbitrary. Therefore the choice $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$ leads to the following solutions:

$$R_1 = \dot{x} \quad R_2 = \dot{y}. \tag{4.3.24}$$

On the other hand the solution of S_1 and S_2 from (4.3.6) and (4.3.7) are then found to be

$$S_1 = -\phi_1 = \frac{\alpha}{x^2}g(u) - \frac{\lambda}{x^3}$$

$$S_2 = -\phi_2 = \frac{\beta}{x^2}f(u) - \frac{\mu}{y^3}, \quad u = \frac{y}{x}$$

Using the above values of R_i and $S_i (i = 1, 2)$ we obtain from (4.3.12) the first integral

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} + N(x, y),$$

where

$$N(x, y) = \int \frac{\alpha}{x^2}g(u)dx + \int \frac{\beta}{x^2}f(u)dy.$$

The condition, $\phi_{1y} = \phi_{2x}$, translates to

$$\alpha g'(u) + 2\beta f(u) + \beta u f'(u) = 0. \tag{4.3.25}$$

Using this condition, $N(x, y)$, may be evaluated and we find that

$$N(x, y) = -\frac{\beta}{x}\Big(\frac{\alpha}{\beta}g(u) + uf(u)\Big).$$

Hence a first integral for the system of second-order equation is given by

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} - \frac{\beta}{x}\Big(\frac{\alpha}{\beta}g(u) + uf(u)\Big). \tag{4.3.26}$$

Let us now look for another solution set of the coupled adjoint equations for R_1 and R_2. It is easily verified that

$$a_1(x, y) = y^2, a_2(x, y) = -xy = b_1(x, y) \text{ and } b_2(x, y) = x^2 \tag{4.3.27}$$

satisfy (4.3.17) and (4.3.20) while (4.3.18) and (4.3.22) are identically satisfied. The remaining equations (4.3.19) and (4.3.21) become identical and reduce to the following equation:

$$3(y\phi_1 - x\phi_2) = (\phi_{2y} - \phi_{1x})xy - \phi_{1y}y^2 + \phi_{2x}x^2. \tag{4.3.28}$$

Substituting the values of $\phi_i (i = 1, 2)$ and their derivatives leads to the following condition on the function f and g, namely:

$$\alpha u g(u) - \beta f(u) = 0, \qquad u = \frac{y}{x}. \tag{4.3.29}$$

From (4.3.27) we derive the following solution for $R_i (i = 1, 2)$:

$$R_1 = -y(x\dot{y} - y\dot{x}) \ \text{ and } \ R_2 = x(x\dot{y} - y\dot{x}). \tag{4.3.30}$$

The corresponding values of $S_i (i = 1, 2)$ are

$$S_1 = \dot{y}(x\dot{y} - y\dot{x}) - \lambda\frac{y^2}{x^3} + \mu\frac{x}{y^2} \ \text{ and } \ S_2 = -\dot{x}(x\dot{y} - y\dot{x}) + \lambda\frac{y}{x^3} - \mu\frac{x^2}{y^3}, \tag{4.3.31}$$

where use has been made of the condition (4.3.29). Hence from (4.3.12) we obtain another first integral given by

$$I = \frac{1}{2}\left[(y\dot{x} - x\dot{y})^2 + \frac{\lambda}{2}\frac{y^2}{x^2} + \frac{\mu}{2}\frac{x^2}{y^2}\right]. \tag{4.3.32}$$

The two first integrals given by (4.3.26) and (4.3.32) will be valid simultaneously provided we can find functions f and g which satisfy (4.3.25) and (4.3.29). It is easily verified that these require the functions f and g to be given by

$$g(u) = \frac{1}{(1 + u^2)^{3/2}} \qquad \text{and} \quad f(u) = \frac{\alpha}{\beta}\frac{u}{(1 + u^2)^{3/2}},$$

respectively. Under the circumstances the system of second-order equations (4.3.13) reduces to the following well known system

$$\ddot{x} + \frac{\alpha x}{(x^2 + y^2)^{3/2}} - \frac{\lambda}{x^3} = 0 \qquad \ddot{y} + \frac{\alpha y}{(x^2 + y^2)^{3/2}} - \frac{\mu}{y^3} = 0,$$

with first integrals

$$I_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} - \frac{\alpha}{\sqrt{x^2 + y^2}}$$

$$I_2 = \frac{1}{2}\left[(y\dot{x} - x\dot{y})^2 + \frac{\lambda}{2}\frac{y^2}{x^2} + \frac{\mu}{2}\frac{x^2}{y^2}\right].$$

A more interesting situation from the physical point of view arises when the functions f and g satisfy condition (4.3.25) but not condition (4.3.29). In that event the system of equations

(4.3.13) admits just one first integral given by (4.3.26), with f and g satisfying (4.3.25). In [98] the authors obtained a system of equations similar in structure to (4.3.13), in the context of the dynamics of stellar systems, with

$$f(u) = 2(1 - ug(u)).$$

Condition (4.3.25) then leads the following differential equation determining $g(u)$:

$$(1 - 2u^2)g'(u) = 2(3ug(u) - 2)$$

and the first integral from (4.3.26) assumes the form (setting all the parameters equal to unity)

$$I = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2x^2} + \frac{1}{2y^2} + \frac{1}{x}(2u + (1 - 2u^2)g(u)), \qquad u = \frac{y}{x}.$$

In fact this first integral serves as Hamiltonian.

In this chapter we have showed that the RS-pair method used by Chandrasekhar *et al* to derive first integrals of second-order ODEs can actually be reformulated in terms of the adjoint symmetry equation of classical Lie symmetry analysis. This illustrates once again the immense power of symmetries in the analysis of ODEs. In the following chapter we extend the very notion of a symmetry as defined by (4.1.1) to include what are known as λ-symmetries and illustrate their use for second and third-order ODEs.

Chapter 5

λ-Symmetries and Integrating Factors of Nonlinear ODEs

5.1 Introduction

In this chapter we discuss λ-symmetries of some second-order equations of the Painlevé-Gambier type and study their relationship with the standard adjoint symmetry equation used for determining the integrating factor of a second-order ordinary differential equation (ODE). This is followed by a brief study of the λ-symmetries of certain special types of third-order ODEs.

Lie symmetry analysis of differential equations provides a powerful and fundamental framework for the exploitation of systematic procedures leading to the integration by quadrature of ordinary differential equations [41, 82, 100]. This was the main motivation of Sophus Lie when he created the theory of Lie groups and Lie algebras. In the last few years special attention has been devoted to a new class of symmetries introduced by Muriel and Romero [65, 66, 67, 68, 70, 74]. These symmetries are neither Lie point nor Lie-Bäcklund symmetries and are called λ-symmetries since they are vector fields which depend upon a function λ. If a system does not have a Lie point symmetry, then Muriel and Romero have demonstrated that many of the processes of reduction of order can be explained via the invariance of the equation under λ-symmetries. In other words the new technique of λ-prolongations together with certain conditions of invariance enables us to introduce the concept of a λ-symmetry and yields a new method of reduction for ordinary differential equations. A generalization of the concept of variational symmetry, based on λ-prolongations, has been studied in [70]. This allows us to construct new methods of reduction for Euler-Lagrange equations.

Pucci and Saccomandi [88] have, on the other hand, identified the most general class of transformation sharing the important properties of standard symmetries concerning reduction of a scalar ODE. The approach to this reduction is based upon differential invariants of telescopic vector fields. Like classical Lie symmetries, if an equation is invariant under a λ-symmetry, one can obtain a complete set of functionally independent invariants and reduce the order of the equation by one. Many applications and extensions of this notion have

been proposed in the last ten years: these include extensions to systems of ODEs, to partial differential equations (PDEs) (cf. [32]), applications to variational principles and theorems of Noether type.
In order to introduce the notion of λ-symmetries let us briefly recollect some of the basic facts about contact forms and the adjoint symmetry equation.

5.2 Contact forms and first integrals

We confine ourselves to the smooth category of manifolds and maps. Let $\pi : Y \to X$ be a smooth vector bundle over a k-dimensional base manifold, with l-dimensional fibres. Suppose that $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_l)$ are the local coordinates on X and Y, respectively, so that sections of Y are prescribed by smooth functions $y = s(x)$. Let $\pi_k : J^k(\pi) \to X$ be the k-order jet bundle associated to π. Any local section $s : X \to Y$ of π generate the section of $J^n\pi$. We denote $j^n(s) : X \to J^n\pi$ as n-jet, which forms a section of the nth-order jet bundle. Let γ_s^n be a graph of section $j^n(s)$ and let $\kappa_n = (x_j, y_i, y_k^i, \cdots, y_{k_1\cdots k_n}) \in \gamma_s^n$ be a special coordinate system. Consider all graphs γ_s^n passing through κ_n. Let $\mathcal{C}_{\kappa_n}$ be the subspace of $T_{\kappa_n}J^n\pi$ spanned by all subspaces $T_{\kappa_n}\gamma_s^n$ of these graphs. The space $\mathcal{C}_{\kappa_n}$ is spanned by the vectors

$$\frac{\partial}{\partial x_j} + y_k^i \frac{\partial}{\partial y_i} + \cdots + y^i_{k_1\cdots k_{n-1}k} \frac{\partial}{\partial y^i_{k_1\cdots k_{n-1}}}.$$

The distribution $\mathcal{C}_n : \kappa_n \to \mathcal{C}_{\kappa_n}$ is called the Cartan distribution on $J^n\pi$. This distribution is defined by a space of differential 1-forms, which are called the Cartan forms. Before going to define this form let us switch to multi-index notation. The induced natural coordinates on $J^n\pi$ are given by $(x, y^{(n)}) = (\cdots x_i \cdots y_J^K \cdots)$, where derivative coordinates y_J^K are written in multi-index form.

A differential form θ on the jet space $J^n\pi$ is called a Cartan form if it is annihilated by all jets $(j^n(s))^*\theta = 0$. In local coordinates every contact one-form on $J^n\pi$ can be expressed as

$$\theta_J^\alpha = dy_J^\alpha - \sum_{i=1}^{k} y_{J,i}^\alpha dx^i, \qquad \alpha = 1, \cdots, l, \;\; 0 \le J < n.$$

In other words, $\chi \in T_{\kappa_n}\gamma_s^n$ belongs to $\mathcal{C}_{\kappa_n}$ if and only if χ is a solution of the linear homogeneous equations in multi-index notation $(dy_J^\alpha - \sum_{i=1}^k y_{J,i}^\alpha dx^i)\chi = 0$, interested readers are referred to [53, 82, 83, 92].

Because we are only concerned with the case of one independent and one dependent variable, the basic contact forms are

$$\theta_0 = (dy - y'dx), \quad \theta_1 = (dy' - y''dx), \cdots, \theta_{n-1} = (dy^{(n-1)} - wdx).$$

A local diffeomorphism $\varphi : J^n\pi \to J^n\pi$ defines a contact transformation of order n if it preserves the contact ideal, i.e., if θ is any contact form on $J^n\pi$, then $\varphi^*\theta$ is also a contact form. Point transformations are those contact transformations that preserve the fibres of the projection $J^n\pi \to \mathbb{R}^2$ for the case of one independent and one dependent variable.

Consider an nth-order ODE

$$\mathcal{E} := \left\{ \Delta(x, y^{(n)}) := y^{(n)} - w(x, y, ..., y^{(n-1)}) = 0 \right\}. \tag{5.2.1}$$

Geometrically $\mathcal{E}$ is interpreted as an hyper surface in the space $J^n(\pi)$ of n-jets of mappings from $\mathbb{R} \to \mathbb{R}$ and any solution of the system is a section of π the nth-order prolongation of which is an integral manifold of the restriction $\mathcal{C}^n|_{\mathcal{E}}$ of the contact distribution to $\mathcal{E}$.

The contact distribution on $\mathcal{E}$ can also be described as the annihilator space of the contact ideal $\mathcal{C}$ generated by the 1-forms

$$dx,\ (dy - y'dx),\ (dy' - y''dx), ..., (dy^{(n-1)} - wdx).$$

It is apparent that these contact forms are null over the solution. One can associate with (5.2.1) the following differential operator (total derivative operator)

$$A = \partial_x + y'\partial_y + \cdots + w\partial_{y^{(n-1)}}. \tag{5.2.2}$$

which annihilates all the contact forms $\theta_i = (dy^{(i)} - y^{(i+1)}dx)$.

Now the statements contained in (4.2.2)-(4.2.13) may be summarized into the following proposition.

Proposition 5.2.1 *Consider an nth-order ODE $y^{(n)} = w(x, y, \cdots, y^{(n-1)})$. Given the contact forms $\theta_1 = (dy - y'dx), \theta_2 = (dy' - y''dx), \cdots, \theta_n = (dy^{(n-1)} - w(x)dx)$, if there exists functions S_i $(i = 0, 1, \cdots (n-1))$ such that $dI = \sum_{i=0}^{n-1} S_i\theta_i$ is exact, then I is a first integral of the ODE provided the S_i satisfy the following set of coupled PDEs*

$$-A[S_k] = (w_{y^{(k)}} S_{n-1} + S_{k-1}), \quad k = n-1, ..., 0. \tag{5.2.3}$$

and

$$\frac{\partial S_{n-1}}{\partial y^{(j)}} = \frac{\partial S_j}{\partial y^{(n-1)}},\ 0 \le j \le n-2. \tag{5.2.4}$$

Illustration

For $n = 1$ we have a single PDE $A[S_0] = -w_y S_0$ which determines the integrating factor S_0. The case of rational w may be treated by the Prelle-Singer (semi-)algorithm [26, 87].

In the case of $n = 2$ with the ODE of the form

$$y'' = w(x, y, y'),$$

and $A = \partial_x + y'\partial_y + w\partial_{y'}$, we have from (5.2.3)

$$A[S_1] = -(w_{y'} S_1 + S_0) \tag{5.2.5}$$

and

$$A[S_0] = -w_y S_1 \tag{5.2.6}$$

together with the integrability condition: $S_{1y} = S_{0y'}$. Clearly, if we can solve for S_0 and S_1, then the first integral may be determined from the relation

$$I(x,y) = \int [S_0 dy + S_1 dy' - (S_0 y' + S_1 w)dx]. \tag{5.2.7}$$

Equations (5.2.5) and (5.2.6) may be solved in a number of ways. As mentioned in the previous chapter one possibility is to make suitable ansätze for S_0 and S_1 (the most commonly employed method is to assume that they are polynomials in y') or we could try to decouple them to get the corresponding adjoint symmetry equation [41]

$$A^2[S_1] + A(w_{y'} S_1) - w_y S_1 = 0. \tag{5.2.8}$$

By solving this equation we can find S_1 and determine S_0 algebraically from (5.2.5) and hence the first integral from (5.2.7).

On the other hand, if we define the ratio $\lambda := -\frac{S_0}{S_1} = -\frac{I_y}{I_{y'}}$, then it follows that

$$I_y + \lambda I_{y'} = 0 \tag{5.2.9}$$

while application of the vector field A to λ as defined above gives

$$A[\lambda] = -\frac{A[S_0]}{S_1} + \frac{A[S_1]}{S_1^2} S_0.$$

Upon using (5.2.5) and (5.2.6) this yields the following equation

$$A[\lambda] + \lambda^2 = w_y + w_{y'}\lambda. \tag{5.2.10}$$

The last two equations are connected to the existence of λ-symmetries as we explain below and has been examined in [69] also. This feature also prompts us to study more closely the case of second and third-order ODEs and to see if one can exploit the notion of integrating factors to generate λ-symmetries and *vice versa.*

5.3 λ-symmetries

A vector field X on Y is given by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

This can be uniquely prolonged to a vector field $X^{[k]}$ in $J^k(\pi)$ by requiring it to preserve the contact structure. A (system of) differential equation(s) $\Delta(x, y^{(1)}, \cdots, y^{(n)}) = 0$, admits X as an exact symmetry if the condition, $X^{[n]}(\Delta)|_{\Delta=0} = 0$, is satisfied.

Like exact symmetries, λ-symmetries also provide a powerful technique for finding invariant solutions. Let λ be a smooth function on $J^1(\pi)$. Then we say that the λ-prolongation to $J^k(\pi)$ of a vector field $X = \xi\partial_x + \eta\partial_y$ on Y is the vector field

$$X^{[\lambda,(k)]} = \xi\partial_x + \eta_{[\lambda,(k)]}\partial_{y^k}$$

with

$$\eta_{[\lambda,(0)]} = \eta, \qquad \eta_{[\lambda,(k)]} = D_x\big(\eta_{[\lambda,(k-1)]}\big) - D_x(\xi)y_k + \lambda\big(\eta_{[\lambda,(k-1)]} - \xi y_k\big).$$

Here

$$D_x = \frac{d}{dx}$$

We say that a vector field $X^{[\lambda,(k)]}$ is a λ-symmetry of $\mathcal{E}$ if and only if $X^{[\lambda,(k)]}$ is tangent to $\mathcal{E}$. Formally the λ-prolongation of a vector field X can be identified as the ordinary prolongation of a nonlocal exponential vector field (see Olver [82], exercise 2.31)

$$\hat{X}^{[n]} = e^{\int \lambda\, dx} X^{[\lambda,(n)]}, \qquad \text{where} \qquad \hat{X} = e^{\int \lambda\, dx} X.$$

In general for an nth-order ODE given by $\Delta(x, y^{(n)}) = 0$, if we suppose $S_0(x, y^{(n-1)})$ to be an integrating factor of (5.2.1) so that

$$S_0(\Delta(x, y^{(n)})) = A(I(x, y^{(n-1)})),$$

then $I(x, y^{(n-1)})$ is a first integral and leads to a reduction in order of the given ODE to $I(x, y^{(n-1)}) = c$, where A is defined by (5.2.2). Note that for brevity here we are denoting $I(x, y, y', ..., y^{(n-1)})$ as $I(x, y^{(n-1)})$. Let $\lambda \in \mathcal{C}^\infty(M^{(k)}),\ 0 \le k \le n-1$ be any solution of the following PDE

$$X^{[\lambda,(n-1)]}I = 0,$$

(assuming that we know a first integral I), where

$$X^{[\lambda,(n-1)]} = \sum_{i=0}^{n-1} (A+\lambda)^i(1)\frac{\partial}{\partial y^{(i)}},$$

denotes the $(n-1)$th-order λ-prolongation of the symmetry generator which has the form $X = \partial_y$, so that the characteristic $Q = \eta - \xi y' = 1$.
On the other hand the symmetry condition is given by

$$X^{[\lambda,(n)]}(y^{(n)} - w) = 0 \quad \text{on} \quad y^{(n)} = w. \tag{5.3.1}$$

This leads to the linearized symmetry condition:

$$(A+\lambda)^n(1) = \sum_{i=0}^{n-1} (A+\lambda)^i(1)\frac{\partial w}{\partial y^{(i)}}. \tag{5.3.2}$$

When $n = 2$, this condition assumes the form

$$A[\lambda] + \lambda^2 = w_y + \lambda w_{y'} \tag{5.3.3}$$

which is identical to (5.2.10). Therefore we can conclude that in case of second-order ODEs λ is always given by $-S_0/S_1$ where S_0 and S_1 are the solutions of the coupled system, (5.2.5) and (5.2.6).

For $n = 3$ the explicit form of the corresponding equation determining λ is

$$A^2[\lambda] + 3\lambda A[\lambda] + \lambda^3 = w_y + \lambda w_{y'} + (A[\lambda] + \lambda^2) w_{y''}. \tag{5.3.4}$$

When a first integral is not known, one can use (5.3.4) to determine λ.

5.3.1 λ-symmetries and the second-order Painlevé-Gambier equations

In this subsection we consider a few of the second-order equations of the Painlevé-Gambier classification and observe certain interesting features of these equations in the context of λ-symmetries. It is seen that when the equations admit at least a single Lie point symmetry the corresponding functional form of λ appears to be a rational function of y' while, when the equation does not admit a single Lie symmetry generator, the corresponding expression for λ is found to be polynomial. Let us consider equation XXVII of the Painlevé-Gambier classification [44], namely

$$y'' = \frac{m-1}{m} y'^2 + \Big(fy + \phi - \frac{m-2}{my}\Big) y' - \frac{mf^2}{(m+2)^2} y^3 + \frac{m(f' - f\phi)}{m+2} y^2 + \psi y - \phi - \frac{1}{my}.$$

Here f, ϕ and ψ are definite rational functions of two analytic functions $q(y)$ and $r(y)$ and of their derivatives. In the particular case of $m = 2$ with $f = -2, \phi = 0$ and $\psi(x) = F(x)$, the canonical equation is

$$y'' = w(x, y, y') = \frac{1}{2}\frac{y'^2}{y} - 2yy' - \frac{y^3}{2} + F(x)y - \frac{1}{2y}. \tag{5.3.5}$$

When we used Maple, we could not find any Lie point symmetries for this equation. The vector field $X = \partial_y$ is a λ-symmetry of (5.3.5) if and only if $\lambda(x, y, y')$ is a solution of the determining equation (5.3.3) so that one has, since $A = \partial_x + y'\partial_y + w\partial_{y'}$,

$$\lambda_x + y'\lambda_y + \Big(\frac{1}{2}\frac{y'^2}{y} - 2yy' - \frac{y^3}{2} + F(x)y - \frac{1}{2y}\Big)\lambda_{y'} + \lambda^2 = -\frac{y'^2}{2y^2} - 2y' - \frac{3y^2}{2} + F(x) + \frac{1}{2y^2} + \lambda\Big(\frac{y'}{y} - 2y\Big). \tag{5.3.6}$$

In order to find some solution of (5.3.6), we assume an ansatz which is linear in y', viz, $\lambda = \alpha(x,y)y' + \beta(x,y)$. Then α and β must satisfy the following equations:

$$\alpha_y + \frac{\alpha}{2y} + \alpha^2 = -\frac{1}{2y^2} + \frac{\alpha}{y}, \tag{5.3.7}$$

$$\beta_y + 2\alpha\beta - \frac{\beta}{y} + \alpha_x = -2, \tag{5.3.8}$$

$$\beta_x - \frac{y^3}{2}\alpha + F(x)y\alpha - \frac{\alpha}{2y} + \beta^2 = F(x) - \frac{3}{2}y^2 + \frac{1}{2y^2} - 2y\beta. \tag{5.3.9}$$

It is clear that $\alpha(x,y) = \frac{1}{y}$ is a particular solution of (5.3.7). With this value of α, equation (5.3.8) becomes

$$\beta_y + \frac{\beta}{y} = -2. \tag{5.3.10}$$

The general solution of (5.3.10) is given by

$$\beta(x,y) = \frac{r(x)}{y} - y, \tag{5.3.11}$$

where $r(x)$ is an arbitrary function of x. When we substitute these values of α and β into (5.3.9), we have

$$\frac{r'(x)}{y} + \frac{r^2(x)}{y^2} = \frac{1}{y^2}, \tag{5.3.12}$$

which implies $r(x) = \pm 1$. Thus there are two λ-symmetries:

$$\lambda_1 = \frac{y'}{y} + \frac{1}{y} - y \;\text{ and }\; \lambda_2 = \frac{y'}{y} - \frac{1}{y} - y. \tag{5.3.13}$$

Next we calculate a first integral $I(x,y,y')$ for the given ODE when it admits λ_i-symmetry with generator $X = \partial_y$. This is accomplished in essentially two steps with a procedure based upon the following theorem due to Muriel and Romero [69].

Theorem 5.3.1 (Muriel-Romero) *(a) If $I(x,y,y')$ is a first integral of $y'' = w(x,y,y')$, then the vector field $X = \partial_y$ is a λ-symmetry of the equation for $\lambda = -I_y/I_{y'}$ and $X^{[\lambda,(1)]}I = 0$.*

(b) Conversely, if $X = \partial_y$ is a λ-symmetry of this equation for some function $\lambda(x,y,y')$, then there exists a first integral $I(x,y,y')$ such that $X^{[\lambda,(1)]}I = 0$.

Note that in this case we have

$$X^{[\lambda,(1)]} = \partial_y + \lambda\partial_{y'} \;\text{ and }\; \left[X^{[\lambda,(1)]}, A\right] = \lambda \cdot X^{[\lambda,(1)]}.$$

As an application of this theorem we note that the latter part of the theorem provides a procedure for determining the first integral, given that the ODE admits a λ-symmetry.

Suppose that $X = \partial_y$ be a $\lambda-$ symmetry for some $\lambda(x, y, y') \in \mathcal{C}^\infty(M^{(1)})$. Let $I(x, y, y')$ denote a nontrivial first integral of $y'' = w(x, y, y')$.
Part(b) of the above theorem, implies $X^{[\lambda,(1)]}I(x, y, y') = I_y + \lambda I_{y'} = 0$, since $X^{[\lambda,(1)]} = \partial_y + \lambda\partial_{y'}$. Also $h(x, y, y') = x$ is another first integral because

$$X^{[\lambda,(1)]}h = (\partial_y + \lambda\partial_{y'})x = 0.$$

Therefore any first integral of $X^{[\lambda,(1)]}$ is necessarily of the form

$$J(x, y, y') = G(x, I(x, y, y')). \tag{5.3.14}$$

Consequently we must look for a common first integral of the vector fields $X^{[\lambda,(1)]}$ and A. In other words one needs to solve the equation

$$A(J) = G_x + A(I).G_I = 0. \tag{5.3.15}$$

Since here

$$\left[X^{[\lambda,(1)]}, A\right] = \lambda \cdot X^{[\lambda,(1)]},$$

it is easy to see that

$$X^{[\lambda,(1)]}I = 0 \qquad \text{implies} \qquad X^{[\lambda,(1)]}A(I) = 0.$$

So $A(I)$ functionally depends upon x and the first integral $I(x, y, y')$, i.e., there exists a function $H(x, I)$ such that $A(I) = H(x, I)$. This allows us to interpret (5.3.15) as as first-order PDE, namely

$$G_x + H(x, I)G_I = 0. \tag{5.3.16}$$

If $G(x, I)$ be its solution, then $J = G(x, I(x, y, y'))$ satisfies

$$A(J) = 0. \tag{5.3.17}$$

We illustrate the procedure with the previous example for which we had two λ-symmetries. In either case we note that a particular solution of

$$X^{[\lambda_i,(1)]}I_i = I_{iy} + \lambda_i I_{iy'} = 0 \quad i = 1, 2, \tag{5.3.18}$$

may be taken in the form

$$I_1 = \frac{y'}{y} + \frac{1}{y} + y, \tag{5.3.19}$$

$$I_2 = \frac{y'}{y} - \frac{1}{y} + y. \tag{5.3.20}$$

Next we calculate $A(I_i)$. This gives

$$A(I_1) = H_1(x, I_1) := F(x) + 1 - \frac{1}{2}I_1^2, \tag{5.3.21}$$

$$A(I_2) = H_2(x, I_2) := F(x) - 1 - \frac{1}{2}I_2^2. \tag{5.3.22}$$

Consequently the first integral for the example (5.3.5) is of the form $J_i = G_i(x, I_i)$ with G_i being a solution of the PDE,

$$G_{ix} + (F(x) \pm 1 - \frac{1}{2}I_i^2)G_{iI_i} = 0, \quad i = 1, 2. \tag{5.3.23}$$

Here the upper(lower) sign corresponds to $i = 1(2)$ respectively. We consider the case of $i = 1$. The Lagrange characteristic is given by

$$\frac{dx}{1} = \frac{dI_1}{F(x) + 1 - \frac{I_1^2}{2}},$$

and implies the following Riccati differential equation

$$\frac{dI_1}{dx} + \frac{I_1^2}{2} = F(x) + 1. \tag{5.3.24}$$

Let W_1 be a particular integral of this equation and write

$$I_1 = \frac{1}{u_1} + W_1. \tag{5.3.25}$$

Substitution of this into (5.3.24) gives the equation

$$\frac{1}{u_1^2}\left(-\frac{du_1}{dx} + \frac{1}{2} + u_1 W_1\right) + \left(\frac{dW_1}{dx} + \frac{1}{2}W_1^2\right) = F(x) + 1.$$

Because W_1 is a particular integral, it follows that

$$\frac{du_1}{dx} - u_1 W_1 = \frac{1}{2}. \tag{5.3.26}$$

As this is a first-order equation, its integrating factor is, $g(x) = e^{-\int W_1 dx}$, and its solution can be obtained by two quadratures in the form

$$u_1 = J_1 s_0(x) + s_1(x), \tag{5.3.27}$$

where

$$s_0(x) = \frac{1}{g(x)} = e^{\int W_1 dx}, \quad s_1(x) = \frac{1}{2g(x)}\int g(x)dx, \tag{5.3.28}$$

and J_1 is a constant. Hence from (5.3.25) we have

$$J_1 = \frac{g(x)}{I_1 - W_1} - \frac{1}{2}\int g(x)dx, \tag{5.3.29}$$

with $g(x)$ and W_1 being the solutions of

$$g'(x) + W_1 g(x) = 0, \tag{5.3.30}$$

$$\frac{dW_1}{dx} + \frac{1}{2}W_1^2 = F(x) + 1, \tag{5.3.31}$$

respectively. Thus corresponding to $i = 1$ the original ODE (5.3.5) admits the first integral,

$$J_1 = \frac{g(x)}{\frac{y'}{y} + \frac{1}{y} + y - W_1} - \frac{1}{2}\int g(x)dx. \tag{5.3.32}$$

In a similar manner corresponding to $i = 2$ one has another independent first integral of (5.3.5) given by

$$J_2 = \frac{h(x)}{\frac{y'}{y} - \frac{1}{y} + y - W_2} - \frac{1}{2}\int h(x)dx, \tag{5.3.33}$$

where

$$h'(x) + W_2 h(x) = 0, \tag{5.3.34}$$

$$\frac{dW_2}{dx} + \frac{1}{2}W_2^2 = F(x) - 1. \tag{5.3.35}$$

Equations with one Lie point symmetry

In the case of ODEs possessing one Lie point symmetry we observe that the calculation of λ-symmetries follows a certain pattern and that the first integrals are comparatively easier to evaluate. We summarize our findings in the Table below.

Table-I Representative list of ODEs from the Painlevé-Gambier classification and their λ-symmetries, integrating factors and first integrals.

Painlevé-Gambier ODE	λ-symmetry	Integrating factor	First integral
III. $y'' = 6y^2 + \frac{1}{2}$	$\frac{6y^2}{y'} + \frac{1}{2y'}$	$2y'$	$y'^2 - 4y^3 - y$
VIII. $y'' = 2y^3 + \beta y + \gamma$	$\frac{2y^3+\beta y+\gamma}{y'}$	y'	$\frac{1}{2}y'^2 - (\frac{y^4}{2} + \beta\frac{y^2}{2} + \gamma y)$
XIX. $y'' = \frac{y'^2}{2y} + 4y^2 + 2y$	$\frac{y'}{2y} + \frac{4y^2+2y}{y'}$	$\frac{y'}{y}$	$\frac{y'^2}{2y} - 2y^2 - 2y$
XXX. $y'' = \frac{y'^2}{2y} + 3\frac{y^3}{2} +$ $+4\alpha y^2 + 2\beta y - \frac{\gamma^2}{2y}$	$\frac{y'}{2y} + \frac{3\frac{y^3}{2}+4\alpha y^2+2\beta y-\frac{\gamma^2}{2y}}{y'}$	$\frac{y'}{y}$	$\frac{y'^2}{2y} - \frac{y^3}{2} - 2\alpha y^2 - 2\beta y - \frac{\gamma^2}{2y}$

One notices that in each case, if the ODE is written as $y'' = w(x, y, y')$, then the corresponding λ-symmetry is given by $\lambda(x, y, y') = w/y'$. Furthermore the first integrals of the vector field A associated with the ODE and the vector field $X^{[\lambda,(1)]}$ are identical. This is actually a consequence of the fact that all the equations from the Painlevé-Gambier classification in Table I admit ∂_x as the Lie point symmetry so that the characteristic $Q := \eta - \xi y' = -y'$. Consequently it may be proved that $X = \partial_y$ is a λ-symmetry for

$$\lambda(y, y') = \frac{A(Q)}{Q} = \frac{A(y')}{y'} = \frac{w(y, y')}{y'}.$$

In fact such equations then always admit a first integral, which does not depend upon the independent variable x, i.e., $I(x, y, y') = I(y, y')$, and which is also a first integral of $X^{[\lambda,(1)]}$. A similar feature is also encountered in the case of third-order equations below.

5.3.2 λ-symmetries for some third-order ODEs

In this section we describe the λ-symmetries of certain third-order ODES. One recalls that to determine the λ-symmetries of any third-order ODE, one has to solve the equation (5.3.4). This is in general a nontrivial exercise and one customarily employs an ansatz for λ. We illustrate this with the following example.

Example 5.3.1

$y''' + \frac{3}{y}y'y'' - 3y'' - \frac{3}{y}y'^2 + 2y' = 0$
Here $w = -(\frac{3}{y}y'y'' - 3y'' - \frac{3}{y}y'^2 + 2y')$. We assume that $\lambda = a(x, y, y')y'' + b(x, y, y')$. Insertion of the above expression for λ into the lhs of (5.3.4) reveals that it is a cubic expression in y'', with the coefficient of y''^3 given by $a^3 + 3aa_{y'} + a_{y'y'}$. On the other hand the rhs of (5.3.4) turns out to be quadratic in y'' so that we have

$$a^3 + 3aa_{y'} + a_{y'y'} = 0. \tag{5.3.36}$$

A particular solution of this equation is obviously given by

$$a(x, y, y') = \frac{1}{y'}.$$

With a being thus determined, upon equating the coefficients of y''^2 in (5.3.4) we find that

$$b_{y'y'} + 3\frac{b_{y'}}{y'} = 0. \tag{5.3.37}$$

This admits the following solution

$$b(x, y, y') = \frac{\xi(x, y)}{y'^2} + \eta(x, y).$$

However, equation of the coefficients of the next lower power, i.e., y'', reveals that $\xi(x, y) = 0$ so that $b(x, y, y') = \eta(x, y)$. Finally from the coefficient of the term independent of y'' we find η to be a solution of the following equation

$$A(\eta) + \eta(\eta + 1) = \eta_x + y'\eta_y + \eta(\eta + 1) = 0. \tag{5.3.38}$$

Clearly η must be independent of y so that finally

$$\lambda(x, y, y', y'') = \frac{y''}{y'} + \eta(x),$$

where $\eta(x)$ is the solution of the equation

$$\eta_x + \eta(\eta + 1) = 0.$$

It is obvious that particular solutions of this are given by $\eta = 0, -1$, respectively, so that the given equation admits two λ symmetries:

$$\lambda_1 = \frac{y''}{y'},$$

$$\lambda_2 = \frac{y''}{y'} - 1.$$

It was shown in [8] that this equation admits a first integral of the form

$$I(x, y, y', y'') = (y'^2 + yy'' - yy')e^{-2x}.$$

This first integral actually corresponds to the choice $\lambda = \lambda_2$ as we illustrate below. Note that, if a first integral is known, then the problem of finding the λ-symmetries reduces to a determination of the solutions of the PDE

$$(A(\lambda) + \lambda^2)I_{y''} + \lambda I_{y'} + I_y = 0, \tag{5.3.39}$$

where the vector field is given by $A = \partial_x + y'\partial_y + y''\partial_{y'} + w\partial_{y''}$ If we assume a linear ansatz in y'' for λ of the form $\lambda = a(x, y, y')y'' + b(x, y, y')$, we find after substituting this into (5.3.39) and equating the coefficients of the different powers of y'' that

$$a_{y'} + a^2 = 0, \tag{5.3.40}$$

$$(a_x + y'a_y) + 3a(1 - \frac{y'}{y}) + b_{y'} + 2ab + \frac{a}{y}(2y' - y) + \frac{1}{y} = 0, \tag{5.3.41}$$

$$a(\frac{3y'^2}{y} - 2y') + (b_x + y'b_y) + b^2 + b(\frac{2y' - y}{y}) - \frac{y'}{y} = 0. \tag{5.3.42}$$

Clearly from the first equation of the above set we have $a = \frac{1}{y'+c(x,y)}$. Substitution of this into the remaining equations and equation of the different powers of y' leads to the following set of equations:

$$b_{y'} = 0, \tag{5.3.43}$$
$$c_y - (2 + \frac{c}{y} + 2b) = 0, \tag{5.3.44}$$
$$c_x - c(2 + \frac{c}{y} + 2b) = 0, \tag{5.3.45}$$
$$b_y + \frac{2}{y}b + \frac{2}{y} = 0, \tag{5.3.46}$$
$$-2 + b_x + cb_y + (b^2 - b) + \frac{c}{y}(2b - 1) = 0, \tag{5.3.47}$$
$$cb_x + c(b^2 - b) = 0. \tag{5.3.48}$$

One may easily check that $b = -1$ and $c = 0$ satisfy the above equations and hence

$$\lambda = \frac{y''}{y'} - 1.$$

On the other hand for $\lambda_1 = y''/y'$ it may be verified that a particular solution of

$$X^{[\lambda,(2)]}I := I_y + \lambda_1 I_{y'} + (A(\lambda_1) + \lambda_1^2)I_{y''} = 0 \tag{5.3.49}$$

is given by

$$I(y, y', y'') = yy'' + y^2 - 3yy' + y'^2. \tag{5.3.50}$$

Furthermore one may easily check that it is also a first integral of the vector field $A = \partial_x + y'\partial_y + y''\partial_{y'} + w\partial_{y''}$ associated with the ODE, i.e., $A(I) = 0$. In other words a solution $I(y, y', y'')$ of (5.3.49) is itself a first integral of the ODE under consideration. This is a consequence of the fact that the ODE admits a translational symmetry, i.e., ∂_x is a Lie point symmetry and $A(\lambda_1) + \lambda_1^2 = w/y'$. Next we present an example from [27] in which use was made of the generalised Sundman transformation to linearize a third-order differential equation [28].

Example 5.3.2

$y''' = \left(\frac{y'y''}{y} + \frac{3}{2}\frac{y'^3}{y^2}\right)$
This equation admits a similar feature to the previous example because upon solving (3.4) with the ansatz $\lambda = a(x, y, y')y'' + b(x, y, y')$ one finds that $\lambda = y''/y'$. To find the first integral we solve (3.39) to get the following particular solutions:

$$I_1 = \frac{y''}{y^2} + \frac{y'^2}{2y^3}, \tag{5.3.51}$$
$$I_2 = y^2y'' - \frac{3}{2}yy'^2. \tag{5.3.52}$$

In either case it is observed that $A(I_k) = 0, \quad k = 1, 2$, so that these are also first integrals of the original ODE. Our final example in this section is taken from [8] and exhibits an interesting feature similar in some respect with the previous result for a second-order ODE.

Example 5.3.3

$y''' = \left(\frac{y''^2}{y'} + \frac{y'y''}{y}\right).$

Here on the assumption that λ is linear in y'', i.e., $\lambda = a(x, y, y')y'' + b(x, y, y')$, one finds after a tedious but straightforward calculation that $\lambda = y''/y'$.

For a third-order equation the PDEs of the system given by (5.2.3) are

$$-A[S_2] = w_{y''}S_2 + S_1,$$

$$-A[S_1] = w_{y'}S_2 + S_0,$$

$$-A[S_0] = w_y S_2.$$

Here S_2 plays the role of the integrating factor since $S_2 = I_{y''}$. If one knows the integrating factor S_2, then defining $\lambda = -S_1/S_2$ we find that

$$\lambda = \left(\frac{A[S_2]}{S_2} + w_{y''}\right). \tag{5.3.53}$$

The integrating factor S_2 can be found from the adjoint symmetry equation following the elimination of S_1 and S_0 from the above set of PDEs. Indeed for the present example one finds that $S_2 = 1/(yy')$ and it follows once again from (5.3.53) that $\lambda = y''/y'$. However, we wish to point out that, unlike the case of second-order ODEs, this formula for λ is not true in general.

In this chapter we have introduced the notion of λ-symmetries which may be considered as a sort of generalisation of the Lie point symmetries and have used them to derive first integrals of second and third-order equations.

In the following chapter we shall pursue further generalisations of the notion of the symmetries of ODEs by considering nonlocal transformations which are also known as Sundman transformations.

Chapter 6

Generalized Sundman transformation and Symmetry

6.1 Introduction

In this chapter we introduce the notion of a generalized Sundman transformation and employ it to obtain certain first integrals of autonomous second-order ordinary differential equations belonging to the Painlevé-Gambier classification [84, 85] and the list of Painlevé-Gambier equation contained in [44]. In particular, this method yields systematically both known and unknown first integrals of a large number of the Painlevé-Gambier equations.
We begin by considering an nth-order ordinary differential equation of the form

$$x^{(n)} = w(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}) \tag{6.1.1}$$

where $x = x(t)$ and $x^{(k)} = d^k x/dt^k$, Formally we define a generalized Sundman transformation for (6.1.1) as follows.

Definition 6.1.1 (Sundman transformation) *A coordinate transformation of the form*

$$X(T) = F(t, x), \quad dT = G(t, x)dt, \quad \frac{\partial F}{\partial x} \neq 0, \quad G \neq 0 \tag{6.1.2}$$

is said to be a generalized Sundman transformation of equation (6.1.1) if differentiable functions F and G are determined such that (6.1.1) is transformed to the autonomous equation

$$X^{(n)} = w_0(X, X', \ldots, X^{(n-1)}), \tag{6.1.3}$$

where $X' = dX/dT$ etc.

This notion of the generalized Sundman transformation, as a kind of nonlocal extension of invertible point transformation was made by Duarte *et al.* Its nonlocal character is apparent from the fact that $T = \int G(t, x(t))\, dt$. If (6.1.3) happens to be a linear ODE,

then we say that the original ODE, (6.1.1), is linearizable. In the event $w_0 = 0$ we say that (6.1.1) has been mapped to the free particle equation.
Closely related to the concept of a generalized Sundman transformation is the notion of an associated Sundman symmetry. This is similar in spirit to the existence of a Lie symmetry under point transformations.
Suppose we have a generalized Sundman transformation (GST)

$$X(T) = F(\tilde{t}, \tilde{x}), \;\; dT = G(\tilde{t}, \tilde{x})d\tilde{t}$$

which maps the equation

$$\tilde{x}^{(n)} = w(\tilde{t}, \tilde{x}, \dot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}) \longmapsto X^{(n)} = w_0(X, X', \ldots X^{(n-1)}).$$

If there exists a transformation of the differentiable functions $F(\tilde{t}, \tilde{x})$ and $G(\tilde{t}, \tilde{x})$, considered as functions of $F(t, x)$ and $G(t, x)$, such that our original differential equation (6.1.1) remains invariant under the transformation, then the transformation defines a Sundman symmetry. Formally it is defined as follows.

Definition 6.1.2 (Sundman symmetry) *A Sundman symmetry for equation (6.1.1) is a transformation of the form*

$$F(\tilde{t}, \tilde{x}) = M(F(t, x), G(x, t)), \qquad G(\tilde{t}, \tilde{x})d\tilde{t} = N(F(t, x), G(t, x))dt, \tag{6.1.4}$$

where M and N are some differentiable functions such that the transformation keeps (6.1.1) invariant. In other words (6.1.1) is transformed to

$$\tilde{x}^{(n)} = w(t; \tilde{x}, \dot{\tilde{x}}, \ddot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}). \tag{6.1.5}$$

If $M(F, G) = F$ and $N(F, G) = G$, then of course, the symmetry is trivial. The set of conditions on the differentiable functions F and G when the differential equation (6.1.1) is mapped to the autonomous differential equation (6.1.3) are referred to as *the Sundman determining equations.*

A Sundman symmetry (6.1.4) is obtained by choosing M and N in such a way that the Sundman determining equations remain invariant. If

$$X = F(\tilde{t}, \tilde{x}), \quad dT = G(\tilde{t}, \tilde{x})d\tilde{t}$$

transforms (6.1.5) to (6.1.3) and

$$X = M(F(t, x), G(t, x)), \quad dT = N(F(t, x), G(t, x))dt$$

also transforms (6.1.1) to (6.1.3), then the composition of these two GSTs leads to the Sundman symmetry (6.1.4) of (6.1.1).

6.2 GST for the Jacobi equation

We begin this section by considering the well-known Jacobi equation. The reason for this is that many of the second-order equations of the Painlevé-Gambier classification may then be regarded as special cases of this rather general equation, as we illustrate below.

The Jacobi equation [46, 79]

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t,x) = 0, \tag{6.2.1}$$

may be transformed to $X'' = 0$ by the transformation (6.1.4) provided the coefficients involved in the transformation satisfy the Sundman determining equations, which are given by the following relations:

$$\frac{1}{2}\phi_x(F,G;t,x) = \frac{F_{xx}}{F_x} - \frac{G_x}{G}, \tag{6.2.2}$$

$$\phi_t(F,G;t,x) = \frac{2F_{xt}}{F_x} - \frac{F_t}{F_x}\frac{G_x}{G} - \frac{G_t}{G}, \tag{6.2.3}$$

$$B(F,G;t,x) = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x}. \tag{6.2.4}$$

Further it admits a Sundman symmetry of the form (6.1.4) if and only if M and N are given by

$$M(F,G) = M(F(t,x)) \text{ and } N(F,G) = G(t,x)\psi(F). \tag{6.2.5}$$

The Sundman symmetry of (6.2.1) is of the form

$$F(\tilde{x},\tilde{t}) = M(F(x,t)), \tag{6.2.6}$$

$$G(\tilde{t},\tilde{x}) = G(t,x)\frac{dM(F(t,x))}{dF}dt, \tag{6.2.7}$$

with no further condition on the differentiable function M. This follows from the following observation. Suppose for the sake of notational convenience we denote

$$F(\tilde{t},\tilde{x}) = \hat{F} \quad \text{and} \quad G(\tilde{t},\tilde{x}) = \hat{G}.$$

The invariance of the Sundman determining equations requires each expression occurring in (6.2.2)-(6.2.4) to be invariant. From (6.2.4) we observe, making use of (6.2.5), that

$$\frac{\hat{F}_{tt}}{\hat{F}_x} - \frac{\hat{G}_t}{\hat{G}}\frac{\hat{F}_t}{\hat{F}_x} = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + \left(\frac{M''(F)}{M'(F)} - \frac{\psi'(F)}{\psi(F)}\right)\frac{F_t^2}{F_x}.$$

The left hand side is clearly an invariant provided

$$\left(\frac{M''(F)}{M'(F)} - \frac{\psi'(F)}{\psi(F)}\right) = 0 \tag{6.2.8}$$

which in turn implies

$$\psi(F) = \frac{dM}{dF}, \tag{6.2.9}$$

where we have chosen the constant of integration to be unity. It may be verified that (6.2.2) and (6.2.3) are also invariant under (6.2.5) provided condition (6.2.9) holds, i.e.,

$$\frac{\hat{F}_{xx}}{\hat{F}_x} - \frac{\hat{G}_x}{\hat{G}} = \frac{F_{xx}}{F_x} - \frac{G_x}{G}$$

$$\frac{2\hat{F}_{xt}}{\hat{F}_x} - \frac{\hat{F}_t}{\hat{F}_x}\frac{\hat{G}_x}{\hat{G}} - \frac{\hat{G}_t}{\hat{G}} = \frac{2F_{xt}}{F_x} - \frac{F_t}{F_x}\frac{G_x}{G} - \frac{G_t}{G}.$$

Case I: When $\phi_t = 0$ and $B(x,t) = 0$ we are lead to the following special case of the Jacobi equation

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 = 0.$$

The determination of the functions F and G will now be explained.

Step I: This involves finding G in term of F

Since $B(t,x) = 0$, from (6.2.4) we can set

$$G = a(x)F_t, \tag{6.2.10}$$

where a is an arbitrary function of x.

Step II: We express F and its derivatives in terms of the coefficient $a(x)$

Since ϕ_t=0, from (6.2.3) and (6.2.10) we have

$$\frac{F_{xt}}{F_x} - \frac{a_x(x)}{a(x)}\frac{F_t}{F_x} - \frac{F_{tt}}{F_t} = 0,$$

i.e.,

$$\frac{\partial}{\partial t}\Big(\frac{F_x}{F_t}\Big) = \frac{a_x(x)}{a(x)}.$$

Integrating this with respect to t we find

$$\frac{F_x}{F_t} = \frac{a_x}{a}t + b(x), \tag{6.2.11}$$

where b is an arbitrary function of x. Finally from (6.2.2) we get

$$\frac{F_x}{G} = c(t)e^{\frac{\phi}{2}} = c(t)K(x), \tag{6.2.12}$$

where c is an arbitrary function of t and

$$e^{\phi/2} = K(x). \tag{6.2.13}$$

Since $\phi_t = 0$, the r.h.s. is independent of t.
Step III: Determination the of coefficients
Using (6.2.10) and (6.2.11) one can show that (6.2.12) can be reduced to

$$\frac{a_x}{a^2}t + \frac{b(x)}{a} = c(t)K(x). \tag{6.2.14}$$

There are two possibilities (a) $c(t) = c_0$ (constant), in this case a is also constant; (b) $c(t) = t$. The latter case is more interesting. Equating the coefficient of t from (6.2.14) we get

$$\frac{a_x}{a^2} = K(x), \tag{6.2.15}$$

which implies

$$a(x) = -\frac{1}{K_1(x) + f} \tag{6.2.16}$$

where

$$K_1(x) = \int K(x)dx \tag{6.2.17}$$

and f is an arbitrary constant. Assuming $f = 0$ one finds

$$a(x) = -\frac{1}{K_1(x)}. \tag{6.2.18}$$

Step IV: Finding F and G
Using (6.2.18) in (6.2.11) and with $b(x) = 0$ we find that

$$\frac{F_x}{F_t} = -\frac{K(x)}{K_1(x)}t. \tag{6.2.19}$$

Using the method of characteristics the general solution of $F(t,x)$ can be expressed in the form

$$F(t,x) = J\left(\frac{K_1(x)}{t}\right), \tag{6.2.20}$$

where $J(\lambda)$ is any arbitrary function of the characteristic coordinate $\lambda = K_1(x)/t$. Hence from (6.2.10) using (6.2.18) with F given by (6.2.20), we easily find that

$$G(t,x) = \frac{1}{t^2}J'(\lambda). \tag{6.2.21}$$

It is interesting to note that, when $J(\lambda) = \lambda$, the nonlocal character of the transformation vanishes for we have

$$X = F(t,x) = \frac{K_1(x)}{t} \text{ and } G(t,x) = \frac{1}{t^2} \text{ so that } dT = \frac{1}{t^2}dt \text{ leading to } T = -\frac{1}{t}. \tag{6.2.22}$$

Step V: Finding first integrals from F and G
As the standard first integrals of the linear ODE, $X'' = 0$, are given by

$$I_1 = X' = \frac{dX}{dT} \text{ and } I_2 = X - TX'$$

respectively, as a result of the GST they make the following forms:

$$I_1 = \frac{F_x}{G}\dot{x} + \frac{F_t}{G} = tK(x)\dot{x} - K_1(x) \tag{6.2.23}$$

and

$$I_2 = X - TX' = F(t,x) - (tK(x)\dot{x} - K_1(x)) \int G(t,x)dt. \tag{6.2.24}$$

In particular, when F and G are given by (6.2.22), I_2 assumes the following simple form

$$I_2 = \dot{x}K(x). \tag{6.2.25}$$

It is important to note that in the following examples we repeatedly use this expression in order to compare the results of our calculations with the known time-independent first integrals given in Ince's book [44].
Secondly, in view of the fact that we have at our disposal two first integrals, it is a straightforward matter to obtain the general solution by eliminating $\dot{x}$ from these expression.

6.2.1 Examples from the Painlevé-Gambier class of equations

Apart from the six Painlevé equations, the remaining 44 second-order ODEs of the Painlevé-Gambier classification scheme possess solutions that can be expressed in terms of elementary functions. These solutions fall into two classes – (a) solutions which are rational in the independent variable and (b) solutions which are expressed in terms of the classical special functions. Since the latter are the solutions of linear equations, the second class of solutions is referred to as the 'linearizable' case, obviously these exist only for special values of the parameters.

In this subsection we focus on equations which do not belong to the six Painlevé transcendents. It may be recalled that Painlevé, Gambier and their pupils found *fifty* second-order ODEs of canonical form, the solutions of which do not have any movable critical singularities, i.e., they possess the Painlevé property. Using the generalized Sundman transformations we have obtained certain new first integrals for the equations XI, XVII, XXXVII, XXXXI and XXXXIII of the Painlevé-Gambier classification, as given in Ince's classic text [44]. The results are presented below.

Example 6.2.1 *(Painlevé-Gambier equation XI)*

The first system we examine is equation number XI of the Painlevé-Gambier classification:

$$\ddot{x} - \frac{1}{x}\dot{x}^2 = 0 \tag{6.2.26}$$

Comparison with the Jacobi equation (6.2.1) reveals that

$$\frac{1}{2}\phi_x = -\frac{1}{x}. \tag{6.2.27}$$

Hence from (6.2.13) we have

$$K(x) = e^{\frac{\phi}{2}} = \frac{1}{x} \tag{6.2.28}$$

and from (6.2.17)

$$K_1(x) = \ln x \tag{6.2.29}$$

Therefore making use of (6.2.22) we find that

$$F = \left(\frac{\ln x}{t}\right)^2 \quad \text{and} \quad G(t,x) = \frac{2\ln x}{t^3} \tag{6.2.30}$$

while from (6.2.23) and (6.2.24) the first integrals for this equation are

$$I_1 = \frac{t}{x}\dot{x} - \ln x \tag{6.2.31}$$

and

$$I_2 = \frac{\dot{x}}{x}. \tag{6.2.32}$$

Notice that whereas the time independent first integral I_2 is mentioned in [44] the remaining first integral I_1 is time dependent and is not stated therein. This is a trivial example in the sense that one could have deduced these results even otherwise. Moreover $G(t,x)$ being a function of t only actually produces a point transformation. But the Sundman symmetry of this simple example is quite interesting.

The Sundman symmetry

To deduce the Sundman symmetry for this equation, it is convenient to assume that, $J(\lambda) = \lambda^2$ for the rest of this subsection so that from (6.2.20) we have

$$F(t,x) = \left(\frac{K_1(x)}{t}\right)^2 = \left(\frac{\ln x}{t}\right)^2. \tag{6.2.33}$$

Since the Sundman symmetry of (6.2.26) is of the form (6.2.6), we assume that

$$\hat{F} = F(\tilde{t},\tilde{x}) = M(F(t,x)).$$

Consequently with F given as in (6.2.33) one finds that

$$\tilde{x} = \exp\left(\tilde{t}\sqrt{M(F)}\right). \tag{6.2.34}$$

On the other hand from (6.2.7), using (6.2.21) to calculate G which now is given by $G(t,x) = 2\ln x/t^3$, we have

$$\hat{G}d\tilde{t} = G\frac{dM(F)}{dF}dt \quad \Rightarrow \quad \frac{\ln\tilde{x}}{\tilde{t}^3}d\tilde{t} = \frac{\ln x}{t^3}\frac{dM}{dF}dt.$$

Upon using (6.2.34) to eliminate $\tilde{x}$ from the l.h.s of the above expression, we obtain the following transformation for the time variable:

$$\tilde{t} = -\left[c + \int \frac{\ln x}{t^3\sqrt{M(F)}}\frac{dM(F)}{dF}dt\right]^{-1}. \tag{6.2.35}$$

Here c is a constant of integration. Substituting this expression into (6.2.34) we get the transformation for the spatial variable, *viz*

$$\tilde{x} = \exp\left(-\frac{\sqrt{M(F)}}{c + \int \frac{\ln x}{t^3\sqrt{M(F)}}\frac{dM(F)}{dF}dt}\right). \tag{6.2.36}$$

Here $M(F)$ is an arbitrary function of F and c is a constant of integration. Equations (6.2.35) and (6.2.36) constitute a Sundman symmetry for the Painlevé-Gambier XI equation.

The above procedure for finding Sundman symmetries may easily be applied to some of the other equations of the Painlevé-Gambier classification. The results for this and some of the other equations of the Painlevé-Gambier classification are summarized in the following Table-1

Table-1
Summary of results of Sundman transformations and symmetries

Painlevé-Gambier	Equation No.	Sundman Transformation	Sundman Symmetry
XI.	$\ddot{x} - \frac{1}{x}\dot{x}^2 = 0$	$F(x,t) = \left(\frac{\log x}{t}\right)^2$ $G(x,t) = \frac{2\ln x}{t^3}$	$\tilde{t} = -\left[c + \int \frac{\log x}{t^3\sqrt{M(F)}} \frac{dM(F)}{dF} dt\right]^{-1}$ $\tilde{x} = \exp\left(-\frac{\sqrt{M(F)}}{c+\int \frac{\log x}{t^3\sqrt{M(F)}} \frac{dM(F)}{dF} dt}\right)$
XVII.	$\ddot{x} - \frac{m-1}{mx}\dot{x}^2 = 0$	$F(x,t) = \left(\frac{mx^{1/m}}{t}\right)^2$ $G(x,t) = 2\frac{mx^{1/m}}{t^3}$	$\tilde{t} = c - \left[m\int \frac{x^{1/m}}{t^3\sqrt{M}} \frac{dM}{dF} dt\right]^{-1}$ $\tilde{x} = \left(\frac{\tilde{t}\sqrt{M(F)}}{m}\right)^m$
XXXVII.	$\ddot{x} - \left\{\frac{1}{2x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0$	$F(x,t) = \left(\frac{1}{t}\log\frac{x^{1/2}-1}{x^{1/2}+1}\right)^2$ $G(x,t) = \frac{2}{t^3}\log\frac{x^{1/2}-1}{x^{1/2}+1}$	$\tilde{t} = c - \left[\int \log\left(\frac{x^{1/2}-1}{x^{1/2}+1}\right) \frac{1}{\sqrt{M}} \frac{dM}{dF} dt\right]^{-1}$ $\tilde{x} = \left(\frac{1+e^{\tilde{t}}\sqrt{M(F)}}{1-e^{\tilde{t}}\sqrt{M(F)}}\right)^2$
XLI.	$\ddot{x} - \frac{2}{3}\left\{\frac{1}{x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0$	$F(x,t) = \frac{K_1^2(x)}{t^2}$ $G(x,t) = \frac{2K_1(x)}{t^3}$ $K_1(x) = -3(-x)^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x)$	
XLIII.	$\ddot{x} - \frac{3}{4}\left\{\frac{1}{x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0$	$F(x,t) = \frac{K_1^2(x)}{t^2}$ $G(x,t) = \frac{2K_1(x)}{t^3}$ $K_1(x) = -4(-x)^{1/4}\ {}_2F_1(3/4, 1/4; 5/4; x)$	

In the above table ${}_2F_1(a, b, c; x)$ is the hypergeometric series which converges for $-1 < x < 1$. For Equations XLI and XLIII it is difficult to obtain explicit expressions for the corresponding symmetries and we do not display them here.

In Table 2 we summarize the results for the time-independent and time-dependent first integrals of the above equations.

Table-2
Summary of First Integrals

	Painlevé-Gambier Equation	Time-dependent F.I	Time-independent F.I
XVII.	$\ddot{x} - \frac{m-1}{mx}\dot{x}^2 = 0$	$tx^{\frac{1-m}{m}}\dot{x} - mx^{\frac{1}{m}}$	$x^{\frac{1-m}{m}}\dot{x}$
XXXVII.	$\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t}{x^{1/2}(x-1)}\dot{x} - \log\frac{x^{1/2}-1}{x^{1/2}+1}$	$-\frac{1}{x^{1/2}(x-1)}\dot{x}$
XLI.	$\ddot{x} - \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t\dot{x}}{x^{\frac{2}{3}}(x-1)^{\frac{2}{3}}} + 3(-x)^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x)$	$\frac{\dot{x}}{x^{2/3}(x-1)^{2/3}}$
XLIII.	$\ddot{x} - \frac{3}{4}\left(\frac{1}{x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t\dot{x}}{x^{\frac{3}{4}}(x-1)^{\frac{3}{4}}} + 4(-x)^{1/4}\ {}_2F_1(3/4, 1/4; 5/4; x)$	$\frac{\dot{x}}{x^{3/4}(x-1)^{3/4}}$

Case B: When $\phi_t = 0 = B_t$
The prototype equation for this case has the generic form

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 + B(x) = 0. \tag{6.2.37}$$

Once again there are a number of equations of the Painlevé-Gambier classification which belong to this category.

6.3 Generalized Sundman Transformation of ODE for the mapping to $X'' + a_0(X) = 0$

In this case we attempt to construct a generalized Sundman transformation (6.1.2) (GST) such that (6.2.37) is mapped to the following equation

$$X'' + a_0(X) = 0, \tag{6.3.1}$$

where $X' = dX/dT$. The exact form of $a_0(X)$ will be specified below. This is possible provided the following conditions hold good (i.e. the Sundman determining equations) for the coefficients of (6.2.37).

$$\frac{1}{2}\phi_x = \frac{F_{xx}}{F_x} - \frac{G_x}{G} \tag{6.3.2}$$

$$0 = 2\frac{F_{xt}}{F_x} - \frac{G_x}{G}\frac{F_t}{F_x} - \frac{G_t}{G} \tag{6.3.3}$$

$$B(x) = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + a_0(F)\frac{G^2}{F_x}. \tag{6.3.4}$$

From (6.3.2) we have

$$\ln F_x - \ln G = \int \frac{1}{2}\phi_x dx - \ln b(t).$$

Here $b(t)$ is an arbitrary constant of integration. It follows that

$$G(t,x) = b(t)e^{-\phi/2}F_x. \tag{6.3.5}$$

Substituting G from (6.3.5) to (6.3.4) we have

$$\frac{F_{tt}}{F_x} - \frac{F_{xt}F_t}{F_x^2} - \frac{\dot{b}(t)}{b(t)}\frac{F_t}{F_x} + a_0(F)b(t)^2e^{-\phi}F_x = B(x). \tag{6.3.6}$$

If we set $b(t) = \beta$, i.e., a constant independent of t and assume

$$\frac{\partial}{\partial t}\left(\frac{F_t}{F_x}\right) = 0, \tag{6.3.7}$$

then (6.3.6) implies

$$a_0(F)\beta^2e^{-\phi}F_x = B(x). \tag{6.3.8}$$

Instead of trying to determine the form of F first, it is more convenient to stipulate $a_0(F)$ and see whether we can satisfy the remaining equations with such a choice of $a_0(F)$. To this end we suppose

$$a_0(F) = \pm F. \tag{6.3.9}$$

Then (6.3.8) yields

$$F^2 = \pm\frac{2}{\beta^2}\int B(x)e^{\phi}dx. \tag{6.3.10}$$

Thus F is a function of x only and as a result it is obvious that (6.3.7) is trivially satisfied. It remains to verify whether such an expression for F is consistent with (6.3.3). Since $b(t) = \beta$ is a constant, we have from (6.3.5),

$$G(t,x) = \beta e^{-\phi/2}F_x = \frac{B(x)e^{\phi/2}}{\left(\pm 2\int B(x)e^{\phi}dx\right)^{1/2}}, \tag{6.3.11}$$

which is clearly independent of t and hence $G_t = 0$. Consequently, since F and G are only functions of x, it follows that (6.3.3) is clearly satisfied. In summary we therefore have the following form of the GST mapping (6.2.37) to the equation $X'' \pm X = 0$, *viz*

$$X = F(x) = \left(\pm\frac{2}{\beta^2}\int B(x)e^{\phi(x)}dx\right)^{1/2}, \quad dT = \frac{B(x)e^{\phi/2}}{\left(\pm 2\int B(x)e^{\phi}dx\right)^{1/2}}dt. \tag{6.3.12}$$

The latter is obviously a nonlocal transformation.

The Sundman symmetry

The Sundman symmetry associated with (6.2.37) is not difficult to deduce. As above, for notational convenience we denote

$$\hat{F} = F(\tilde{t}, \tilde{x}) \quad \text{and} \quad \hat{G} = G(\tilde{t}, \tilde{x}).$$

To ensure invariance of the Sundman determining equations, namely (6.3.2)-(6.3.4), we assume

$$\hat{F} = M(F) \;\; \text{and} \;\; \hat{G} = G(t, x)\psi(F). \tag{6.3.13}$$

The functional forms of M and ψ are determined by demanding invariance of the Sundman determining equations. Invariance of (6.3.2) leads to

$$\psi(F) = K\frac{dM(F)}{dF},$$

where K is a constant of integration, which may be set to unity, so that

$$\psi(F) = M'(F). \tag{6.3.14}$$

Invariance of (6.3.4) then leads to the equation

$$\frac{dM}{dF} = \frac{a_0(F)}{a_0(M)}$$

whence it follows, with $a_0(F) = \pm F$, that

$$M = \pm\sqrt{F^2 + c}, \tag{6.3.15}$$

where c is a constant of integration. Note that, if $c = 0$, then we get a trivial symmetry. The functional form of ψ is therefore given by

$$\psi(F) = \pm\frac{F}{\sqrt{F^2 + c}}. \tag{6.3.16}$$

With M and ψ given by (6.3.15) and (6.3.16) respectively, one can easily verify that the final Sundman determining equation, namely (6.3.3), is identically satisfied. Thus in summary we have the following Sundman symmetry for (6.2.37)

$$F(\tilde{t}, \tilde{x}) = \pm\sqrt{F^2(t, x) + c} \;\; \text{and} \quad G(\tilde{t}, \tilde{x})d\tilde{t} = \pm G(t, x)\frac{F}{\sqrt{F^2 + c}}dt. \tag{6.3.17}$$

In the following we consider only the case in which the GST maps equations of the Painlevé-Gambier classification belonging to the class of (6.2.37) to a harmonic oscillator equation

$$X'' + X = 0. \tag{6.3.18}$$

Note that a first integral for (6.3.18) is obviously

$$X'^2 + X^2 = I_1. \tag{6.3.19}$$

Example 6.3.1 *(Painlevé-Gambier equation XXI)*

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0. \tag{6.3.20}$$

Here $\frac{1}{2}\phi_x = -\frac{3}{4x}$ implying $\phi = \ln x^{-3/2}$ and $B(x) = -3x^2$. As a result from (6.3.12) taking the positive square root we find $F(x) = \frac{2i}{\beta}x^{\frac{3}{4}}$ and it turns out that $G = \frac{3i}{2}\sqrt{x}$. Hence the Sundman transformation has the explicit form

$$X = F(x) = \frac{2i}{\beta}x^{\frac{3}{4}}, \;\; dT = \frac{3i}{2}\sqrt{x}\; dt. \tag{6.3.21}$$

When the first integral (6.3.19) is evaluated in terms of the preceding transformation, it reproduces the result in [44].
Table 3 contains a summary of the Sundman symmetry for some of the Painléve-Gambier equations falling under Case-B

Table-3
Summary of Sundman Symmetry

	Painlevé-Gambier Equation	Sundman Symmetry
XVIII.	$\ddot{x} - \frac{1}{2x}\dot{x}^2 - 4x^2 = 0$	$\tilde{x} = \frac{1}{2i}\sqrt{c\beta^2 - 4x^2}$ $\tilde{t} = A + \int \frac{(2ix)^{3/2}}{(c\beta^2 - 4x^2)^{3/4}}dt$
XXI.	$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0$	$\tilde{x} = (\frac{1}{2i})^{4/3}(c\beta^2 - 4x^{3/2})^{2/3}$ $\tilde{t} = A + \int \frac{(2i)^{5/3}x^{5/4}}{(c\beta^2 - 4x^{3/2})^{5/6}}dt$
XXII.	$\ddot{x} - \frac{3}{4x}\dot{x}^2 + 1 = 0$	$\tilde{x} = \frac{16}{(c\beta^2 - 4x^{-1/2})^2}$ $\tilde{t} = A + \int \frac{8ix^{-3/4}}{(c\beta^2 - 4x^{-1/2})^{3/2}}dt$
XIX.	$\ddot{x} - \frac{1}{2x}\dot{x}^2 - (4x^2 + 2x) = 0$	$\tilde{x} = \frac{-1+\sqrt{4x^2+4x+1-c\beta^2}}{2}$ $\tilde{t} = A + \int \frac{\sqrt{2x}(2x+1)}{\sqrt{4x^2+4x+1-c\beta^2}\sqrt{\sqrt{4x^2+4x+1-c\beta^2}-1}}dt$

6.4 Parametric Solutions

As remarked earlier, when we have two first integrals for a second order ODE, then its general solution may be obtained simply by eliminating the first derivatives from the two first integrals. However, the problem of finding a sufficient number of first integrals is itself a non trivial exercise. In most instances, one is lucky if there exists even a single first integral. In such cases a parametric solution of the ODE can often be constructed by integrating the first integral in terms of a parameter. using the technique first developed by euler *et. al.* [27, 28] we shall now present parametric solutions of some Painléve-Gambier equations.
To explain the procedure let us consider a first-order equation of the form

$$F\left(x(t), \frac{dx}{dt}\right) = 0. \tag{6.4.1}$$

Let $x(t) = f(\tau)$, $\frac{dx}{dt} = g(\tau)$ and $\tau = \tau(t)$ where f and g satisfy the relation $F(f(\tau), g(\tau)) = 0$ with τ being a parameter. Since

$$\frac{dx}{dt} = \frac{df}{d\tau}\frac{d\tau}{dt}$$

so

$$g(\tau) = \frac{df}{d\tau}\frac{d\tau}{dt}$$

, i.e.,

$$\int dt = \int \frac{df}{d\tau}\frac{1}{g(\tau)}d\tau + C, \tag{6.4.2}$$

where C is an integrating constant. The general solution (parametric) of (6.4.1) is then given by

$$x(\tau) = f(\tau), \tag{6.4.3}$$

$$t(\tau) = \int \frac{df}{d\tau}\frac{1}{g(\tau)}d\tau + C, \tag{6.4.4}$$

$$F(f(\tau), g(\tau)) = 0. \tag{6.4.5}$$

Using this method we can integrate (6.3.19) with respect to the parameter τ to obtain the general solution of (6.3.18) in the form

$$X(\tau) = \sqrt{I_1 - \tau^2} \tag{6.4.6}$$

$$T(\tau) = C_1 - \arcsin\left(\frac{\tau}{\sqrt{I_1}}\right), \tag{6.4.7}$$

with I_1 and C_1 being the arbitrary constants of integration.

The general solution of (6.3.20) is then obtained by using the transformation (6.3.21) together with the parametric solutions (6.4.6) and (6.4.7) and is given by

$$x(\tau) = \left(\frac{\beta}{2i}\right)^{4/3}(I_1 - \tau^2)^{2/3}, \tag{6.4.8}$$

$$t(\tau) = -\frac{4}{3\beta}\left(\frac{\beta}{2i}\right)^{1/3}\int \frac{d\tau}{(I_1-\tau^2)^{5/6}} + C_2. \tag{6.4.9}$$

where I_1 and C_2 are arbitrary constants.
In the Table 4 we present the parametric solutions for some of the other equations of the Painlevé-Gambier classification scheme, obtained by using the above method.

Table-4
Summary of parametric solutions

	Painlevé-Gambier Eqn.	Sundman Transformation	Parametric solution
XIX.	$\ddot{x} - \frac{1}{2x}\dot{x}^2 - (4x^2+2x) = 0$	$F(x,t) = \frac{2i}{\beta}(x^2+x)^{1/2}$ $dT = \frac{\beta x^{1/2}(2x+1)}{2\sqrt{x^2+x}}dt$	$x(\tau) = \frac{1}{2}(-1+\sqrt{\beta^2\tau^2+1-\beta^2 I_1})$ $t(\tau) = \frac{\beta}{\sqrt{2}}\int \frac{d\tau}{((\beta^2\tau^2+1-\beta^2 I_1)\sqrt{\beta^2\tau^2+1-\beta^2 I_1}-1)^{1/2}} + C_2$
XVIII.	$\ddot{x} - \frac{1}{2x}\dot{x}^2 - 4x^2 = 0$	$F(x,t) = \frac{2i}{\beta}x$ $dT = 2i\sqrt{x}dt$	$x(\tau) = \frac{\beta}{2i}\sqrt{I_1-\tau^2}$ $t(\tau) = -\frac{1}{\sqrt{2i\beta}}\int\frac{d\tau}{(I_1-\tau^2)^{3/4}} + C_2$
XXII.	$\ddot{x} - \frac{3}{4x}\dot{x}^2 + 1 = 0$	$F(x,t) = \frac{2i}{\beta}x^{-1/4}$ $dT = -\frac{i}{2}x^{-1/2}dt$	$x(\tau) = \frac{16}{\beta^4}\frac{1}{\sqrt{I_1-\tau^2}}$ $t(\tau) = c_2 + \frac{8i}{\beta}\int\frac{d\tau}{(I_1-\tau^2)^{3/2}}$

In this chapter we introduce the concept of nonlocal transformations defined by means of the generalized Sundman transformation. We have also introduced the notion of Sundman symmetries which may be viewed as the counterpart of the Lie symmetries in the context of such nonlocal transformations. Furthermore we have derived certain parametric solutions for some of the Painlevé-Gambier equations by exploiting the Sundman transformation.

In the next chapter we will further generalize the nonlocal character of these transformations assuming that the transformation $(x,t) \to (X,T)$ is nonlocal in both the variables.

Chapter 7

First integral for time-dependent higher-order Riccati equation by nonholonomic or a nonlocal transformation

7.1 Introduction

The time-independent second-order Riccati equation (SORE), also sometimes known as the Painlevé-Ince equation, plays an important role in dynamical systems. This equation was studied from a geometric perspective in [5] and shown to admit two alternative Lagrangian formulations, with both Lagrangians belonging to a nonnatural class. The Lie point symmetries of the SORE are known to have an algebra identical to that of the eight-parameter group $\mathrm{SL}(3, \mathbf{R})$ [61]. Since the free particle also possesses a similar symmetry algebra it is therefore not surprising that under an appropriate transformation the SORE may be transformed into that of the free particle.

In [6], the authors studied the second, third and fourth-order cases of the hierarchy of Riccati equations and have shown the existence of Darboux functions and generators of time-dependent constants of motion.
In this chapter we present a time-dependent generalization of the second-order Riccati equation

$$\ddot{x} + 3\beta\, x\, \dot{x} + \beta^2 x^3 = 0. \tag{7.1.1}$$

which is taken in the form

$$\ddot{x} + 3h(t)x\dot{x} + h^2 x^3 + \dot{h}(t)\, x^2 = 0. \tag{7.1.2}$$

Clearly when the coefficient h is a constant (7.1.2) reduces to the usual second-order Riccati equation (7.1.1). Before pursuing the issue of deriv ing a first integral of the

time-dependent second-order Riccati equation TDSORE in (7.1.2) it is pertinent to note the following features of the equation. Unlike its time independent counterpart of (7.1.1) the TDSORE is not a bi-Lagrangian system. Furthermore it is actually a truncated version of the Gambier equation [31].
Gambier in course of his classification of integrable second-order differential equations solved the following equation, which is listed as Equation XXVII of the Painlevé-Gambier series as given in Ince's book [44] and occurs as Equation 15 in Gambier's minimal list of 24 second-order equations with the Painlevé property. The Gambier equation (see [4] for a relatively recent update) is given by

$$\ddot{x} = \frac{n-1}{n}\frac{\dot{x}^2}{x} + a\frac{n+2}{n}x\dot{x} + b\dot{x} - \frac{n-2}{n}\frac{\dot{x}}{x}\sigma - \frac{a^2}{n}x^3 + (\dot{a}-ab)x^2 + (cn - \frac{2a}{n}\sigma)x - b\sigma - \frac{\sigma^2}{nx}, \quad (7.1.3)$$

Here a, b and c are functions of the independent variable, σ is a constant which can be scaled to 1 unless it happens to be 0 and n is an integer. If we set $b = c = 0$ and assume that $n = 1$ and $\sigma = 0$, then Gambier 1 reduces to a time-dependent second-order Riccati equation which can be mapped to Gambier 14 of the Gambier's minimal list.

Moreover the TDSORE can be shown to arise from a Riccati sequence which may be introduced as bellow.
Let $h(t)$ be an arbitrary differentiable function and $\mathbb{D}_R$ denote the following differential operator

$$\mathbb{D}_R := \frac{d}{dt} + h(t)\,x\,,$$

to be called the 'Riccati differential operator'. Next consider the sequence obtained by applying such a differential operator to the function x in an iterative way. For example when

$$\begin{aligned}
n = 1, \qquad & \mathbb{D}_R x = \left(\frac{d}{dt} + h(t)\,x\right) x = \dot{x} + h(t)\,x^2, \\
n = 2, \qquad & \mathbb{D}_R^2 x = \left(\frac{d}{dt} + h(t)\,x\right)^2 x = \ddot{x} + 3h(t)\,x\dot{x} + h^2(t)\,x^3 + \dot{h}(t)\,x^2, \\
n = 3, \qquad & \mathbb{D}_R^3 x = \left(\frac{d}{dt} + h(t)\,x\right)^3 x = \dddot{x} + 4h(t)\,x\ddot{x} + 6h^2(t)\,x^2\dot{x} + 3h(t)\,\dot{x}^2 + h^3(t)\,x^4 \\
& \qquad +5\dot{h}(t)\,x\dot{x} + 3h(t)\,\dot{h}(t)\,x^3 + \ddot{h}(t)\,x^2, \\
n = 4 \qquad & \mathbb{D}^4 x = \left(\frac{d}{dt} + h(t)\,y\right)^4 x = \ddddot{x} + 5h(t)\,x\dddot{x} + 10h(t)\,\dot{x}\ddot{x} + 15h^2(t)\,x\,\dot{x}^2 + 10h^2(t)x^2\ddot{x} \\
& +10h^3(t)\,x^3\dot{x} + h^4(t)\,x^5 + \dot{h}(t)\,(9x\ddot{x} + 8\dot{x}^2) + 26h(t)\,\dot{h}(t)\,x^2\dot{x} + 7\ddot{h}(t)\,x\dot{x} + 3\dot{h}^2(t)x^3 \\
& +4h(t)\,\ddot{h}(t)\,x^3 + \dddot{h}(t)\,x^2 + 6h^2(t)\,\dot{h}(t)\,x^4
\end{aligned} \quad (7.1.4)$$

and analogous expressions for higher values of n which turn out to be quite lengthy.
The equation

$$R^{(k)}(x, \ldots, x^{(k)}) = \mathbb{D}_R^k x = 0\,, \quad k = 0, 1, \ldots \quad (7.1.5)$$

defines a Riccati equation with variable coefficients of order k. Note that $R^{(0)}(x) = x$, and the particular Riccati equation $R^{(1)}(x, \dot{x}) = 0$ obtained for $k = 1$ is the standard Riccati equation but with a variable coefficient $h(t)$,

$$\dot{x} + h(t)\,x^2 = 0. \tag{7.1.6}$$

It is thus obvious that the TDSORE corresponds to the second member of the above sequence.
To return to the issue of deriving a first integral for the TDSORE we shall make use of nonlocal transformation [50, 51, 52, 55, 86] to linearize the equation and there after identify an appropriate first integral. In this process we wish to illustrate the effectiveness of such transformations. In fact their efficacy will be even more evident when we take up the case of third-order ODEs. In addition we will also consider a generalization of the TDSORE and examine its relation with Sugai equation [102].

7.2 Nonholonomic transformations and first integrals of time dependent second-order Riccati equation

In the previous chapter it was shown that Sundman transformations are often quite useful for the determination of first integrals of second and higher-order ordinary differential equations (ODEs), [25, 27, 28]. Such transformations, it will be recalled, are partially nonlocal in character. Here we shall further generalize the nonlocal character of the transformation by assuming both the new variables X and T are given by nonlocal expressions. In this sense they are the complete opposite of point transformations.
Consider a second-order ordinary differential equation

$$\ddot{x} = w(t, x, \dot{x}), \tag{7.2.1}$$

where $w(t, x, \dot{x})$ is linear in $\dot{x}$ then formally it may be recast as

$$\ddot{x} + f(x,t)\dot{x} + g(x,t) = 0, \tag{7.2.2}$$

which may be viewed as a kind of time dependent version of the Liénard equation.
suppose we seek a nonlocal transformation of $(t, x) \mapsto (T, X)$ of the form

$$dX = A(x,t)dx + B(x,t)dt, \tag{7.2.3}$$

$$dT = C(x,t)dx + D(x,t)dt. \tag{7.2.4}$$

such that the ODE (7.2.2) is transformed to the autonomous linear equation [86]

$$\frac{d^2X}{dT^2} = 0. \tag{7.2.5}$$

Our first objective is to determine the differentiable functions A, B, C and D which enable such a linearization to be made for the particular case of (7.1.2). The nonlocal nature of the above transformation may be ensured by demanding that

$$A_t \neq B_x, \quad C_t \neq D_x. \tag{7.2.6}$$

It is obvious that if such a transformation exists then in terms of the new variables we immediately obtain a first integral

$$\frac{dX}{dT} = \text{constant}. \tag{7.2.7}$$

However from (7.2.7) it follows that such a first integral is clearly dependent on time since when expressed in terms of the original variables x and t, it is given by

$$I(t, x, \dot{x}) = \frac{A(x,t)\dot{x} + B(x,t)}{C(x,t)\dot{x} + D(x,t)} \tag{7.2.8}$$

clearly defines a time-dependent first integral of (7.2.2).

The crucial question is whether one can derive a nonlocal transformation which enables such a first integral to be identified for a given second-order equation. In answer to this question, we note that if (7.2.8) is indeed a first integral of (7.2.2) then we must have $dI/dt = 0$. This in turn leads to the following condition:

$$\Delta(x,t)\ddot{x} + (CA_x - AC_x)\dot{x}^3 + (C(A_t + B_x) - A(C_t + D_x) + DA_x - BC_x)\dot{x}^2 +$$
$$+ (CB_t - BC_t + DA_t - AD_t + DB_x - BD_x)\dot{x} + (DB_t - BD_t) = 0, \tag{7.2.9}$$

where

$$\Delta(x,t) := A(x,t)D(x,t) - B(x,t)C(x,t).$$

Comparison with (7.2.2) shows that first of all we must have

$$CA_x = AC_x \text{ which implies } C(x,t) = a(t)A(x,t), \tag{7.2.10}$$

since there is no term proportional to $\dot{x}^3$; while the vanishing of the coefficient of $\dot{x}^2$ implies

$$C(A_t + B_x) - A(C_t + D_x) + DA_x - BC_x = 0. \tag{7.2.11}$$

In view of (7.2.10) we may rewrite $\Delta(x,t)$ as

$$\Delta(x,t) = A(x,t)(D(x,t) - a(t)B(x,t)).$$

Next we make a simplifying assumption *viz*,

$$D(x,t) := a(t)B(x,t) + b(x,t)A(x,t), \quad \text{so that} \quad \Delta(x,t) = A^2(x,t)b(x,t). \tag{7.2.12}$$

Here $a(t)$ and $b(x,t)$ are functions to be determined. Under these circumstances (7.2.3) and (7.2.4) reduce to the following

$$dX = A(x,t)[dx + S(x,t)dt], \tag{7.2.13}$$

$$dT = a(t)A(x,t)[dx + (S(x,t) + \frac{b}{a})dt], \tag{7.2.14}$$

where

$$S(x,t) := B(x,t)/A(x,t)$$

. In view of (7.2.10) and (7.2.12) the condition (7.2.11) simplifies to

$$\dot{a} + b_x = 0. \tag{7.2.15}$$

Returning now to (7.2.9) we require the coefficients of $\dot{x}$ in (7.2.2) to satisfy

$$f(x,t) = \frac{1}{\Delta}\left[CB_t - BC_t + DA_t - AD_t + DB_x - BD_x\right], \tag{7.2.16}$$

and

$$g(x,t) = \frac{1}{\Delta}\left[DB_t - BD_t\right]. \tag{7.2.17}$$

Using (7.2.10) and (7.2.12) one can rewrite (7.2.16) as

$$f(x,t) = S_x - \left(\frac{2\dot{a} + b_x}{b}\right)S - \frac{b_t}{b}, \tag{7.2.18}$$

while (7.2.17) becomes

$$g(x,t) = S_t - \left(\frac{\dot{a}S + b_t}{b}\right)S. \tag{7.2.19}$$

Thus when there exists a function $S(x,t)$ such that (7.2.18) and (7.2.19) are satisfied then it follows from (7.2.13) and (7.2.14) that there exists a first integral of the form

$$I(t,x,\dot{x}) = \frac{\dot{x} + S}{a(\dot{x} + S) + b}. \tag{7.2.20}$$

Before presenting our main result, let us quickly see whether the above procedure works for the following modified Emden type equation [5, 11].

Example 7.2.1

Notice that if we choose $S(x,t) = \beta x^2$, $b(x,t) = -kx$ and $a(t) = kt$ then substitution into (7.2.18) and (7.2.19) leads to the equation

$$\ddot{x} + 3\beta x\dot{x} + \beta^2 x^3 = 0.$$

Its associated first integral, by the formula stated in (7.2.20), is

$$I(t,x,\dot{x}) = \frac{\dot{x} + \beta x^2}{kt(\dot{x} + \beta x^2 - x/t)}.$$

7.2.1 Generalized time-dependent Riccati equation

Since the TDSORE emerges as a truncated version of the Gambier equation we consider (7.1.3) with minor changes of notation.

$$\ddot{x} = \frac{n-1}{n}\frac{\dot{x}^2}{x} + \alpha\frac{n+2}{n}x\dot{x} + \beta\dot{x} - \frac{n-2}{n}\frac{\dot{x}}{x}\sigma - \frac{\alpha^2}{n}x^3 + (\dot{\alpha}-\alpha\beta)x^2 + (\gamma n - \frac{2\alpha}{n}\sigma)x - \beta\sigma - \frac{\sigma^2}{nx}, \tag{7.2.21}$$

where $\gamma = \frac{\dot{\beta}}{2} - \frac{\beta^2}{4}$.
If we assume $\sigma = 0$ and $n = 1$ then the above system reduces to the generalized time-dependent Riccati equation or Sugai equation [102]. It is evident that the Sugai equation includes as special cases the TDSORE for the particular choice $\beta = 0$ while the Gambier V equation, namely

$$\ddot{x} = -3x\dot{x} + \beta(t)x - x^3 + \beta(t)x^2, \tag{7.2.22}$$

corresponds to the specific choice $n = 1, \sigma = 0,\ \alpha(t) = -1$ and $\gamma = 0$. One of the main results of this chapter is the following proposition.

Proposition 7.2.1 *A time-dependent first integral of the variable coefficient second-order equation*

$$\ddot{x} - [3h(t)x + r(t)]\dot{x} + h^2(t)x^3 - [\dot{h}(t) - h(t)r(t)]x^2 + \lambda(t)x + \left(\frac{r^2(t)}{4} - \frac{\dot{r}(t)}{2}\right)x = 0 \tag{7.2.23}$$

is given by the function

$$I(t,x,\dot{x}) = \frac{\dot{x}+S}{a(t)(\dot{x}+S) - \dot{a}x} \tag{7.2.24}$$

where $S(x,t) = \left(\frac{\ddot{a}}{\dot{a}} - r(t)\right)\frac{x}{2} - h(t)x^2$ *and* $\lambda(t)$ *is given by the Schwarzian derivative*

$$\lambda(t) = \frac{1}{2}\left[\frac{\dddot{a}}{\dot{a}} - \frac{3}{2}\frac{\ddot{a}^2}{\dot{a}^2}\right].$$

Proof: The proof essentially revolves around finding the function $S(x,t)$. Using (7.2.15) to simplify (7.2.18) it follows that S must satisfy the following:

$$f(x,t) = -(3h(t)x + r(t)) = S_x - \left(\frac{\dot{a}}{b}\right)S - \frac{b_t}{b} \tag{7.2.25}$$

$$g(x,t) = S_t - \left(\frac{\dot{a}S + b_t}{b}\right)S \tag{7.2.26}$$

where $g(x,t) = h^2(t)x^3 - [\dot{h}(t) - h(t)r(t)]x^2 + \lambda(t)x + \left(\frac{r^2(t)}{4} - \frac{\dot{r}(t)}{2}\right)x$. A particular solution of (7.2.15) is clearly given by

$$b(x,t) = -\dot{a}x, \tag{7.2.27}$$

where we have set the constant of integration to be zero. Next we make the following ansatz for $S(x,t)$, *viz*

$$S(x,t) = S_2(t)x^2 + S_1(t)x + S_0(t).$$

Upon substitution of this into the right side of (7.2.25) and after equating the coefficients of the different powers of x we get

$$S_0(t) = 0, \quad S_1(t) = \frac{1}{2}\left(\frac{\ddot{a}}{\dot{a}} - r(t)\right) \text{ and } S_2(t) = -h(t),$$

so that

$$S(x,t) = \left(\frac{\ddot{a}}{\dot{a}} - r(t)\right)\frac{x}{2} - h(t)x^2. \tag{7.2.28}$$

It is easy to verify that this expression for S gives the required form of the function $g(x,t)$ when substituted in (7.2.26).

Proposition 7.2.1 *A time-dependent first integral of the variable coefficient second-order equation*

$$\ddot{x} + 3h(t)x\dot{x} + h^2(t)x^3 + \lambda(t)x + \dot{h}(t)x^2 = 0 \tag{7.2.29}$$

is given by the function

$$I(t,x,\dot{x}) = \frac{\dot{x} + S}{a(t)(\dot{x} + S) - \dot{a}x} \tag{7.2.30}$$

where $S(x,t) = h(t)x^2 + \frac{\ddot{a}}{2\dot{a}}x$ *and* $\lambda(t)$ *is given by the Schwarzian derivative*

$$\lambda(t) = \frac{1}{2}\left[\frac{\dddot{a}}{\dot{a}} - \frac{3}{2}\frac{\ddot{a}^2}{\dot{a}^2}\right].$$

Proof: The proof follows from the previous proposition by setting $r(t) = 0$ and replacing $h(t) \longrightarrow -h(t)$.

Proposition 7.2.2 *If* $a(t)$ *be such that the function* $R(t) := \ddot{a}/\dot{a}$ *satisfies a first-order Riccati equation then the equation (7.2.29) can be mapped to a standard variable coefficient second-order Riccati equation.*

Proof: The second-order variable coefficient Riccati equation is given by

$$\ddot{x} + 3h(t)x\dot{x} + h^2(t)x^3 + \dot{h}(t)x^2 = 0$$

It is easy to see that when $\lambda(t) = 0$ then (7.2.29) reduces to a second-order Riccati equation with variable coefficients. The vanishing of $\lambda(t)$ leads to the first-order Riccati equation, namely

$$R_t - \frac{1}{2}R^2 = 0. \tag{7.2.31}$$

In a similar manner it can be shown that a first integral for the Gambier V equation (7.2.22) is given by

$$I(x,\dot{x},t) = \frac{\dot{x}+x^2}{\beta(t)(\dot{x}+x^2) - x\beta(t)^2/2}$$

when $\beta(t)$ is a solution of the first-order Riccati equation (7.2.31).
It is evident that once a first integral is obtained one can easily read off the nonlocal transformation from its numerator and denominator respectively in view of (7.2.8).

In the following section we further illustrate the above procedure by considering the case of third-order ODEs.

7.3 Nonholonomic Transformations for Third-Order Differential Equations

The linearization problem in case of third-order ODEs has been studied from the perspective of point and contact transformations in [30]. However, continuing in the same spirit as above, we consider here a general third-order differential equation (TODE) of the form

$$\dddot{x} + a_0(x,t)\ddot{x} + g_2(x,t)\dot{x}^2 + g_1(x,t)\dot{x} + g_0(x,t) = 0, \tag{7.3.1}$$

and search for a nonlocal transformation such that it is mapped to the following equation

$$X'''(T) = 0, \tag{7.3.2}$$

(here $X' = \frac{dX}{dT}$), by the nonlocal transformation

$$dX = A(x,t)dx + B(x,t)dt, \quad dT = H(x,t)dt. \tag{7.3.3}$$

Note that here we have retained the flavour of the sundman transformation for the 'time-part' of the transformation to ensure relative simplification. It is a matter of straightforward computation to show that the TODE (7.3.1) is mapped to (7.3.2) by the transformation (7.3.3) provided its coefficients satisfy the following equations:

$$2\frac{A_t}{A} + \frac{B_x}{A} - \frac{B}{A}\frac{H_x}{H} - 3\frac{H_t}{H} = a_0(x,t), \tag{7.3.4}$$

$$3\frac{A_x}{A} - 4\frac{H_x}{H} = 0, \tag{7.3.5}$$

$$\frac{A_{xx}}{A} - \frac{H_{xx}}{H} - 3\frac{H_x}{H}\frac{A_x}{A} + 3\left(\frac{H_x}{H}\right)^2 = 0, \tag{7.3.6}$$

$$2\frac{A_{xt}}{A} - 2\frac{H_{xt}}{H} + \frac{B_{xx}}{A} - \frac{H_{xx}}{H}\frac{B}{A} - 3\frac{H_t}{H}\frac{A_x}{A} + 6\frac{H_x}{H}\frac{H_t}{H} - 3\frac{H_x}{H}\frac{A_t}{A} - 3\frac{B_x}{A}\frac{H_x}{H} + 3\frac{B}{A}\left(\frac{H_x}{H}\right)^2 = g_2(x,t), \tag{7.3.7}$$

$$\frac{A_{tt}}{A} - \frac{H_{tt}}{H} + 2\frac{B_{xt}}{A} - 2\frac{B}{A}\frac{H_{xt}}{H} - 3\frac{H_t}{H}\frac{A_t}{A} + 3\left(\frac{H_t}{H}\right)^2 - 3\frac{H_t}{H}\frac{B_x}{A} + 6\frac{B}{A}\frac{H_x}{H}\frac{H_t}{H} - 3\frac{H_x}{H}\frac{B_t}{A} = g_1(x,t), \tag{7.3.8}$$

$$\frac{B_{tt}}{A} - \frac{H_{tt}}{H}\frac{B}{A} - 3\frac{H_t}{H}\frac{B_t}{A} + 3\frac{B}{A}\left(\frac{H_t}{H}\right)^2 = g_0(x,t). \tag{7.3.9}$$

Thus given a TODE so that the explicit form of the coefficients $a_0(x,t)$ and g_i $i = 0, ..., 2$ are known, the set of equations (7.3.4 -7.3.9) constitute an over determined set for the three unknown functions A, B and H. Therefore, if upon solving the above set of equations (7.3.4 -7.3.9) one can deduce the functions A, B and H then the linearizing transformation can be determined and consequently equations of the form (7.3.1) may be linearized to the free particle equation.

It is obvious that a first integral of (7.3.2) is given by

$$I_1(t, x, \dot{x}, \ddot{x}) = X'' = \text{constant}. \tag{7.3.10}$$

The explicit form of the first integral can be immediately worked out from the transformation (7.3.3) and has the following appearance,

$$X'' = \frac{1}{H^3}\left[HA\ddot{x} + (HA_x - AH_x)\dot{x}^2 + ((HA_t - AH_t) + (HB_x - BH_x))\,\dot{x} + (HB_t - BH_t)\right],$$

which may be written as

$$X'' = \frac{1}{H}\left[\frac{A}{H}\ddot{x} + \left(\frac{A}{H}\right)_x \dot{x}^2 + \left(\left(\frac{A}{H}\right)_t + \left(\frac{B}{H}\right)_x\right)\dot{x} + \left(\frac{B}{H}\right)_t\right] = \text{constant}. \tag{7.3.11}$$

Having explained the general idea behind the construction of a linearizing transformation for an equation of the form (7.3.1), we proceed to the determination of the unknown functions A, B and H. From (7.3.5), we have

$$H(x,t) = \alpha(t)A^{3/4}, \tag{7.3.12}$$

where $\alpha(t)$ is an arbitrary function of t. Eliminating H from (7.3.6) leads to the following equation determining the function $A(x,t)$, namely

$$\frac{A_{xx}}{A} - \frac{3}{2}\left(\frac{A_x}{A}\right)^2 = 0, \tag{7.3.13}$$

which admits the solution

$$A(x,t) = \frac{\gamma(t)}{(2 - x\beta(t))^2}. \tag{7.3.14}$$

Here β and γ are arbitrary functions of t. Next eliminating H from (7.3.4) we have

$$\left(\frac{B}{A}\right)_x + \frac{1}{4}\frac{A_x}{A}\left(\frac{B}{A}\right) = \frac{1}{4}\frac{A_t}{A} + 3\frac{\dot{\alpha}}{\alpha} + a_0$$

which has the formal solution,

$$B(x,t) = A^{3/4}(x,t)\left[\int A^{1/4}\left(\frac{A_t}{4A} + 3\frac{\dot{\alpha}}{\alpha} + a_0\right)dx + \delta(t)\right]. \tag{7.3.15}$$

Note that since α, β, γ and δ are arbitrary functions of t we may choose them to be constants in order to simplify the calculations. Secondly, having completed the determination of the unknown functions A, B and H involved in our nonlocal transformation, it remains to examine their compatibility with equations (7.3.7-7.3.9). In the following we consider the case when the functions α, β, γ and δ assume the following specific values.

Case (i) $\alpha = \beta = \gamma = 1$ *and* $\delta = 0$

With these values one finds that

$$A(x,t) = \frac{1}{(2-x)^2}, \quad H(x,t) = \frac{1}{(2-x)^{3/2}}, \quad \frac{B}{A} = \frac{1}{A^{1/4}}\int^x A^{1/4} a_0(s,t)ds. \tag{7.3.16}$$

Consequently from (7.3.7-7.3.9) we arrive at the following relations:

$$g_2(x,t) = a_{0x} - \frac{1}{2}a_0\left(\frac{A_x}{A}\right) - \frac{3}{4}\left(\frac{A_x}{A}\right)_x\left[\frac{1}{A^{1/4}}\int^x A^{1/4}a_0(s,t)ds\right], \tag{7.3.17}$$

$$g_1(x,t) = 2a_{0t} - \frac{3}{4}\left(\frac{A_x}{A}\right)\left[\frac{1}{A^{1/4}}\int^x A^{1/4}a_{0t}(s,t)ds\right], \tag{7.3.18}$$

$$g_0(x,t) = \left[\frac{1}{A^{1/4}}\int^x A^{1/4}a_{0tt}(s,t)ds\right]. \tag{7.3.19}$$

In general requiring the right hand sides of the above equations to match the given values of $g_i(x,t), i = 0, ..., 2$, may be too stringent a requirement so that alternatively we could choose to define the g_i by these very relations and thereby derive suitable third-order ODEs admitting the first integrals of the form (7.3.11). Furthermore if we assume

$$a_0(x,t) = h(t)f(x)$$

and define a function

$$F(x) := \frac{1}{A^{1/4}}\int^x A^{1/4}f(s)ds,$$

so that

$$\frac{B}{A} = h(t)F(x),$$

then the expressions for g_i become

$$g_0(x,t) = \ddot{h}(t)F(x), \tag{7.3.20}$$

$$g_1(x,t) = \dot{h}(t)\left[2f(x) - \frac{3F(x)}{2(2-x)}\right], \tag{7.3.21}$$

$$g_2(x,t) = h(t)\left[f'(x) - \frac{f(x)}{(2-x)} + \frac{3F(x)}{2(2-x)^2}\right]. \tag{7.3.22}$$

Note that the explicit form of $F(x)$ in view of (7.3.16), is given by

$$F(x) = (2-x)^{1/2}\int^x \frac{f(s)}{(2-s)^{1/2}}ds.$$

Therefore a third-order equation of the form

$$\dddot{x} + h(t)f(x)\ddot{x} + [g_2(x,t)\dot{x}^2 + g_1(x,t)\dot{x} + g_0(x,t)] = 0$$

with g_0, g_1 and g_2 given by the equations (7.3.20-7.3.22) may be linearized to $X''' = 0$ by the nonholonomic transformation

$$dX = \frac{1}{(2-x)^2}[dx + h(t)F(x)dt], \quad dT = \frac{1}{(2-x)^{3/2}}dt. \tag{7.3.23}$$

Its associated first integral may be obtained from (7.3.11) and is then given by the following expression

$$I(x,t,\dot{x},\ddot{x}) = (2-x)\ddot{x} + \frac{1}{2}\dot{x}^2 + h(t)\left((2-x)F'(x) + \frac{F(x)}{2}\right)\dot{x} + (2-x)F(x)\dot{h} = \text{constant}. \tag{7.3.24}$$

Case (ii) $\alpha = 1, \gamma = 4$ *and* $\beta = \delta = 0$
This case is considerably simply because when $\beta = 0$ and $\gamma = 4$ it follows that $A(x,t) = H(x,t) = 1$ while from (7.3.15) we have $B = \int^x a_0(s,t)ds = h(t)\int^x f(s)ds$ (assuming $a_0 = h(t)f(x)$). Moreover the expressions for g_i, as stated above, reduce to the following:

$$g_2(x,t) = a_{0x} = h(t)f'(x), \quad g_1(x,t) = 2\dot{h}f(x), \quad g_0(x,t) = \ddot{h}\int^x f(s)ds := \ddot{h}F_1(x). \tag{7.3.25}$$

The explicit nature of the transformation in this case is interesting, since it does not involve any change in the time coordinate,

$$dX = dx + h(t)F_1(x)dt, \quad dT = dt. \tag{7.3.26}$$

The corresponding first integral is now given by

$$I_1(x,t,\dot{x},\ddot{x}) = \left[\ddot{x} + h(t)F_1'(x)\dot{x} + \dot{h}(t)F_1(x)\right] = \text{constant}. \tag{7.3.27}$$

In this chapter we have computed the first integrals of the time-dependent second-order Riccati equation and its generalization by using the method of nonholonomic transformations. It appears that unlike Sundman transformation this method leads to considerable computational simplification. In the latter half of the chapter we have calculated the first integral in certain particular cases of suitably defined third-order nonlinear equation which may be viewed as a kind of generalization of the second-order Liénard type equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Although the general class of third-order equations may not always be amiable to such an analysis the utility of looking for nonlocal transformations to unearth first integrals can be an interesting and fruitful exercise

In the next chapter we will consider the Jacobi Last Multiplier which was introduced in Chapter 2 and examine its role in the context of Lagrangian and Hamiltonian dynamics.

Chapter 8

The Jacobi Last Multiplier, Integrating Factors and the Lagrangian formulation of differential equations of the Painlevé-Gambier Classification

In section 2.7 we introduced the Jacobi Last Multiplier (JLM) and dwelt on some of its essential properties. In this chapter we shall describe certain applications of the JLM to second-order ODEs in the context of Lagrangian dynamics.
In a series of recent papers Leach, Nucci and Tamizhmani (for example, [76, 77, 78, 79] and references therein) have investigated the relation between integrating factors and the Hessian. It appears that this connection has a long history, which can be traced to Jacobi's attempts to obtain the last multiplier [45, 46]. In 1874 Lie [56, 57] showed that point symmetries could be used to determine Jacobi's last multiplier (JLM). The explicit nature of the relation between the JLM and Hessian was clarified by Rao in a article [60] and is also mentioned in Whittaker's book on analytical dynamics [105]. Unlike the Hamiltonian structure of the six Painlevé equations, which have received much attention [81], the Lagrangian formulation has not been sufficiently nurtured. In a recent paper Wolf and Brand [106] proposed Lagrangian for Painlevé VI.

8.1 Lagrangians and the last Multiplier

For a second-order ODE

$$y'' = w(x, y, y'), \tag{8.1.1}$$

which admits a Lagrangian function $L(x, y, y')$. The Euler-Lagrange equation states that

$$\frac{d}{dx}\Big(\frac{\partial L}{\partial y'}\Big) - \frac{\partial L}{\partial y} = 0. \tag{8.1.2}$$

Expanding the total derivative operator we get

$$\frac{\partial^2 L}{\partial x \partial y'} + y' \frac{\partial^2 L}{\partial y \partial y'} + w \frac{\partial^2 L}{\partial y'^2} - \frac{\partial L}{\partial y} = 0, \tag{8.1.3}$$

where we have made use of the equation (8.1.1). If we have take a partial derivative with respect to y' then we obtain

$$\frac{\partial}{\partial x}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) + \frac{\partial^2 L}{\partial y \partial y'} + y' \frac{\partial}{\partial y}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) + \frac{\partial w}{\partial y'}\frac{\partial^2 L}{\partial y'^2} + w \frac{\partial}{\partial y'}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) - \frac{\partial^2 L}{\partial y' \partial y} = 0$$

which implies

$$\frac{\partial}{\partial x}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) + y' \frac{\partial}{\partial y}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) + w \frac{\partial}{\partial y'}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) + \frac{\partial w}{\partial y'}\frac{\partial^2 L}{\partial y'^2} = 0$$

Or,

$$\frac{d}{dx}\Big(\frac{\partial^2 L}{\partial y'^2}\Big) + \frac{\partial w}{\partial y'}\frac{\partial^2 L}{\partial y'^2} = 0. \tag{8.1.4}$$

A comparison with the equation

$$\frac{\partial M}{\partial x} + \frac{\partial M y'}{\partial y} + \frac{\partial M w}{\partial y'} = 0, \tag{8.1.5}$$

which is the equation defining the last multiplier M of the equation (8.1.1), then shows that last multiplier is given by

$$M = \frac{\partial^2 L}{\partial y'^2}. \tag{8.1.6}$$

On the other hand, given a system of first order equations

$$y_k' = f_k(x, y), \quad y = (y_1, y_2, ... y_n),$$

the JLM is a solution of the equation

$$\frac{d \log M}{dx} + \sum_{k=1}^{n} \frac{\partial f_k}{\partial y_k} = 0.$$

It follows that, if a solution of this equation is obtained, then from a knowledge of the JLM one can construct the Lagrangian function as

$$L(x, y, y') = \int \Big(\int M dy' \Big) + f_1(x, y) y' + f_2(x, y). \tag{8.1.7}$$

8.2 Lagrangians for the Painlevé-Gambier Equations

A large number of second-order ODEs in the Painlevé-Gambier classification system belong to the following class of equations, namely

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x) = 0. \tag{8.2.1}$$

Writing this equation in the form

$$\ddot{x} = \mathcal{F}(t, x, \dot{x}) = -\left[\frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x)\right],$$

the Jacobi Last Multiplier M for (8.2.1) is given by the solution of

$$\frac{d}{dt}\log M = -\frac{\partial \mathcal{F}}{\partial \dot{x}}. \tag{8.2.2}$$

In the present case we have

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} = \exp\left[\phi(t, x)\right]. \tag{8.2.3}$$

By (8.1.7) we then obtain the Lagrangian as

$$L(t, x, \dot{x}) = \frac{1}{2}e^{\phi(t,x)}\dot{x}^2 + f_1(t, x)\dot{x} + f_2(t, x). \tag{8.2.4}$$

To determine the unknown functions, f_1 and f_2, we substitute this Lagrangian into the Euler-Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{8.2.5}$$

and use (8.2.1) to get

$$f_{1t} - f_{2x} = e^{\phi}B(t, x).$$

Then making a gauge transformation $f_1 = G_x$ and $f_2 = G_t + f_3(t, x)$ allows us to satisfy the last equation when

$$f_3(t, x) = -\int e^{\phi}B(t, x)dx. \tag{8.2.6}$$

Consequently the final Lagrangian for (8.2.1) assumes the form

$$L(t, x, \dot{x}) = e^{\phi(t,x)}\frac{\dot{x}^2}{2} - \int e^{\phi}B(t, x)dx + \frac{dG}{dt}. \tag{8.2.7}$$

The total derivative term obviously is of little consequence and may safely be discarded. The conjugate momentum is defined by

$$p = \frac{\partial L}{\partial \dot{x}} = e^{\phi}\dot{x} \text{ and implies } \dot{x} = e^{-\phi}p.$$

This leads to the Hamiltonian

$$H = e^{-\phi}\frac{p^2}{2} + \int e^{\phi}B(t,x)dx,$$

by the usual Legendre transformation. It is clear that the Lagrangian obtained in the above manner is a non standard one.
One can attempt to bring it closer to the standard form by means of the transformation

$$\dot{y} = e^{\phi/2}\dot{x} \quad \text{or} \quad y(t,x) = \int e^{\phi(t,x)/2}dx. \tag{8.2.8}$$

We illustrate this by a specific example in the sequel.

Example 8.2.1

The Painlevé-Gambier equation XXI
This equation has the form

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0. \tag{8.2.9}$$

The Jacobi Last Multiplier is given by $M = x^{-3/2}$ and the corresponding Lagrangian is

$$L_{21} = x^{-3/2}\frac{\dot{x}^2}{2} + 2x^{3/2}. \tag{8.2.10}$$

The associated Hamiltonian H_{21} provides a first integral, (i.e.,$\frac{dH_{21}}{dt} = 0$) namely

$$H_{21} = x^{-3/2}\frac{\dot{x}^2}{2} - 2x^{3/2}. \tag{8.2.11}$$

It is interesting to note that L_{21} and H_{21} both have a 'wrong relative sign'. Consider the transformation

$$x \mapsto y = 4x^{1/4} \quad \text{so that} \quad \dot{y} = x^{-3/4}\dot{x}. \tag{8.2.12}$$

Under such a transformation the Lagrangian L_{21} assumes the more familiar form

$$L_{21}(t,y,\dot{y}) = \left[\frac{1}{2}\dot{y}^2 + \left(2(y/4)^6\right)\right],$$

but continues to have a 'wrong relative sign'.

8.2.1 Lagrangians of the Painlevé Transcendents

The most interesting of the fifty Painlevé-Gambier equations are those which are not replaceable by a simpler equation or combination of simpler equations i.e. irreducible and serve to define new transcendents. These irreducible six equations are those numbered IV, IX, XIII, XXXI, XXXIX, and L [44]. It is convenient to tabulate and renumber them,

$$P_I : \ \ddot{x} = 6x^2 + t, \tag{8.2.13}$$

$$P_{II} : \ \ddot{x} = 2x^3 + xt + \alpha, \tag{8.2.14}$$

$$P_{III} : \ \ddot{x} - \frac{1}{x}\dot{x}^2 + \frac{1}{t}\dot{x} - \frac{1}{t}(\alpha x^2 + \beta) - \gamma x^3 - \frac{\delta}{x} = 0, \tag{8.2.15}$$

$$P_{IV} : \ \ddot{x} - \frac{1}{2x}\dot{x}^2 - \left[\frac{3}{2}x^3 + 4tx^2 + 2(t^2 - \alpha)x + \frac{\beta}{x}\right] = 0, \tag{8.2.16}$$

$$P_V : \ \ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 + \frac{1}{t}\dot{x} - \left[\frac{(x-1)^2}{t^2}(\alpha x + \frac{\beta}{x}) + \frac{\gamma x}{t} + \frac{\delta x(x+1)}{x-1}\right] = 0, \tag{8.2.17}$$

$$P_{VI} : \ \ddot{x} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\dot{x}^2 + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\dot{x} \tag{8.2.18}$$

$$- \frac{(x-1)(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha + \frac{\beta t}{x^2} + \frac{\gamma(t-1)}{(x-1)^2} + \frac{\delta t(t-1)}{(x-t)^2}\right] = 0. \tag{8.2.19}$$

In the following we derive their Lagrangians and Hamiltonians by making use of the Jacobi Last Multiplier.

The Painlevé I equation

The P_I equation may be written as

$$\ddot{x} - 6x^2 - t = 0. \tag{8.2.20}$$

Comparison with (8.2.1) shows that $\phi_x = 0$ and $\phi_t = 0$ which yields for the last multiplier $M = \exp\phi = 1$. Then from (8.2.7) we obtain

$$L_I = \frac{\dot{x}^2}{2} + 2x^3 + xt \tag{8.2.21}$$

and the Hamiltonian as

$$H_I = \frac{p^2}{2} - 2x^3 - xt. \tag{8.2.22}$$

The Painlevé II equation

The P_{II} equation may be written as

$$\ddot{x} - 2x^3 - xt - \gamma = 0. \tag{8.2.23}$$

Comparison with (8.2.1) shows that $\phi_x = 0$ and $\phi_t = 0$ which yields for the last multiplier $M = \exp\phi = 1$. Then from (8.2.7) we obtain

$$L_I = \frac{\dot{x}^2}{2} + \frac{x^4}{2} + \frac{tx^2}{2} + \gamma x. \tag{8.2.24}$$

and the Hamiltonian as

$$H_I = \frac{p^2}{2} - \frac{x^4}{2} - \frac{tx^2}{2} - \gamma x. \tag{8.2.25}$$

The Painlevé III equation

The P_{III} equation may be written as

$$\ddot{x} - \frac{1}{x}\dot{x}^2 + \frac{1}{t}\dot{x} + B(t,x) = 0, \tag{8.2.26}$$

where $B(t,x) = -[\frac{1}{t}(\alpha x^2 + \beta) + \gamma x^3 + \frac{\delta}{x}]$. Comparison with (8.2.1) shows that $\phi_x = -2/x$ and $\phi_t = 1/t$ which yields for the last multiplier $M = \exp\phi = t/x^2$. Then from (8.2.7) we obtain

$$L_{III} = \frac{t}{x^2}\frac{\dot{x}^2}{2} + \alpha x - \frac{\beta}{x} + t\left(\frac{\gamma x^2}{2} - \frac{\delta}{2x^2}\right) \tag{8.2.27}$$

and the Hamiltonian as

$$H_{III} = \frac{x^2}{t}\frac{p^2}{2} + \left(\frac{\beta}{x^2} - \alpha x\right) + \frac{t}{2}\left(\frac{\delta}{x^2} - \gamma x^2\right). \tag{8.2.28}$$

The Painlevé IV equation

The P_{IV} equation may be written as

$$\ddot{x} - \frac{1}{2x}\dot{x}^2 + B(t,x) = 0, \tag{8.2.29}$$

where

$$B(t,x) = -\left[\frac{3}{2}x^3 + 4tx^2 + 2(t^2-\alpha)x + \frac{\beta}{x}\right].$$

Unlike the previous two Painlevé equations, here we have $\phi_t = 0$ so that the last multiplier is now time independent. Indeed for the P_{IV} equation we have $M = 1/x$ while the corresponding Lagrangian is

$$L_{IV} = \frac{1}{x}\frac{\dot{x}^2}{2} + \left[\beta\ln|x| + (t^2-\alpha)x^2 + \frac{4}{3}tx^3 + \frac{3}{8}x^4\right]. \tag{8.2.30}$$

The associated Hamiltonian is

$$H_{IV} = \frac{xp^2}{2} - \left[\beta\ln|x| + (t^2-\alpha)x^2 + \frac{4}{3}tx^3 + \frac{3}{8}x^4\right]. \tag{8.2.31}$$

The Painlevé V equation

The P_V equation may be written as

$$\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 + \frac{1}{t}\dot{x} + B(t,x) = 0, \tag{8.2.32}$$

where

$$B(t,x) = -\left[\frac{(x-1)^2}{t^2}(\alpha x + \frac{\beta}{x}) + \frac{\gamma x}{t} + \frac{\delta x(x+1)}{x-1}\right].$$

Following the same procedure as before we obtain for the Jacobi Last Multiplier

$$M = \frac{t}{x(x-1)^2}$$

and the Lagrangian

$$L_V = \frac{t}{x(x-1)^2}\frac{\dot{x}^2}{2} + \frac{1}{t}\left(\alpha x - \frac{\beta}{x}\right) - \frac{\gamma}{x-1} - \delta\frac{tx}{(x-1)^2}. \tag{8.2.33}$$

The corresponding Hamiltonian is

$$H_V = \frac{x(x-1)^2}{t}\frac{p^2}{2} - \frac{1}{t}\left(\alpha x - \frac{\beta}{x}\right) + \frac{\gamma}{x-1} + \delta\frac{tx}{(x-1)^2}. \tag{8.2.34}$$

The Painlevé VI equation

The P_{VI} equation is perhaps one of the most well-studied equations of the Painlevé class. It may be written as

$$\ddot{x} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\dot{x}^2 + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\dot{x} + B(t,x) = 0, \tag{8.2.35}$$

where

$$-B(t,x) = \frac{(x-1)(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha + \frac{\beta t}{x^2} + \frac{\gamma(t-1)}{(x-1)^2} + \frac{\delta t(t-1)}{(x-t)^2}\right].$$

In this case we have

$$\phi_x = -\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right) \text{ and } \phi_t = \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)$$

so that the last multiplier is given by

$$M = e^{\phi} = \frac{t(t-1)}{x(x-1)(x-t)}. \tag{8.2.36}$$

The Lagrangian for the P_{VI} equation is found to be

$$L_{VI}(t,x,\dot{x}) = \frac{t(t-1)}{x(x-1)(x-t)}\frac{\dot{x}^2}{2} + \int \frac{t(t-1)}{x(x-1)(x-t)}(-B(t,x))dx + \frac{dG}{dt}$$

$$L_{VI}(t,x,\dot{x}) = \frac{t(t-1)}{x(x-1)(x-t)}\frac{\dot{x}^2}{2} + \frac{\alpha x}{t(t-1)} - \frac{\beta}{x(t-1)} - \frac{\gamma}{t(x-1)} - \frac{\delta}{x-t} + \frac{dG}{dt}. \quad (8.2.37)$$

Let p be the conjugate momentum. With

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{t(t-1)}{x(x-1)(x-t)}\dot{x}$$

the corresponding Hamiltonian is

$$H_{VI} = \frac{t(t-1)}{x(x-1)(x-t)}\frac{p^2}{2} - \frac{\alpha x}{t(t-1)} + \frac{\beta}{x(t-1)} + \frac{\gamma}{t(x-1)} + \frac{\delta}{x-t}. \quad (8.2.38)$$

8.3 Equations of the Liénard type

In a series of interesting papers Chandrasekhar *et al* have studied many nonlinear equations of the oscillator type, using an extension of the Prelle-Singer method [8, 9, 10]. We investigate below one such generic equation of the Liénard type,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad (8.3.1)$$

from the perspective of the Jacobi Last Multiplier.

8.3.1 Lagrangian for second-order equations of Liénard type

From (8.2.2) the last multiplier for the equation (8.3.1) is given by $M = \exp(\int f(x)dt)$. Following [78] we introduce a new variable v by setting

$$\int f(x)dt = \log\left(v^{-\alpha^{-1}}\right) \quad (8.3.2)$$

which implies

$$\dot{v} + \alpha f(x)v = 0, \quad (8.3.3)$$

with α being a non zero scalar to be determined. As a result we have

$$M = v^{-\alpha^{-1}}. \quad (8.3.4)$$

Indeed, if we can map the original equation, (8.3.1), to the first-order equation (8.3.3) in terms of the variable v, then a suitable Lagrangian can be easily deduced. It is obvious that v must be linear in $\dot{x}$. In fact it is shown in [78] that such a map exists and is given by

$$v = \dot{x} + \frac{g}{\alpha f} \quad (8.3.5)$$

provided f and g satisfy the condition

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \alpha(1-\alpha)f. \tag{8.3.6}$$

From (8.3.4), since $M = \partial^2 L/\partial \dot{x}^2$, we find that

$$L = \frac{1}{(2-\alpha^{-1})(1-\alpha^{-1})} v^{2-\alpha^{-1}} + f_1 v + f_2. \tag{8.3.7}$$

Substituting this into the Euler-Lagrange equation leads to the condition

$$f_{1t} - f_{2x} = \frac{d}{dx}\left(f_1 \frac{g}{\alpha f}\right),$$

which may be satisfied by setting $f_1 = G_x$ and $f_2 = G_t + f_3$ yielding

$$f_{3x} = -\frac{d}{dx}(G_x \frac{g}{\alpha f}) \Rightarrow \quad f_3 = -G_x \frac{g}{\alpha f}.$$

The simple choice $G_x = 0$, i.e., $f_1 = 0$ gives, $f_3 = 0$ and $f_2 = dG/dt$. Thus

$$L = \frac{1}{(2-\alpha^{-1})(1-\alpha^{-1})} \left(\dot{x} + \frac{g}{\alpha f}\right)^{2-1/\alpha} + \frac{dG}{dt}, \quad \alpha \neq 0, \frac{1}{2}, 1. \tag{8.3.8}$$

We can rescale the Lagrangian to get rid of the inconsequential scalar factors and also drop the total time derivative term to get it into the neater form

$$L = \left(\dot{x} + \frac{g}{\alpha f}\right)^{2-1/\alpha}. \tag{8.3.9}$$

This Lagrangian, being invariant under time translation, admits a Noether symmetry with corresponding conserved quantity or first integral (disregarding overall scalar factors)

$$I = \left(\dot{x} + \frac{g}{\alpha f}\right)^{1-1/\alpha} \frac{(\alpha-1)f\dot{x} - g}{f}. \tag{8.3.10}$$

Example 8.3.1

We consider a generic equation of nonlinear oscillator type given by

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + (k_3 x^{2q+1} + k_4 x^{q+1} + k_5 x) = 0. \tag{8.3.11}$$

This includes many subcases depending upon the choice of the k_i, which are parameters. The case $q = 0$ corresponds to a damped harmonic oscillator, while $q = 1$ corresponds to the force-free Helmholtz oscillator. Substituting f and g from (8.3.11) into the condition (8.3.6), we obtain the following equations from the different coefficients of x.

$$\alpha(1-\alpha) = (q+1)\frac{k_3}{k_1^2} \tag{8.3.12}$$

$$\alpha(1-\alpha) = \frac{k_5}{k_2^2} \tag{8.3.13}$$

$$k_1 k_4 + k_2 k_3 (2q+1) = \alpha(1-\alpha) k_1^2 k_2 \tag{8.3.14}$$

$$k_1 k_5 (1-q) + k_2 k_4 (1+q) = 3\alpha(1-\alpha) k_1 k_2^2. \tag{8.3.15}$$

Equating (8.3.12) and (8.3.13) we find that

$$q + 1 = \frac{k_1^2 k_5}{k_2^2 k_3}. \tag{8.3.16}$$

Using this value of q in the remaining eqns. (8.3.14) and (8.3.15) while eliminating α by means of (8.3.13), we get

$$k_5 = \frac{k_2}{k_1^2}(k_1 k_4 - k_2 k_3). \tag{8.3.17}$$

The constant α is determined from the quadratic equation (8.3.13) and is given by

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_5}{k_2^2}}\right), \tag{8.3.18}$$

where k_5 is given in (8.3.17). Given q there exists another relation between the k_i $(i = 1, ..., 5)$ derivable from (8.3.16) and (8.3.17), *viz*

$$\frac{k_1 k_4}{k_2 k_3} = q + 2. \tag{8.3.19}$$

Thus of the five parameters k_i $(i = 1, ..., 5)$ only three are independent and to summarize we have the following relations :

$$k_4 = \frac{k_2 k_3}{k_1}(q+2)$$

$$k_5 = \frac{k_2^2 k_3}{k_1}(q+1)$$

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_3}{k_1^2}(q+1)}\right).$$

Special cases

When $q = 0$, we have $k_1k_4 = 2k_2k_3$ and $k_5 = k_2^2k_3/k_1^2$. Consequently $\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_3}{k_1^2}}\right)$ and the equation $\ddot{x} + (k_1 + k_2)\dot{x} + (k_3 + k_4 + k_5)x = 0$, which is simply the damped harmonic oscillator, has Lagrangian

$$L = \left(\dot{x} + \frac{(k_3 + k_4 + k_5)x}{\alpha(k_1 + k_2)}\right)^{2-1/\alpha}.$$

When $q = 1$, we have $k_1k_4 = 3k_2k_3$ and $k_5 = 2k_2^2k_3/k_1^2$ while $\alpha = \frac{1}{2}(1 \pm \sqrt{1 - 8k_3/k_1^2})$. The Lagrangian for the equation, $\ddot{x} + (k_1x + k_2)\dot{x} + k_3(x^3 + 3k_2/k_1x^2 + 2k_2^2/k_1^2x) = 0$ is

$$L = \left\{\dot{x} + \frac{k_3}{\alpha k_1}(x^2 + 2k_2/k_1x)\right\}^{2-1/\alpha}.$$

From this Lagrangian one can easily compute the conjugate momentum to obtain the corresponding Hamiltonian.

8.4 A system of second-order coupled equations

The extension of the above technique to a system of second-order ODEs is also possible under certain conditions. We describe below the formulation as presented in [80]. In the case of a system of n degrees of freedom the Lagrangian $L = L(t, q, \dot{q})$, where $q = \{q_1, \cdots, q_n\}$ and $\dot{q} = \{\dot{q}_1, \cdots, \dot{q}_n\}$ define the generalized coordinates and corresponding velocities, we may define the ijth Jacobi Last Multiplier by

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \ldots, n. \tag{8.4.1}$$

It is assumed that the equations of motion:

$$\ddot{q}_k = w_k(t, q, \dot{q}), \quad k = 1, \ldots, n \tag{8.4.2}$$

are derivable from the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \ldots, n. \tag{8.4.3}$$

It is evident that the conjugate momenta are

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = p_j(t, q, \dot{q}), \quad j = 1, \ldots, n$$

which implies

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial t} + \sum_{k=1}^{n}\left(\dot{q}_k \frac{\partial p_j}{\partial q_k} + w_k \frac{\partial p_j}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_j}.$$

This means

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)+\sum_{k=1}^{n}\left(\dot{q}_k\frac{\partial^2 L}{\partial q_k\partial \dot{q}_j}+w_k\frac{\partial p_j}{\partial \dot{q}_k\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j},\quad j=1,\ldots,n. \tag{8.4.4}$$

Differentiating (8.4.4) with respect to $\dot{q}_i$ and using the definition of the Last Multiplier given in (8.4.1) we find

$$\frac{\partial M_{ij}}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ij})+\frac{\partial}{\partial \dot{q}_k}(w_k M_{ij})\right)+\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_i}M_{kj}-\frac{\partial w_k}{\partial \dot{q}_k}M_{ij}\right)+\frac{\partial^2 L}{\partial q_i\partial \dot{q}_j}-\frac{\partial^2 L}{\partial \dot{q}_i\partial q_j}=0. \tag{8.4.5}$$

Interchanging i and j in (8.4.5) we get

$$\frac{\partial M_{ji}}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ji})+\frac{\partial}{\partial \dot{q}_k}(w_k M_{ji})\right)+\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_j}M_{ki}-\frac{\partial w_k}{\partial \dot{q}_k}M_{ji}\right)+\frac{\partial^2 L}{\partial q_j\partial \dot{q}_i}-\frac{\partial^2 L}{\partial \dot{q}_j\partial q_i}=0. \tag{8.4.6}$$

Adding (8.4.5) and (8.4.6) and making use of the fact that $M_{ij}=M_{ji}$ we have

$$\frac{\partial M_{ij}}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ij})+\frac{\partial}{\partial \dot{q}_k}(w_k M_{ij})\right)+\sum_{k=1}^{n}\left(\frac{1}{2}\left(\frac{\partial w_k}{\partial \dot{q}_i}M_{kj}+\frac{\partial w_k}{\partial \dot{q}_j}M_{ki}\right)-\frac{\partial w_k}{\partial \dot{q}_k}M_{ij}\right)=0. \tag{8.4.7}$$

It is evident that M_{ij} satisfies the defining relation (2.7.29) for the JLM whenever

$$\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_i}M_{kj}+\frac{\partial w_k}{\partial \dot{q}_j}M_{ki}\right)=2\sum_{k=1}^{n}\frac{\partial w_k}{\partial \dot{q}_k}M_{ij}\ \text{ for each }\ k=1,\ldots,n. \tag{8.4.8}$$

A trivial way to ensure this condition is satisfied is to assume the w_k's to be velocity independent, i.e.,

$$\frac{\partial w_k}{\partial \dot{q}_l}=0\quad\text{for all }\ k,l=1,\ldots,n.$$

On the other hand, when $i=j$, the last two terms in (8.4.5) cancel leaving

$$\frac{\partial M_{ii}}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ii})+\frac{\partial}{\partial \dot{q}_k}(w_k M_{ii})\right)+\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_i}M_{ki}-\frac{\partial w_k}{\partial \dot{q}_k}M_{ii}\right)=0. \tag{8.4.9}$$

Here also M_{ii} satisfies (2.7.29) when the last sum of (8.4.9) vanishes, which may be ensured by choosing the w_k's to be velocity independent. Under these circumstances all the M_{ij}'s satisfy the equation

$$\frac{\partial M_{ii}}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ii})+\frac{\partial}{\partial \dot{q}_k}(w_k M_{ii})\right)=0, \tag{8.4.10}$$

as they should, provided $\partial w_k/\partial \dot{q}_j = 0$ for all $k, j = 1, \ldots, n$. With this assumption equations (8.4.7) and 8.4.10) always admit the solution M_{ij} = constant. The following examples illustrate how simple choices of M_{ij} can be made to obtain the Lagrangians of second-order ODEs satisfying the above velocity-independent criterion.

Example 8.4.1

Consider the system

$$\ddot{x} + \frac{\alpha}{x^2} g(y/x) - \frac{\lambda}{x^3} = 0, \qquad \ddot{y} + \frac{\beta}{x^2} f(y/x) - \frac{\mu}{y^3} = 0.$$

Here $w_1(x,y) = -\alpha g(y/x)/x^2 + \lambda/x^3$ and $w_2(x,y) = -\beta f(y/x)/x^2 + \mu/y^3$ respectively. On the other hand α, β, λ and μ are arbitrary parameters while g and f are functions with argument $u = y/x$. Notice that w_1 and w_2 are independent of the velocities. The Jacobi Last Multiplier for this system is therefore a solution of the equation,

$$\frac{\partial M}{\partial t} + \frac{\partial (M\dot{x})}{\partial x} + \frac{\partial (M\dot{y})}{\partial y} + \frac{\partial (M w_1)}{\partial \dot{x}} + \frac{\partial (M w_2)}{\partial \dot{y}} = 0, \tag{8.4.11}$$

and admits constant solutions. We choose them as follows:

$$M_{xy} = M_{yx} = 0 \ \text{ and } \ M_{xx} = M_{yy} = 1. \tag{8.4.12}$$

These yield the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + h_1(t,x,y)\dot{x} + h_2(t,x,y)\dot{y} + h_3(t,x,y). \tag{8.4.13}$$

Substitution of this into the Euler-Lagrange equations for x and y gives, upon using the original equations of motion,

$$h_{1t} - h_{3x} + w_1 + (h_{1y} - h_{2x})\dot{y} = 0 \tag{8.4.14}$$

$$h_{2t} - h_{3y} + w_2 + (h_{2x} - h_{1y})\dot{x} = 0. \tag{8.4.15}$$

Equating the coefficients of $\dot{x}$ and $\dot{y}$ respectively we get the following set of equations:

$$h_{1y} - h_{2x} = 0 \ \text{ which implies } \ h_1 = G_x, \quad h_2 = G_y \ \text{ and} \tag{8.4.16}$$

$$h_{1t} - h_{3x} + w_1 = 0 \tag{8.4.17}$$

$$h_{2t} - h_{3y} + w_2 = 0. \tag{8.4.18}$$

These in turn give

$$h_{3x} = G_{xt} + w_1 \ \text{ or } \ h_3 = G_t + \int w_1 dx + r(y) \tag{8.4.19}$$

$$h_{3y} = G_{yt} + w_2 \text{ or } h_3 = G_t + \int w_2 dy + s(x). \tag{8.4.20}$$

Consistency for h_3 requires that

$$h_{3xy} = h_{3yx}$$

and translates into the requirement that $w_{1y} = w_{2x}$. This imposes the following condition on the functions f and g which define the second-order system:

$$\frac{\alpha}{\beta}g'(u) + uf'(u) + 2f(u) = 0, \quad \text{where } u = \frac{y}{x}.$$

One can rewrite this as

$$\frac{\alpha}{\beta}ug'(u) + \frac{d}{du}(u^2 f(u)) = 0. \tag{8.4.21}$$

When we use the explicit forms of w_1 and w_2 and make use of the last condition, the form of the functions $r(y)$ and $s(x)$ occurring in (8.4.19) and (8.4.20) may be fixed and the functional form of h_3 is found to be

$$h_3(t,x,y) = G_t - \left[\frac{\alpha}{2x^2} + \frac{\mu}{2y^2} - \frac{1}{x}\left(\alpha g(y/x) + \beta\frac{y}{x}f(y/x)\right)\right].$$

Therefore the Lagrangian is given by

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left[\frac{\alpha}{2x^2} + \frac{\mu}{2y^2} - \frac{1}{x}\left(\alpha g(y/x) + \beta\frac{y}{x}f(y/x)\right)\right] + \frac{dG}{dt}. \tag{8.4.22}$$

Again the total derivative term, being of little physical significance in the classical case, may be safely discarded.
It is interesting to note that the above second-order system, though similar in some respects to equations of the Ermakov system, is not merely a mathematical artifact. It is similar in structure to the system studied in [98] in the context of the dynamics of stellar systems.
A similar exercise may be carried out for the following.

Example 8.4.2 ***(Generalized Van der Waals Potential)***

$$\ddot{x} = -\left(2\gamma x + \frac{x}{r^3}\right) = w_1(x,y)$$
$$\ddot{y} = -\left(2\gamma\beta^2 y + \frac{y}{r^3}\right) = w_2(x,y) \text{ where } r = \sqrt{x^2+y^2}$$

and γ, β are parameters. In this case the Lagrangian is given by

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left[\gamma(x^2 + \beta^2 y^2 - \frac{1}{r}\right] + \frac{dG}{dt}. \tag{8.4.23}$$

Similarly for the

Example 8.4.3 ***(Henon-Heiles system)***

$$\ddot{x} = -(Ax + 2\alpha xy)$$
$$\ddot{y} = -(By + \alpha x^2 - \beta y^2), \tag{8.4.24}$$

the Lagrangian is given by

$$L(t, x, \dot{x}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \left(A\frac{x^2}{2} + B\frac{y^2}{2} + \alpha x^2 y - \beta\frac{y^3}{3}\right) + \frac{dG}{dt}. \tag{8.4.25}$$

In this chapter we have discussed applications of the Jacobi Last Multiplier for the deduction of Lagrangian functions for the second-order ODEs of the Painlevé-Gambier classification. We have specifically deduced the Lagrangians/ Hamiltonians for the six Painlevé equations as also other prototype equations of the Painlevé-Gambier classification. In addition we have used the above technique to analyse a particular class of coupled second-order equations. Besides the well-known Henon-Heiles system we have also obtained the Lagrangian for a relatively less studied systems occurring in the context of stellar dynamics. The Lagrangians discussed here are found to admit a Noetherian symmetry, with an associated first integral, which are the Hamiltonians of the equations concerned.

Chapter 9

Conclusion

In this work we have focused on a number of features associated with ordinary differential equations. Foremost among these is the issue of obtaining first integrals for ODEs. To this end we have exploited several tools such as Darboux polynomials, the Prelle-Singer semi algorithm and generalised Sundman transformations. Much of our work has also been devoted to the Painlevé-Gambier class of equations as stated in Ince's classic text. We also obtained first integrals for a generalized Raychaudhuri equations which has appeared in string theory.

We have succeeded in unravelling the relationship between the so called extended Prelle-Singer method and the adjoint symmetry equation. Thereby setting at rest any speculation regarding the efficacy of Lie symmetries. In addition we have extended the analysis of Lie Symmetries by including a chapter on the so called λ-symmetries and have used them to analyse some of the Painlevé-Gambier equations; especially equations III, VIII, XIX and XXX. Extension to third-order equations have also been considered in a limited number of cases.

As an extension of the generalised Sundman transformation we have introduced the notion of nonlocal transformations for both dependent and independent variables and have demonstrated their applications in the linearization of second and higher-order differential equations. In the process the problem of obtaining first integrals for such equations have also been tackled.

The concept of Jacobi's last multiplier, though it has been in existence for more than a hundred years, appears not to be too well known to the wider community of Physicists or Mathematicians. It is hoped that though this work the notion of the Jacobi Last Multiplier especially its use in the context of Lagrangian dynamics will be appreciated. Besides focusing its original role in determining the integrability of a system of first-order ODEs we have also brought to light its relationship with Lie symmetries as well as its role in the deduction of first integrals. As mentioned earlier the Painlevé-Gambier equations form a recurring theme of our analysis of ODEs and consequently it is only appropriate that we should end this work with a derivation of Lagrangians and Hamiltonians of the six Painlevé transcendents making use of Jacobi Last multipliers.

Bibliography

[1] Aguirre M and Krause J, *Sl(3,R) as the group symmetry transformation for all one-dimensional linear system,* J. Math. Phys. **29** (1988), 9-15.

[2] Anco S C and Bluman G, *Integrating factors and first integrals for ordinary differential equations,* Euro. J. of Applied Mathematics. **9** (1998), 245-259.

[3] Bianchi L, *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*, Pisa: Enrico Spoerri; 1918.

[4] Bountis T C, Ramani A, Grammaticos B and Dorizzi B, *On the complete and partial integrability of non-Hamiltonian systems*, J. Phys. A **128** (1984), no. 1-2, 268–288.

[5] Cariñena J F, Rañada M F and Santander M, *Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability*, J. Math. Phys. **46** (2005), 062703.

[6] Cariñena J F, Guha P and Rañada M F, *A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion*, Journal of Physics Conference Series **175** (2009), 012009.

[7] Cartan E, Les problems d'équivalence. Oeuvres complétes, vol. 2 (1953), pp. 1311-1334.

[8] Chandrasekar V K, Senthilvelan M and Lakshmanan M, *Extended Prelle-Singer method and integrability/solvability of a class of nonlinear nth order ordinary differential equations*, J. Nonlinear Math. Phys. **12** (2005), suppl. 1, 184–201.

[9] Chandrasekar V K, Senthilvelan M and Lakshmanan M, *A unification in the theory of linearization of second-order nonlinear ordinary differential equations*, J. Phys. A **39** (2006), no. 3, L69–L76.

[10] Chandrasekar V K, Senthilvelan M and Lakshmanan M, *A simple and unified approach to identify integrable nonlinear oscillators and systems*, J. Math. Phys. **47** (2006), no. 2, 023508, 37 pp.

[11] Chandrasekhar V K, Senthilvelan M, Kundu A and Lakshmanan M, *A nonlocal connection between certain linear and nonlinear ordinary differential equations/oscillators*, J. Phys. A Math. Gen **39**(2006), 9743-9754.

[12] Chandrasekhar V K, Senthilvelan M, Lakshmanan M, *On the complete integrability and linearization of nonlinear ordinary differential equations. II. Third order equations*, Proc. R. Soc A **762** (2006), 1831-1852.

[13] Chavarriga J and Grau M, *Some open problems related to 16th Hilbert problem*, Scientia, Series A: Mathematical Sciences, Vol. **9** (2003), 1-26.

[14] Christopher C, *Liouvillian first integrals of second order polynomial differential equations*, Electron. J. Differential Equations No. **49** (1999), 7 pp. (electronic).

[15] Christofer C and Llibre J, *Algebraic aspects of integrability for polynomial systems*, Qualitative theory of Dynamical sys. **1** (1999), 71-95.

[16] Christofer C and Llibre J, *Integrability via invariant algebraic curves for planar polynomial differential systems*, Ann. of Diff. Eqns. **16** (2000), 5-19.

[17] Clarkson P A, *Painlevé equations-nonlinear special functions*, Lecture notes on mathrematics, **1883** (2006), springer, Berlin, 331-411.

[18] Collins C B, *Two-dimensional homogeneous polynomial vector fields with common factors*, J. Math. Anal. Appl. **181** (1994), no. 3, 836–863.

[19] Dasgupta A, Nandan H and Kar S, *Kinematics of flows on curved deformable media*, Int. J. Geom. Meth. Mod. Phys. **6** (2009), 645-666.

[20] Davis H, *Introduction to nonlinear differential and integral equations*, Dover New York, 1960.

[21] Darboux G, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré.*, Bull. Sci. Math. (2) **2** (1878), 60–96, 123–144, 151–200.

[22] Dorizzi B, Grammaticos B, Hietarinta J, Ramani A, Schwartz F, *New integrable three-dimensional quartic potentials*, Phys. Lett. A **1**16 (1986), no. 9, 432–436.

[23] Duarte L G S, Duarte S E S, da Mota L A C and Skea J E F, *Solving second-order ordinary differential equations by extending the Prelle-Singer method*, J. Phys. A **34** (2001), no. 14, 3015–3024.

[24] Duarte L G S, Duarte S E S, da Mota L A C, *Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution*, J. Phys. A **35** (2002), no. 4, 1001–1006.

[25] Duarte L G S, Moreira I C and Santos F C, *Linearization under non-point transformations*, J.Phys. A **27** (1994), L739-L743.

[26] Duarte L G S and da Mota L A C P, *Prelle-Singer Approach and Liouvillian Solutions: An Important Result*, J. Math. Phys. **50** (2009), 013514.

[27] Euler N and Euler M, *Sundman symmetries of nonlinear second-order and third-order ordinary differential equations*, J. Nonlinear Math. Phys. **11** (2004), no. 3, 399–421.

[28] Euler N, Wolf T, Leach P G L and Euler M, *Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case* $X''' = 0$, Acta Appl. Math. 76 (2003), no. 1, 89–115.

[29] Euler M and Euler N, Strömberg, A., Ström, E. *Transformation between a generalized Emden-Fowler equation and the first Painlevé transcendent*, Math. Methods Appl. Sci. **30** (2007), no. 16, 2121–2124.

[30] Exton H, *On nonlinear ordinary differential equations with fixed critical points*, Rend de Mat **6(6)** (1973),419-62.

[31] Gambier B, *Sur les équations différentielles du second ordre et du premier degré dont l'intgrale générale est á points critique fixés*, Acta. Math. **33** (1910), 1-55.

[32] Gaeta G, *Twisted symmetries of differential equations*, J. Nonlinear Math. Phys. 16 (2009), 107.

[33] Giacomini H J, Repetto C E and Zandron O P, *Integrals of motion for three dimensional non Hamiltonian dynamical systems,* J.Phys. A Math. Gen. **24** (1991), 4567-4574.

[34] Giacomini H J and Neukirch S, *Integrals of motion and the shape of the attractor for the Lorenz model*, Phys. Lett. A **227** (1997), no. 5-6, 309–318;

[35] Giacomini H J and Neukirch S, *Number of limit cycles of the Liénard equation*, Phys. Rev. E (3) **56** (1997), no. 4, 3809–3813.

[36] Gordon T J, *On the symmetries and invariants of the harmonic oscillator,* J. Phys. A. **19**(1986), 183-189.

[37] Goriely A, *Integrability and nonintegrability of dynamical systems*, Advanced Series in Nonlinear Dynamics, 19. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xviii+415 pp.

[38] Grammaticos B, Moulin-Ollagnier J, Ramani A, Strelcyn A and Wojciechowski S, *Integrals of quadratic ordinary differential equations in* R^3*: the Lotka-Volterra system*, Phys. A **163** (1990), no. 2, 683–722.

[39] Hietarinta J, *Direct methods for the search of the second invariant*, Phys. Rep. **147** (1987), no. 2, 87–154.

[40] Hirsch M W and Smale S, *Differential equation, Dynamical Systems and Linear Algebra*, (Academic, New York, 1974).

[41] Hydon P E, *Symmetry Methods for Differential Equations*, Cambridge Univ. Press. Cambridge, UK. 1999.

[42] Ibragimov N H, S V Meleshko S V and Suksern S *Linearization of fourth-order ordinary differential equations by point transformations*, J. Phys. A: Math. Theor. **39**(2006), 15135-15145.

[43] Ibragimov N H and Meleshko S V, *Linearization of third-order differential equation by point and contact transformations*, J. Math. Anal. Appl. **308** (2005), 266-289.

[44] Ince E L, *Ordinary Differential Equations*, New York: Dover, reprinted 1956.

[45] Jacobi C G J, *Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica,* Giornale Arcadico di Scienze, Lettere ed Arti **99** (1844), 129-146.

[46] Jacobi C G J, *Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi,*J. Reine Angew. Math **27** (1844), 199-268, *Ibid* **29**(1845), 213-279 and 333-376.

[47] Jordan D W and Smith P,*Nonlinear Ordinary Differential Equations* , Oxford University Press: Reprinted 1979.

[48] Jouanolou J P, *Équations de Pfaff algébriques.* (French) [Algebraic Pfaffian equations] Lecture Notes in Mathematics, 708. Springer, Berlin, 1979. v+255 pp.

[49] Kar S and Sengupta S, *The Raychaudhuri equations: a brief review*, arXiv:gr-qc/0611123.

[50] Kennedy J, *The hydrogen atom as an example of quantization with a constraint*, Proc. R. Irish Acad. A **82** (1982), 1-8.

[51] Kleinert H, *Nonholonomic Mapping Principle for Classical and Quantum Mechanics in Spaces with Curvature and Torsion*, arXiv:gr-qc/0203029v1.

[52] Kleinert H, Chapter 10 of *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* pp. 1-1547, World Scientific, Singapore 2009.

[53] Krasilshchik I S, Vinogradov A M, *Symmetries and conservation laws for differential equations of mathematical physics*, Amer. Math. Soc., Providence, RI, 1999.

[54] Kus M, J. Phys. A: Math. Gen. **15**(1983), L389.

[55] Kustaanheimo P and Stiefel E, *Perturbation theory of Kepler motion based on spinor regularization*, J. Reine Angew. Math. **218** (1965), 204–219.

[56] Lie S, *Verhandlungen der Gesselschaft der Wissenschaften zu Christinia* (1874).

[57] Lie S, *Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x,y, die eine Gruppe von Transformationen gestatten*, Math. Ann. **32** (1888), 213-281.

[58] Lie S, *Arch. Mat. Nat.* **8**(1883), 371-427 (reprinted in Lie's Gessammelte Abhanlundgen, 1924, 5, paper XIY, pp 362-427).

[59] Llibre J and Pereira J V, *Multiplicity of invariant algebraic curves and Darboux integrability,* Arxiv.math/0009020v1[Math.Ds].

[60] Madhava Rao B S, *On the reduction of dynamical equationto the Lagrangian form*, Proc. Benaras Math. Soc.(N.S.) **2** (1940), 53-59.

[61] Mahomed F M and Leach P G L, *Lie algebras associated with second-order ordinary differential equations,* J. Math. Phys. **30** (1989), 2770-2777.

[62] Mahomed F M and Leach P G L, *The Lie algebra sl(3,R) and linearization,* Quaestiones Math. **12** (1989), 121-139.

[63] Man Y K and MacCallum M A H, *A rational approach to the Prelle-Singer algorithm*, J. Symb. Comput. **11** (1997), 1-11.

[64] Meleshko S V, *On linearization of third-order ordinary differential equations*, J. Phys. A: Math. Theor. **39**(2006), 15135-15145.

[65] Muriel C and J L Romero, *New method of reduction for ordinary differential equations*, IMA J. Appl. Math. **66** (2001), 111125.

[66] Muriel C and J L Romero, C^∞ *symmetries and nonsolvable symmetry algebras*, IMA J. Appl. Math. **66** (2001), 47798.

[67] Muriel C and J L Romero, *Integrability of equations admitting the nonsolvable symmetry algebra* $so(3,r)$, Stud. Appl. Math. **109** (2002), 33752.

[68] Muriel C and Romero J L, C^∞ *symmetries and reduction of equations without Lie-point symmetries*, J. Lie Theory **13** (2003), 16788.

[69] Muriel C and Romero J L, *First integrals, integrating factors and* λ*-symmetries of second-order differential equations*, J. Phys. A: Math. Theoret. **42** (2009), 365207.

[70] Muriel C, Romero J L and Olver P J, Variational C^{∞}-symmetries and EulerLagrange equations, Journal of Differential Equations **222** (2006), 164-184.

[71] Muriel C and Romero J L, *Second-order ordinary differential equations and first integrals of the form $A(t,x)\dot{x}+B(t,x)$,* J. Nonlinear Math. Phys. **16** (2009), 209-222.

[72] Muriel C and Romero J L, *Nonlocal transformations and linearizations of second-order ordinary differential equations,* J. Phys.A: Math. Theor **43** (2010), 434025 (13pp).

[73] Muriel C and Romero J L, *Second-order ordinary differential equations and first integral of the form $C(t) + 1/(A(t,x)\dot{x} + B(t.x))$*, J. Nonlinear Math. Phys. **18** Suppl. 1 (2011), 237-250.

[74] Muriel C and Romero J L, *λ-symmetries and integrating factors*, J. Nonlinear Math. Phys. **15** (2008), 290-299.

[75] Nakpim W and Meleshko S V, *Linearization of third order ordinary differential equations by generalized Sundman transformation: The case $X'''{}^{+}\alpha X = 0$*, C. Nonlinear Sci Numer Simulat. **15** (2010), 1717-1723.

[76] Nucci M C and Leach P G L, *Jacobi's last multiplier and the complete symmetry group of the Euler-Poinsot system,* J. Nonlin. Math. Phys. **9-s2** (2002), 110-121.

[77] Nucci M C and Leach P G L, *Jacobi's last multiplier and symmetries for the Kepler problem plus a lineal story,* J. Phys. A, Math. Gen. **37** (2004), 7743-7753.

[78] Nucci M C, Tamizhmani K M, *Lagrangian for dissipative nonlinear oscillators: the method of Jacobi last multiplier,* J. Nonlinear Math. Phys. **17** (2010), 167.

[79] Nucci M C, Tamizhmani K M, *Using an old method of Jacobi to derive Lagrangians: a nonlinear dynamical system with variable coefficients*, *arXiv:0807.2791.*

[80] Nucci M C and Leach P G L, *Jacobi's last multiplier and Lagrangians for multidimensional systems*, Journal of mathematical Physics **49** (2008), 073517.

[81] Okamoto K, *On the τ-function of the Painlevé equations*, Physica D **2** (1981), 525-535.

[82] Olver P J,*Equivalence, Invariants and Symmetry,* New York: Cambridge University Press, 1995.

[83] Olver P J, *Application of Lie Groups to Differential Equations* 1986 (Berlin: Springer).

[84] Painlevé P, *Memoire sur les quations différentielles dont l'intégrale générale est uniforme*, Bull. Soc. Math. Phys. France 28 (1900), 201-261.

[85] Painlevé P, *Sur les quations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Math. **25** (1902), 1-85.

[86] Pelster A. and Kleinert H, *Transformation Properties of Classical and Quantum Laws under Some Nonholonomic Spacetime Transformations*, arXiv:quant-ph/9608037v1.

[87] Prelle M J and Singer M F, *Elementary first integrals of differential equations,* Trans. Amer. Math. Soc. **279** (1983), no. 1, 215–229.

[88] Pucci E and Saccomandi G, *On the reduction methods for ordinary differential equations*, J. Phys. A: Math. Gen. **35** (2002), 614555.

[89] Sarlet W, Prince G E and Crampin M, *Adjoint symmetries for time-independent second-order equations,* J.Phys.A. **23**(1990), 1335-1347.

[90] Sarlet W, Cantrun F and Crampin M, *Psedo-symmetries, Noether's theorem and the adjoint equations,* J.Phys.A. **20**(1987), 1365-1376.

[91] Sarlet W, Mahomed F M and Leach P G L, *Symmetries of non-linear differential equations and linearization,* J.Phys.A. **20** (1987), 277-292.

[92] Saunders D J, *The geometry of jet bundles*, LMS lecture note series 142; CUP, Cambridge, 1989.

[93] Shilnikov A and Kolomiets M *Method of the qualitative theory for the Hindmarsh-Rose model: A case study. A tutorial*, Int. Journal of Bifrcation and Chaos, **18**, No. 8 (2008), 2141-2168.

[94] D. Schlomiuk, *Elementary first integrals and algebraic invariant curves of differential equations*, Exposition. Math. 11 (1993), no. 5, 433–454.

[95] Shtokhamer R, *Solving first-order differential equations using the Prelle-Singer algorithm*, Technical report 88-09, Center for Mathematical Computation, University of Delaware (1988).

[96] Singer M F, *Liouvillian first integrals of differential equations,* Trans. Amer. Math. Soc. **333** (1992), no. 2, 673–688.

[97] Sookme S and Meleshko S V, *Conditions for linearization of a projectable system of two second-order ordinary differential equations*, J. Phys. A: Math. Theor. **41**(2008), 402001 (7pp).

[98] Sridhar S and Nityananda R, *Undamped oscillations of collisionless systems: spheres, spheroids and discs*, J. Astrophys. Astron. **10**(1989), 279-93.

[99] Steeb W H, *Invertible Point Transformation and Nonlinear Differential Equations.* World Scientific, Singapore 1993, ISBN 981-02-1355-7.

[100] Stephani H, *Differential equations-Their solution using symmetries*, Ed. M.MacCullum, Cambridge Univ (1989).

[101] Strelcyn J M and Wojciechoski S, *A method of finding integrals for three-dimensional systems*, Phys. Letts. A, 133 (4, 5)(1988), 207-212.

[102] Sugai I, *Riccati's nonlinear differential equation*, Am. Math. Monthly **67** (1960), 134-139.

[103] Sundman, K. F. *Mémoire sur le problém des trois corps*, Acta Math. **36** (1912–1913), 105–179.

[104] Tresse A M, Détermination des Invariants Ponctules de l'Équation Différentielle ordinaire du seconde ordre $y'' = w(x, y, y')$ (Leipzig; Hirzel)(1896).

[105] Whittaker E T, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, 1988.

[106] Wolf T and Brand A, *Examples of the investigation of differential equations with modularized programs. Algorithms and software for symbolic analysis of nonlinear systems*, Math. Comput. Modelling **25** (1997), 133-139.

Reprints

Publications included in the Thesis

Journal of Nonlinear Mathematical Physics Volume 15, Number 4 (2008), 365–382 Article

Solutions of some second order ODEs by the extended Prelle-Singer method and symmetries

A Ghose Choudhury [a], *Partha Guha* [b] *and Barun Khanra* [c]

[a] *Department of Physics, Surendranath College,*
24/2 Mahatma Gandhi Road, Calcutta-700009, India
E-mail: a_ghosechoudhury@rediffmail.com

[b] *Max Planck Institute for Mathematics in the Sciences*
Inselstrasse 22, D-04103 Leipzig, Germany and
S.N. Bose National Centre for Basic Sciences
JD Block, Sector III, Salt Lake, Kolkata - 700098, India
E-mail: partha.guha@mis.mpg.de

[c] *Sailendra Sircar Vidyalaya, 62A Shyampukur Street*
Calcutta-700 004, India
E-mail: barunkhanra@rediffmail.com

Received July 18, 2008; Accepted in revised form November 28, 2008

Abstract

In this paper we compute first integrals of nonlinear ordinary differential equations using the extended Prelle-Singer method, as formulated by Chandrasekar *et al* in J. Math. Phys. 47 (2), 023508, (2006). We find a new first integral for the Painlevé-Gambier XXII equation. We also derive the first integrals of generalized two-dimensional Kepler system and the Liénard type oscillators.

1 Introduction

The problem of finding first integrals of ordinary differential equations (ODE) has a long and interesting history which may be traced to the seminal works of Darboux and Lie in the latter half of the nineteenth century. For an nth-order ODE, a first integral is an expression involving the independent variable, the dependent variable and its derivatives to order $n-1$. In fact, if r such first integrals are known for an nth-order ODE, then the latter may be reduced to an $(n-r)$th-order ODE. The familiar case of a simple harmonic oscillator, is an example of a second-order ODE with a first integral given by the mechanical energy. The latter enables us to reduce the problem to solving a first-order ODE.

Dynamical systems, described by nonlinear oscillators, are a common occurrence in many areas of physics and the applied sciences. The main difficulty involved in solving

such ODEs is that they are often nonlinear, involve several degrees of freedom, and are usually coupled to one other in a non-trivial way. Moreover, these equations are generally non Hamiltonian in nature and describe the time evolution of physical processes which are usually dissipative in character.
The techniques involved in finding first integrals of systems of one or more ODEs generally make use of integrating factors, which are functions multiplying each of the ODEs to yield a first integral.

In 1878 Darboux showed that one can construct an integrating factor (and first integrals) of a planar polynomial differential system, if there exists a sufficient number of invariant algebraic curves (real or complex). On the other hand for first-order scalar ODEs, S. Lie devised a method for constructing an integrating factor from each admitted point symmetry. Then, after almost a century, a major breakthrough in the construction of an algorithm for solving first-order ordinary differential equations was put forward by Prelle and Singer [1] in 1983. The method is a semi algorithmic procedure for solving nonlinear first-order ordinary differential equations of the form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{1.1}$$

when $P(x,y)$ and $Q(x,y)$ are coprime polynomials. The Prelle-Singer (PS) method provides the form of the integrating factor when the solution of the associated system of differential equations is expressible in terms of elementary functions. Their work has been influential in providing some of the fundamental algebraic results required for the automated solution of ODEs using computer algebra. An extension of this method provides the form of an integrating factor when the solution is expressible in terms of Liouvillian functions. Recently Duarte *et al* [10, 11] have extended the technique to second-order ODEs. Essentially, their objective was to look for a wider class of possible integrating factors. To this end, they succeeded in adding the derivatives of some rational functions, to the previously known linear combinations of logarithmic derivatives.

Most recently Lakshmanan and his coworkers have generalized and used the extended Prelle-Singer method to obtain the first integrals and general solutions for a class of nonlinear equations [6, 7, 9]. They have also devised a procedure to construct a transformation which removes the time-dependent part from the first integral and provides the general solution by quadrature [8]. This procedure is shown to have a wider applicability through several examples.

In this paper we use the extended Prelle-Singer method to derive the first integrals of the Painlevé-Gambier class of ODEs. We derive a new first integral for the Painlevé-Gambier XXII equation. Using this method we also show how the known first integrals of a large class of equations, of a specific form, in the Painlevé-Gambier classification may be deduced. In addition we analyze a Liénard type equation, (second order Riccati equation) and a generalized two-dimensional Kepler system.

The organization of the paper is as follows. In Section 2 we briefly recollect the basic results involved in Darboux integrability and give a brief introduction to the Prelle-Singer method, including the relevant definitions and results related to PS method. In Section 3, we review the extended PS method, as developed by Chandrasekar *et al.* In Section 4 we discuss applications of the extended PS method to ODEs of the Painlevé-Gambier classification. Section 5 contains a discussion on second-order Liénard type equations. Finally, in Section 6 we briefly consider applications to systems of second-order ODEs and illustrate it with an example of a generalized two-dimensional Kepler system. We finish our paper with a modest outlook.

2 Preliminaries

Let us consider planar polynomial differential systems

$$\dot{x} = Q(x,y) \text{ and } \dot{y} = P(x,y), \tag{2.1}$$

where $P(x,y) = \sum_{i=0}^{m} P_i(x,y), Q(x,y) = \sum_{i=0}^{m} Q_i(x,y)$ are coprime polynomials in $\mathbb{C}$ such that max $\{deg\, P, deg\, Q\} = m$ and $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous components of degree i. This differential system (2.1) may be described either by the vector field

$$D = Q(x,y)\frac{\partial}{\partial x} + P(x,y)\frac{\partial}{\partial y}, \tag{2.2}$$

or the differential form

$$\omega = Pdx - Qdy.$$

The corresponding phase flow is given by the solution of the first-order ordinary differential equation

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}. \tag{2.3}$$

Definition 2.1. Let U be an open subset of $\mathbb{K}^2$. We say that a nonconstant function $I : U \to \mathbb{K}$ is a first integral of a vector field D on U, if and only if, $D|_U(I) = 0$.

The tangents to the trajectories of a planar polynomial differential system are defined everywhere [12]. If $f(x,y) = 0$ is the equation of an *invariant curve*, its tangent must coincide with the tangents of the trajectories. In other words, the gradient to f, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and (Q,P) must be orthogonal over the curve $f(x,y) = 0$:

$$\dot{f} = (Q\frac{\partial f}{\partial x} + P\frac{\partial f}{\partial y})_{f=0} = 0.$$

Definition 2.2. An invariant curve $f(x,y) = 0$ is called an *algebraic curve* or Darboux polynomial of degree m when $f(x,y)$ is a polynomial of degree m.

Definition 2.3. Let D be the vector field associated with a differential equation. A curve $f(x,y) = 0$ is an *invariant algebraic curve* if $D[f]/f$ is a polynomial. The latter polynomial $\lambda_f = D[f]/f$ is usually called the *cofactor* of the invariant algebraic curve or *Darboux polynomial.*

2.1 Darboux method

The Darboux method of constructing integrating factors and first integrals of planar ODEs, relies essentially on the existence of invariant algebraic curves (or Darboux polynomials) [3]. Furthermore, the maximum degree of the invariant algebraic curves is bounded [4]. Suppose the vector field D admits s distinct invariant algebraic curves $f_i \quad i = 1, ..., s$.
(a) If there are $n_i \in \mathbb{C}$ not all zero, such that

$$\sum_{i=1}^{s} n_i \lambda_{f_i} = 0 \text{ then the function } \prod_{i=1}^{s} f_i^{n_i} \text{ is a first integral of the vector field } D. \tag{2.4}$$

(b) If there exists $n_i \in \mathbb{C}$ not all zero, such that

$$\sum_{i=1}^{s} n_i \lambda_{f_i} = -\text{div}D \text{ then the function } \prod_{i=1}^{s} f_i^{n_i} \text{ is an integrating factor of } D. \tag{2.5}$$

These results form the essential content of Darboux integrability.

2.1.1 Extension of the Darboux method

If $f, g \in \mathbb{C}[x, y]$, then $e = \exp(\frac{g}{f})$ is an exponential factor of the vector field D of degree d if $D(e)/e$ is a polynomial of degree at most $d - 1$. Thus there are two major kinds of first integrals (1) Rational and (2) Darbouxian $\Rightarrow f^{\nu}(\exp(\frac{h}{g}))^{\mu} \quad \nu, \mu \in \mathbb{C}$.

2.2 The Prelle-Singer method

In 1983 Prelle and Singer [1, 2] devised a procedure which could not only determine polynomial first integrals but more importantly could be applied to systems admitting rational first integrals.

Suppose

$$\dot{x} = Q(x, y) \text{ and } \dot{y} = P(x, y) \tag{2.6}$$

is a system of first order ODEs. The vector field associated with this system is

$$D = Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}. \tag{2.7}$$

Since this is a first-order ODE, it is integrable on an open subset U of $\mathbb{K}^2$, if there exists a first integral of the system on U.

Definition 2.4. A non zero function, $R : U \to \mathbb{K}$, is an integrating factor of a vector field D on U if and only if $D(R) = -div\,(D) \cdot R$ on U.

When an integrating factor is known, we can compute by quadrature, a first integral of the system up to a constant. Let us therefore assume that, we have identified a sufficient number of Darboux polynomials f_i satisfying

$$D[f_i] = \lambda_i f_i, \tag{2.8}$$

where the λ_i are cofactors. From (2.5), we have

$$\frac{D[R]}{R} = \sum_i n_i \frac{D[f_i]}{f_i} = -(Q_x + P_y). \tag{2.9}$$

Clearly Q_x and P_y are polynomials since Q and P are themselves polynomials; and therefore it is necessary that f_i divides $D[f_i]$. If we manage to find such Darboux polynomials, then all that remains is to determine the numbers n_i such that (2.9) is satisfied. This can be achieved by equating terms of various orders $x^\alpha y^\beta$ on either side and finding a consistent set of values for the n_i. The problem lies in determining the f_i. The Prelle-Singer method provides a semi algorithm for determining these whenever there exists a first integral which is an elementary function. This involves establishing bounds of different orders on the f_i. For example, we start with $N = 1$ and assume $f = \alpha x + \beta y + \gamma$; next we check for what values of α, β and γ, f divides $D[f]$. If we fail to find such an f, we go to the next level and set $N = 2$ try $f = \alpha x^2 + 2\beta xy + \gamma y^2 + \delta x + \epsilon y + \mu$ and find a particular combination which divides $D[f]$ and so on. It is clear that the process is semi algorithmic by its very nature.

While the Prelle-Singer method allows for the determination of first integrals for many planar systems, it is not applicable to linear first-order ODEs having exponential integrating factors. In [10, 11] Duarte *et al* have extended the PS method to include such situations. Subsequently in a series of papers [6, 8, 9] Chandrasekar *et al* have uncovered rational and even non rational first integrals for a large class of oscillator type equations, by appropriately modifying and also extending the basic idea behind the PS procedure.

3 The Extended Prelle-Singer method

Consider a second-order ODE of the generic form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{3.1}$$

In the existing literature such equations are called Lienard type ODEs and include a number of important physical systems:

1. $f(x) = k, \quad g(x) = w_0^2 x, \ \Rightarrow \ \ddot{x} + k\dot{x} + w_0^2 x = 0$ *damped harmonic oscillator.*
2. $f(x) = \alpha x, \quad g(x) = \beta x^3, \ \Rightarrow \ \ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$ *Modified Emden equation.*
3. $f(x) = \alpha + \beta x^2, \quad g(x) = -\gamma x + x^3, \ \Rightarrow \ \ddot{x} + (\alpha + \beta x^2)\dot{x} - (\gamma x + x^3) = 0$ *Duffing Van der Pol oscillator.*
4. $f(x) = (k_1 x^q + k_2), \ g(x) = k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x$, where $q \in \mathbb{R}$

The last case includes many systems like the anharmonic oscillator force free Helmholtz and Duffing oscillator as special cases. In [9], the authors have studied this system for $q =$ arbitrary and deduced a number of new completely integrable cases.

3.1 Formulation

Let us briefly review the method used by the authors to deduce first integrals of oscillator type systems under very general conditions [7, 8, 9]. According to these calculations the equation of motion for the second-order ODE is written in the form:

$$\ddot{x} = \phi(x, \dot{x}). \tag{3.2}$$

This may be recast as a system of first order ODEs

$$\dot{x} = y, \quad \dot{y} = \phi(x, y) \tag{3.3}$$

or as a pair of differential one forms:

$$Sdx = Sydt \tag{3.4}$$

$$dy = \phi dt. \tag{3.5}$$

Here S is an unknown function of x, y which must be determined. Addition of (3.4), (3.5) leads to

$$(Sy + \phi)dt = Sdx + dy.$$

Assuming R to be an integrating factor of this equation we have upon multiplication

$$R(Sy + \phi)dt - RSdx - Rdy = 0, \tag{3.6}$$

which implies that if $I(t, x, y)$ be the corresponding first integral such that

$$I_t dt + I_x dx + I_y dy = 0$$

we must have

$$I_t = R(Sy + \phi), \quad I_x = -RS, \quad I_y = -R. \tag{3.7}$$

The compatibility of these equations requires

$$I_{xy} = I_{yx}, I_{tx} = I_{xt} \text{ and } I_{ty} = I_{yt}. \tag{3.8}$$

From these conditions it is straightforward to derive the following equations

$$D[R] = -((RS) + \phi_y R), \tag{3.9}$$

$$D[RS] = -\phi_x R. \tag{3.10}$$

Two subcases may be distinguished,
A: when $I_t = 0$, that is when the system is conservative and
B: when $I_t \neq 0$ for a non conservative system.
In case of the former, it is easy to see that $S = -\frac{\phi}{y}$. Therefore one needs to determine only the unknown function R, which is the required integrating factor. We shall analyze case A first, since it is somewhat simpler, and postpone a discussion of the latter.

For case A, (3.9) simplifies to

$$D[R] = (\frac{\phi}{y} - \phi_y)R, \tag{3.11}$$

with

$$D = y\partial_x + \phi\partial_y.$$

Substituting the ansatz

$$R = \frac{y}{T(x,y)} \tag{3.12}$$

causes (3.11) to simplify further and it reduces to

$$D[T] = yT_x + \phi T_y = \phi_y T. \tag{3.13}$$

Let us consider an example to illustrate the method developed thus far.
Example 1: Consider the equation

$$\ddot{x} + \frac{1}{2}\psi_x \dot{x}^2 + \psi_t \dot{x} + B(t,x) = 0.$$

This is equivalent to the system of equations

$$\dot{x} = y$$

$$\dot{y} = \phi(t,x,y) = -\left[\frac{1}{2}\psi_x \dot{x}^2 + \psi_t \dot{x} + B(t,x)\right]$$

so that

$$\phi_y = -(\psi_x y + \psi_t) = -D[\psi].$$

Thus (3.13) becomes $D[\log T + \psi] = 0$ which implies $T = K\exp(-\psi)$. Hence $R = \frac{y}{K}\exp(\psi) = -I_y$ implies $I = -\frac{e^{\psi}}{K}\frac{y^2}{2} + \frac{J(x)}{K}$, where K is a numerical constant. On the other hand $I_x = -RS$ implies $J'(x) = e^{\psi}(-\psi_t y - B(t,x))$. Clearly one must have $\psi_t = 0$ and $B(t,x) = B(x)$ for a time independent first integral. In that case we obtain

$$I(x,y) = -\frac{1}{K}\left[e^{\psi}\frac{y^2}{2} + \int^x e^{\psi}B dx\right]. \tag{3.14}$$

Such a first integral occurs, therefore for all equations having the generic form

$$\ddot{x} + \frac{1}{2}\psi_x \dot{x}^2 + B(x) = 0, \tag{3.15}$$

and may be a treated as a formula for deriving an time independent first integral.

4 First integrals of Painlevé-Gambier equations

It will be evident that the above method may be applied, in principle to a number of equations of the Painlevé-Gambier classification. We introduce a slight change of notation, for easy reference and illustrate this below.

4.1 Painlevé-Gambier XII equation

Let us consider the Painlevé-Gambier XII equation

$$y'' = \frac{1}{y}y'^2 + \alpha y^3 + \beta y^2 + \gamma + \frac{\delta}{y}$$

Comparison with (3.15) above indicates that $\frac{1}{2}\psi_y = -\frac{1}{y}$ and hence $e^{\psi} = y^{-2}$, while $B(x,y) = -\left[\alpha y^3 + \beta y^2 + \gamma + \frac{\delta}{y}\right]$. Then (3.14) yields the following first integral

$$y'^2 = \alpha y^4 + 2\beta y^3 - 2\gamma y - \delta + K_1 y^2$$

where we have set $K_1 = -2KI(y, y')$. We have checked that *all* the known x- independent first integrals of the Painlevé-Gambier classification [13], can be obtained from (3.14).

It is of interest to know, whether there exists other first integrals, depending perhaps on the independent variable x, for equations having a first integral given by the above formula. This brings us actually to a discussion of case B, i.e. ($I_t \neq 0$) of the previous section.

4.2 Painlevé-Gambier XXII equation

We illustrate next, the existence of an x dependent first integral for equation XXII of the Painlevé-Gambier classification:

$$\frac{d^2y}{dx^2} = \frac{3y'^2}{4y} - 1. \tag{4.1}$$

A known first integral of this equation is

$$K = \left(\frac{y'^2 - 4y}{4y^{3/2}}\right), \tag{4.2}$$

which may be obtained from (3.14).
From (3.9) and (3.10), we have

$$D[R] = -(S + \phi_{y'})R \tag{4.3}$$

$$D[RS] = -\phi_y R, \tag{4.4}$$

as a result of our change in notation. Here $D = \partial_x + y'\partial_y + \phi\partial_{y'}$ with

$$\phi(x, y, y') = \frac{3y'^2}{4y} - 1 = \phi_0(y)y'^2 - 1, \quad \phi_0(y) = \frac{3}{4y}. \tag{4.5}$$

Closer inspection of equations (4.3) and (4.4) reveals that they are a pair of coupled first order equations in the variables R and RS respectively. Assuming them to admit rational solutions of the form

$$R = \frac{f}{g} \text{ and } RS = \frac{h}{g} \;\Rightarrow\; S = \frac{h}{f}, \tag{4.6}$$

these equations become

$$gD[f] - fD[g] = -(h + \phi_{y'} f) \cdot g \tag{4.7}$$

$$gD[h] - hD[g] = -\phi_y f \cdot g. \tag{4.8}$$

From a leading order analysis of the above equations, assuming $f \sim y'^{\alpha}$, $h \sim y'^{\gamma}$ and $g \sim y'^{\beta}$ and with ϕ as in (4.5), it follows that $\gamma = \alpha + 1$ with β being arbitrary. This suggests the following ansatz for the y' dependence of the functions f, g and h namely:

$$f(y, y') = f_0 + f_1 y' + f_2 y'^2,$$

$$h(y, y') = h_0 + h_1 y' + h_2 y'^2 + h_3 y'^3,$$

$$g(y, y') = g_0 + g_1 y' + g_2 y'^2 + g_3 y'^3 + g_4 y'^4. \tag{4.9}$$

Substituting them into (4.7) and equating different powers of y' leads to the set of equations:

$$-g_0 f_1 + f_0 g_1 = -h_0 g_0, \tag{4.10}$$

$$g_0 F_1 - f_0 G_1 = -\{(h_1 + 2\phi_0 f_0) g_0 + h_0 g_1\}, \tag{4.11}$$

$$-g_2 f_1 + g_1 F_1 + g_0 F_2 + f_2 g_1 - f_1 G_1 - f_0 G_2 = -\{(h_2 + 2\phi_0 f_1) g_0 + (h_1 + 2\phi_0 f_0) g_1 + h_0 g_2\}, \tag{4.12}$$

$$\begin{aligned} &-g_3 f_1 + g_2 F_1 + g_1 F_2 + g_0 F_3 - f_2 G_1 - f_1 G_2 - f_0 G_3 \\ &= -\{(h_3 + 2\phi_0 f_2) g_0 + (h_2 + 2\phi_0 f_1) g_1 + (h_1 + 2\phi_0 f_0) g_2 + h_0 g_3\}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} &-g_4 f_1 + g_3 F_1 + g_2 F_2 + g_1 F_3 - f_2 G_2 - f_1 G_3 - f_0 G_4 = -\{(h_3 + 2\phi_0 f_2) g_1 + \\ &(h_2 + 2\phi_0 f_1) g_2 + (h_1 + 2\phi_0 f_0) g_3 + h_0 g_4\}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &g_4 F_1 + g_3 F_2 + g_2 F_3 - f_2 G_3 - f_1 G_4 - f_0 G_5 \\ &= -\{(h_3 + 2\phi_0 f_2) g_2 + (h_2 + 2\phi_0 f_1) g_3 + (h_1 + 2\phi_0 f_0) g_4\}, \end{aligned} \tag{4.15}$$

$$g_4 F_2 + g_3 F_3 - f_2 G_4 - f_1 G_5 = -\{(h_3 + 2\phi_0 f_2) g_3 + (h_2 + 2\phi_0 f_1) g_4\}, \tag{4.16}$$

$$g_4 f_{2y} - f_2 g_{4y} = -h_3 g_4. \tag{4.17}$$

where

$$F_1 = f_{0y} - 2f_2$$

$$F_2 = f_{1y} + \phi_0 f_1$$

$$F_3 = f_{2y} + \phi_0 f_2 \tag{4.18}$$

and

$$G_1 = g_{0y} - 2g_2,$$

$$G_2 = g_{1y} + \phi_0 g_1 - 3g_3,$$

$$G_3 = g_{2y} + 2\phi_0 g_2 - 4g_4$$

$$G_4 = g_{3y} + 3\phi_0 g_3$$

$$G_5 = g_{4y} + 4\phi_0 g_4. \tag{4.19}$$

On the other hand from (4.8) we obtain the following equations:

$$-h_0g_1 + g_0h_1 = 0, \tag{4.20}$$

$$h_0G_1 - g_0H_1 = 0, \tag{4.21}$$

$$-h_2g_1 + h_1G_1 + h_0G_2 + g_2h_1 - g_1H_1 - g_0H_2 = \phi_{0y}f_0g_0 \tag{4.22}$$

$$-h_3g_1 + h_2G_1 + h_1G_2 + h_0G_3 + g_3h_1 - g_2H_1 - g_1H_2 - g_0H_3 = \phi_{0y}(f_1g_0 + f_0g_1), \tag{4.23}$$

$$h_3G_1 + h_2G_2 + h_1G_3 + h_0G_4 + g_4h_1 - g_3H_1 -$$
$$g_2H_2 - g_1H_3 - g_0H_4 = \phi_{0y}(f_2g_0 + f_1g_1 + f_0g_2), \tag{4.24}$$

$$h_3G_2 + h_2G_3 + h_1G_4 + h_0G_5 - g_4H_1 - g_3H_2 - g_2H_3 - g_1H_4 = \phi_{0y}(f_2g_1 + f_1g_2 + f_0g_3), \tag{4.25}$$

$$h_3G_3 + h_2G_4 + h_1G_5 - g_4H_2 - g_3H_3 - g_2H_4 = \phi_{0y}(f_2g_2 + f_1g_3 + f_0g_4), \tag{4.26}$$

$$h_3G_4 + h_2G_5 - g_4H_3 - g_3H_4 = \phi_{0y}(f_2g_3 + f_1g_4), \tag{4.27}$$

$$h_3G_5 - g_4H_4 = \phi_{0y}f_2g_4, \tag{4.28}$$

where

$$H_1 = h_{0y} - 2h_2,$$
$$H_2 = h_{1y} + \phi_0h_1 - 3h_3,$$
$$H_3 = h_{2y} + 2\phi_0h_2$$
$$H_4 = h_{3y} + 3\phi_0h_3. \tag{4.29}$$

To solve the system of first order coupled PDEs given by (4.10)-(4.17) and (4.20)-(4.28) we observe that, one can satisfy one half of each set identically, by making a second ansatz, namely

$$f_{odd} = g_{odd} = h_{even} = 0. \tag{4.30}$$

It then follows that

$$H_1 = H_3 = G_2 = G_4 = 0,$$

and from (4.20) we find

$$h_1 = 0. \tag{4.31}$$

Taking this in to account we are now left with the following equations from the set (4.10)-(4.17):

$$g_0(f_{0y} - 2f_2) - f_0(g_{0y} - 2g_2) = -2\phi_0f_0g_0, \tag{4.32}$$

$$g_2(f_{0y} - 2f_2) + g_0(f_{2y} + 2\phi_0f_2) - f_2(g_{0y} - 2g_2) - f_0(g_{2y} + 2\phi_0g_2 - 4g_4)$$
$$= \{(h_3 + 2\phi_0f_2)g_0 + 2\phi_0f_0g_2\} \tag{4.33}$$

$$g_4(f_{0y} - 2f_2) + g_2(f_{2y} + 2\phi_0f_2) - f_2(g_{2y} + 2\phi_0g_2 - 4g_4) - f_0(g_{4y} + 4\phi_0g_4)$$
$$= \{(h_3 + 2\phi_0f_2)g_2 + 2\phi_0f_0g_4\} \tag{4.34}$$

$$g_4f_{2y} - f_2g_{4y} = h_3g_4. \tag{4.35}$$

On the other hand from the set of equations (4.20)-(4.28), with $h_1 = 0$ we obtain the following four equations:

$$3h_3 = \phi_{0y} f_0, \tag{4.36}$$

$$h_3(g_{0y} + g_2 - 3\phi_0 g_0) - g_0 h_{3y} = \phi_{0y}(f_2 g_0 + f_0 g_2), \tag{4.37}$$

$$h_3(g_{2y} - g_4 - \phi_0 g_2) - g_2 h_{3y} = \phi_{0y}(f_2 g_2 + f_0 g_4), \tag{4.38}$$

$$h_3(g_{4y} + \phi_0 g_4) - g_4 h_{3y} = \phi_{0y} f_2 g_4. \tag{4.39}$$

Since $\phi_0 = \frac{3}{4y}$ it follows $\phi_{0y} = -\frac{\phi_0}{y}$ and upon rearranging (4.39), we have

$$h_3 g_{4y} - g_4 h_{3y} = -\phi_0 g_4 (h_3 + \frac{f_2}{y}).$$

Making the assumption that the coefficients of the highest powers of y' in the expressions for g, h are constants, say $g_4 = \mu$ and $h_3 = \nu$ so that $g_{4y} = h_{3y} = 0$, one obtains the following relation determining the coefficient of the y'^2 in f:

$$h_3 + \frac{f_2}{y} = 0 \Rightarrow f_2 = -\nu y. \tag{4.40}$$

While from (4.36) we obtain

$$f_0 = -4\nu y^2. \tag{4.41}$$

The remaining equations (4.37) and (4.38) determine the coefficients g_0 and g_2, from solutions of the following coupled linear equations:

$$g_{0y} - \frac{3}{y} g_0 = 2g_2, \tag{4.42}$$

$$g_{2y} - \frac{3}{2y} g_2 = 4\mu. \tag{4.43}$$

These conditions are consistent with the set of equations (4.32)-(4.35), as may be verified. Furthermore, the solutions of (4.42) and (4.43) are easy to construct and are given by

$$g_2 = -8\mu y \text{ and } g_0 = 16\mu y^2.$$

Hence we finally obtain

$$R = \frac{f}{g} = \frac{-\nu y(4y + y'^2)}{\mu(y'^2 - 4y)^2} \text{ and } S = \frac{h}{f} = \frac{\nu y'^3}{-\nu y(4y + y'^2)}. \tag{4.44}$$

It is now straightforward to obtain the corresponding first integral as

$$I(x, y, y') = \frac{\nu}{4\mu}\left(x - \frac{4yy'}{y'^2 - 4y}\right). \tag{4.45}$$

5 Lienard type equations

We discuss next equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{5.1}$$

Instead of writing this as a system of first-order equations of the usual form

$$\dot{x} = y$$

$$\dot{y} = \phi(x, y)$$

where $\phi = -(f(x)y + g(x))$, we re-write it as

$$\dot{x} = v - r\frac{g(x)}{f(x)} \tag{5.2}$$

$$\dot{v} = -\frac{1}{r}f(x)v, \tag{5.3}$$

subject to the condition

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \frac{1}{r}\left(1 - \frac{1}{r}\right)f(x), \quad r \neq 0, 1. \tag{5.4}$$

Here r is a parameter. In order to determine a first integral for the system (5.2)and (5.3), we follow the same formulation as outlined in section (3.1) and demand that the one form

$$R[S(v - r\frac{g}{f}) - \frac{1}{r}fv]dt - RSdx - Rdv = 0, \tag{5.5}$$

be exact. This means there exists a function $I(t, x, v)$ such that

$$I_t = R[S(v - r\frac{g}{f}) - \frac{1}{r}fv]$$

$$I_x = -RS \quad \text{and} \quad I_v = -R. \tag{5.6}$$

If we are interested in a time independent first integral, so that $I_t = 0$, we immediately obtain

$$S = \frac{fv}{r\left(v - r\frac{g}{f}\right)}. \tag{5.7}$$

From the compatibility of (5.6), using the above expression for S, we have the following equation for determining the integrating factor R, *viz*

$$R_x + \frac{fv/r}{\left(r\frac{g}{f} - v\right)}R_v = -\frac{g}{\left(r\frac{g}{f} - v\right)^2}R. \tag{5.8}$$

Next we make the ansatz

$$R = \frac{\left(r\frac{g}{f} - v\right)}{T(x, v)}, \tag{5.9}$$

and after inserting it into (5.8), obtain the following equation for determining $T(x,v)$,

$$X[T] := \left(r\frac{g}{f} - v\right)\frac{\partial T}{\partial x} + \frac{fv}{r}\frac{\partial T}{\partial v} = fT. \tag{5.10}$$

As Chandrasekar *et al* have shown, it is not necessary to obtain the general solution of (5.10). Any particular solution of it is sufficient to determine a first integral, when it exists. In principle this leads to a considerable simplification, which must not be underestimated. For the problem of determining a particular solution of T, we shall use the technique of Darboux polynomials. Notice that if $f(x)$ be a polynomial, then in view of (5.4), we conclude that g/f must also be a polynomial. For the vector field X as defined in (5.10) we find that

$$X[h_1] = X[v] = \frac{f}{r}h_1 \tag{5.11}$$

and

$$X[h_2] = X\left[\frac{g}{f} - \frac{(r-1)}{r(r-2)}v\right] = \frac{(r-1)}{r}fh_2. \tag{5.12}$$

In other words, $h_1 = v$ and $h_2 = \frac{g}{f} - \frac{(r-1)}{r(r-2)}v$ are Darboux polynomials of the vector field X with cofactors $\lambda_1 = \frac{f}{r}$ and $\lambda_2 = \frac{(r-1)}{r}f$ respectively. Consequently, for $T(x,v) = h_1^{n_1}h_2^{n_2}$, we can find rational numbers such that $X[T] = fT$ namely $n_1 = n_2 = 1$. Thus we have the following particular solution of (5.10):

$$T(x,v) = v\left(\frac{g}{f} - \frac{(r-1)}{r(r-2)}v\right). \tag{5.13}$$

This completes the determination of the integrating factor R as

$$I_v = -R = -\frac{rg/f - v}{v/r(rg/f - (r-1)v/(r-2))} \quad \text{and} \quad I_x = -RS = \frac{f}{rg/f - (r-1)v/(r-2)}. \tag{5.14}$$

The corresponding first integral is given by

$$I(x,v) = \log\left[\frac{\left(r\frac{g}{f} - \frac{r-1}{r-2}v\right)}{v^{r-1}}\right]^{\frac{r}{r-1}}, \quad r \neq 0,1,2, \tag{5.15}$$

which essentially means that

$$C(x,v) = \left[\frac{\left(r\frac{g}{f} - \frac{r-1}{r-2}v\right)}{v^{r-1}}\right], \quad r \neq 0,1,2 \tag{5.16}$$

is a constant of motion.

Of course one could have obtained this first integral in a much more simpler way, by observing that (5.10) admits a solution $T = v^r$. This in turn gives $R = (rg/f - v)/v^r$ and $RS = -fv^{1-r}/r$ from which one gets the first integral (5.16).

5.1 A Liénard type nonlinear oscillator – the second order Riccati equation

We illustrate the above method with a well known example:

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0. \tag{5.17}$$

Here $f(x) = \alpha x$ and $g(x) = \beta x^3$. The condition (5.4) gives a quadratic equation for the parameter r, with solution

$$\frac{1}{r} = \frac{1}{2}\left[1 \pm \sqrt{1 - 8\beta/\alpha^2}\right].$$

If we choose the value of r, then this solution determines a relation between the parameters α and β of the equation; conversely, given the parameters it fixes the value of r . For example, the choice $r = 3$ yields $\beta = \frac{\alpha^2}{9}$. Thus setting $\alpha = 3k$ we have $\beta = k^2$ and the equation becomes

$$\ddot{x} + 3kx\dot{x} + k^2x^3 = 0. \tag{5.18}$$

This particular form is often called the second Riccati equation (and is also the Painlevé-Gambier equation VI with q(Z)=0 of [13]). Its first integral from (5.16) is

$$C_1(x, v) = \frac{kx^2 - 2v}{v^2}. \tag{5.19}$$

The phase flow for the equation, under these circumstances, as determined from (5.2) and (5.3) is

$$\frac{dv}{dx} = \frac{2kxv}{kx^2 - 2v},$$

which may be separated by using the above expression for C_1 *viz*

$$\frac{dv}{dx} = \frac{2kx}{C_1 v} \quad \Rightarrow \quad \frac{1}{2}C_1 v^2 - kx^2 = K_2$$

where K_2 is an integration constant.

If we desire to express $C_1(x, v)$ in terms of x and the actual velocity $\dot{x}$, then we simply eliminate v using (5.2) to get

$$C_1(x, \dot{x}) = -\frac{2\dot{x} + kx^2}{(\dot{x} + kx^2)^2},$$

which coincides with the results in [5]. In fact it has been shown by Cariñena *et al* that this first integral plays the role of the Hamiltonian for (5.18).

6 A generalized 2D- Kepler system

In [7], the authors considered a system of second order ODE's of the generic form

$$\ddot{x} = \frac{P_1}{Q_1} = \phi_1 \text{ and } \ddot{y} = \frac{P_2}{Q_2} = \phi_2$$

where it is assumed that $\phi_i(i=1,2)$ depend on $t, x, \dot{x}, y, \dot{y}$ in general. They illustrate the general procedure and finish off with the following example of the two-dimensional Kepler problem.

$$\ddot{x} = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\ddot{y} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}. \tag{6.1}$$

Their analysis yielded the following first integrals:

$$I_1 = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{\sqrt{x^2+y^2}}$$

$$I_2 = y\dot{x} - x\dot{y}$$

$$I_3 = \dot{x}(y\dot{x} - x\dot{y}) - \frac{y}{\sqrt{x^2+y^2}}. \tag{6.2}$$

corresponding to the Hamiltonian, the angular momentum and the Runge Lenz vector respectively.
We shall consider a system which is similar to this, but of the form:

$$\ddot{x} = -\frac{x(x^2+b)}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{xg_1(x,y)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\ddot{y} = -\frac{y(3x^2+2y^2+b)}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{yg_2(x,y)}{(x^2+y^2)^{\frac{3}{2}}} \tag{6.3}$$

where

$$g_1(x,y) = (x^2+b), \text{ and } g_2(x,y) = (3x^2+2y^2+b). \tag{6.4}$$

As shown in [7], if I be a first integral of the coupled system such that

$$dI = I_t dt + I_x dx + I_y dy + I_{\dot{x}} d\dot{x} + I_{\dot{y}} d\dot{y} = 0$$

and if we write the coupled system of equations as:

$$(\phi_1 + S_1\dot{x})\, dt - S_1 dx - d\dot{x} = 0 \tag{6.5}$$

$$(\phi_2 + S_2\dot{y})\, dt - S_2 dy - d\dot{y} = 0 \tag{6.6}$$

then we must have

$$I_t = R_1(\phi_1 + S_1\dot{x}) + R_2(\phi_2 + S_2\dot{y})$$

$$I_x = -R_1 S_1$$

$$I_y = -R_2 S_2$$

$$I_{\dot{x}} = -R_1$$

$$I_{\dot{y}} = -R_2. \tag{6.7}$$

Here R_1, R_2 represent the respective integrating factors of the system of equations (6.5, 6.6). Compatibility of the set of equations (6.7) then yields the following:

$$D[S_1] = -\phi_{1x} - \frac{R_2}{R_1}\phi_{2x} + \frac{R_2}{R_1}S_1\phi_{2\dot{x}} + S_1\phi_{1\dot{x}} + S_1^2$$

$$D[S_2] = -\phi_{2y} - \frac{R_1}{R_2}\phi_{1y} + \frac{R_1}{R_2}S_2\phi_{1\dot{y}} + S_2\phi_{2\dot{y}} + S_2^2$$

$$D[R_1] = (R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}} + R_1S_1)$$

$$D[R_2] = -(R_2\phi_{2\dot{y}} + R_1\phi_{1\dot{y}} + R_2S_2)$$

$$S_1R_{1y} = R_1S_{1y} + S_2R_{2x} + R_2S_{2x}$$

$$R_{1x} = \frac{\partial}{\partial\dot{x}}(R_1S_1), \ R_{2y} = \frac{\partial}{\partial\dot{y}}(R_2S_2)$$

$$R_{1y} = \frac{\partial}{\partial\dot{x}}(R_2S_2), \ R_{2x} = \frac{\partial}{\partial\dot{y}}(R_1S_1), \ R_{1\dot{y}} = R_{2\dot{x}}$$

Here D represents the vector field

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \phi_1\frac{\partial}{\partial\dot{x}} + \phi_2\frac{\partial}{\partial\dot{y}}$$

The problem is basically to find solutions (particular) satisfying these equations. The explicit details of how these may be simplified and reduced to a more manageable form are contained in [7]. For the system (6.3) one particular solution is the following:

$$R_1 = \dot{x}, \quad R_2 = \dot{y}, \quad S_1 = \frac{x(x^2+b)}{\dot{x}(x^2+y^2)^{\frac{3}{2}}}, \quad S_2 = \frac{y(3x^2+2y^2+b)}{\dot{y}(x^2+y^2)^{\frac{3}{2}}}.$$

With these values of R_i, S_i $(i = 1, 2)$ we obtain the following first integral:

$$I(x, y, \dot{x}, \dot{y}) = -\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{x^2 + 2y^2 - b}{\sqrt{x^2+y^2}}\right]. \tag{6.8}$$

In fact it is easy to verify that this integral is actually the Hamiltonian. However, we have not yet been able to deduce the analogs of the angular momentum or the Lenz vector for this case.

7 Conclusion

In this article we have shown how the extended Prelle-Singer (PS) method, as developed by Lakshmanan and his coworkers, can be used to deduce first integrals of certain special classes of second order nonlinear ODEs. In particular, we have focused on the Painlevé-Gambier type ODEs. We have found an additional first integral for the Painlevé-Gambier XXII equation. This appears to be a new result. Using this method we have also derived a formula for the first integrals of a particular sub-class of equations of the Painlevé-Gambier classification. In addition, we have used a novel transformation to analyze the second order Riccati equation (a special case of the Painlevé-Gambier VI equation). Finally we have applied the extended PS method to derive a first integral, of a modified form of the 2D Kepler problem, which is an example of a system of second-order ODEs.

It would be interesting to extend the analysis to the third and higher order nonlinear oscillator equations. Hopefully the proposed method would work for both scalar and multicomponent equations of arbitrary order. Using the first integrals we expect to study appropriate Lagrangians and Hamiltonians. In fact a quantized description can be developed using these Hamiltonian forms which can be mapped onto known quantum mechanical toy models for the damped systems.

Finally we have observed that the method described here for determination of first integrals is closely related to finding solutions of the adjoint symmetry equation. In our forthcoming paper we hope to give a detail mapping and an algorithmic formulation for these class of systems.

ACKNOWLEDGMENT We are grateful to Professor M. Lakshmanan for his interest and encouragement. PG wishes to thank Professors Pepin Cariñena, Jarmo Hietarinta and Basil Grammaticos for enlightening discussions and many valuable inputs. In addition AGC wishes to acknowledge the support provided by the S. N. Bose National Centre for Basic Sciences, Kolkata in the form of an Associateship. Finally, AGC and PG wish to thank IMI-IISc programme on Nonlinear Dynamics for giving them an wonderful opportunity to start working on this project.

References

[1] M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations.* Trans. Amer. Math. Soc. 279 (1983), no. 1, 215–229.

[2] M.F. Singer, *Liouvillian first integrals of differential equations.* Trans. Amer. Math. Soc. 333 (1992), no. 2, 673–688.

[3] G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré.*, Bull. Sci. Math. (2) 2 (1878), 60–96, 123–144, 151–200.

[4] J.P. Juanolou, *Équations de Pfaff algébriques.* (French)[Algebraic Pfaffian equations] Lecture Notes in Mathematics, 708, Springer, Berlin, 1979. v+255 pp.

[5] J.F. Cariñena, M.F. Rañada and M. Santander, *Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability.* J. Math. Phys. 46 (2005), no. 6, 062703, 18 pp.

[6] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan,*On the general solution of the modified Emden-type equation* $\ddot{x}+\alpha x\dot{x}+\beta x^3=0$.J. Phys. A 40 (2007), 4717-4727.

[7] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *Extended Prelle-Singer method and integrability/solvability of a class of nonlinear nth order ordinary differential equations.* J. Nonlinear Math. Phys. 12 (2005), suppl. 1, 184–201.

[8] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *A unification in the theory of linearization of second-order nonlinear ordinary differential equations.* J. Phys. A 39 (2006), no. 3, L69–L76.

[9] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, *A simple and unified approach to identify integrable nonlinear oscillators and systems.* J. Math. Phys. 47 (2006), no. 2, 023508, 37 pp.

[10] L.G.S. Duarte, S. E. S. Duarte, L. A. C. da Mota and J. E. F. Skea, *Solving second-order ordinary differential equations by extending the Prelle-Singer method.* J. Phys. A 34 (2001), no. 14, 3015–3024.

[11] L.G.S. Duarte, S. E. S. Duarte, L. A. C. da Mota, *Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution.* J. Phys. A 35 (2002), no. 4, 1001–1006.

[12] A. Goriely, *Integrability and nonintegrability of dynamical systems.* Advanced Series in Nonlinear Dynamics, 19. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xviii+415 pp.

[13] E.L. Ince, *Ordinary differential equations*, Dover, New York. 1956.

JOURNAL OF MATHEMATICAL PHYSICS **50**, 102502 (2009)

Determination of elementary first integrals of a generalized Raychaudhuri equation by the Darboux integrability method

A. Ghose Choudhury,[1,a)] Partha Guha,[2,b)] and Barun Khanra[3,c)]
[1]*Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta 700009, India*
[2]*Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany and S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata 700098, India*
[3]*Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Calcutta 700 004, India*

(Received 27 May 2009; accepted 3 September 2009; published online 23 October 2009)

The Darboux integrability method is particularly useful to determine first integrals of nonplanar autonomous systems of ordinary differential equations, whose associated vector fields are polynomials. In particular, we obtain first integrals for a variant of the generalized Raychaudhuri equation, which has appeared in string inspired modern cosmology. © *2009 American Institute of Physics.*
[doi:10.1063/1.3243455]

I. INTRODUCTION

The problem of solving ordinary nonlinear differential equations is a challenging area in nonlinear dynamics. For a two dimensional system the existence of a first integral completely determines its phase portrait. It is well known that such systems do not exhibit chaos because of the Poincaré–Bendixson theorem.[1] According to this theorem, for a two dimensional system of ordinary differential equations (ODEs), which is real analytic and defined in a simply connected domain, any *compact* limit set of the system is either a fixed point, a cycle, or a union of fixed points and connections, i.e., a polycycle. In three dimension this is no longer true. In the case of nonplanar systems, the problem of determining first integrals is a nontrivial task, in general, and various methods have been introduced for studying the existence of such first integrals. However, except for some special cases[2] there are few known satisfactory general methods for their determination.[3–6] In 1878 Darboux[7] initiated the theory of planar polynomial differential systems, and his work provided a link between algebraic geometry and the search of first integrals.[8,19] He demonstrated how to construct first integrals of polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$. The extension of Darboux theory of integrability to polynomial systems in $\mathbb{R}^n$ and $\mathbb{C}^n$ (for $n \geq 3$) was given by Jouanolou.[9] This yielded the notion of what is today known as Darboux integrability (cf. Refs. 10 and 11). Research in this area which lies at the crossroads of ODE theory with algebraic geometry and differential algebra has deep implications for the problem of the *center*, as well as, for Hilbert's 16th problem on limit cycles. In an interesting survey, Schlomiuk[12] has described the early ideas of Darboux and related them to the influential paper of Prelle and Singer.[13] Among other results, Prelle and Singer showed that if a system of differential equations has an elementary first integral then it must be computable from invariant algebraic curves. In fact, their paper is so influential that the present version of the updated Darboux integrability is also known as extended Prelle–Singer method. Nevertheless, it is Darboux integrability that lies at the very heart of the

[a)]Electronic mail: a_ghosechoudhury@rediffmail.com.
[b)]Author to whom correspondence should be addressed. Electronic mail: partha.guha@mis.mpg.de.
[c)]Electronic mail: barunkhanra@rediffmail.com.

notions of Liouville integrability and the Prelle–Singer theorem, which are essentially built on its foundation.

Lastly, it has to be mentioned that the classical and powerful method of symmetry analysis, as formulated by Lie, is an important tool for finding solutions of differential equations and includes various methods for determining first integrals.[14] We describe the modified Darboux theory of integrability for polynomial ODEs in three and more dimensions. We demonstrate in this paper that the Darboux method of integrability is one of the best known methods for finding first integrals of polynomial ODEs. Using this theory we study the existence of first integrals for a generalized Raychaudhuri equation, which has appeared in modern string inspired cosmology.

The *organization* of the paper is as follows. In Sec. II we discuss the basic definitions, background, and the Prelle–Singer method. This section is of pedagogical nature, since the methods of Darboux integrability or the Prelle–Singer technique are not very well known outside the mathematics community. In Sec. III, we examine a system related to the Raychaudhuri equation; appropriate for studying kinematics of a deformable media in a two dimensional nonflat spacetime; and obtain the first integrals of this equation.

II. PRELIMINARIES

Consider a system of two first-order ODEs of the form

$$\frac{dx_1}{dt} = X_1(t,x_1,x_2),$$

$$\frac{dx_2}{dt} = X_2(t,x_1,x_2). \tag{2.1}$$

A solution of (2.1), namely, $x_1=x_1(t), x_2=x_2(t)$, assuming the values $x_1(0), x_2(0)$ at $t=t_0$ say, defines in space a certain curve, which passes through the point $P_0(t_0,x_1(0),x_2(0))$, and is called an integral curve of the system (2.1).

In geometrical terms the Cauchy problem amounts to finding the integral curve of (2.1) passing through the given point P_0. An alternative interpretation of the solution of (2.1) treats t as a parameter and $x_1=x_1(t), x_2=x_2(t)$ as the parametric equation of a curve in the x_1-x_2 plane called the *phase plane*. The projection of the integral curve on the phase plane then gives the trajectory of the system. However, while from the integral curve one can define the phase trajectory uniquely, the converse is not true, in general.

If the right hand side of (2.1) is not explicitly dependent on t then the system is said to be autonomous; otherwise it is called a nonautonomous system.

Definition 2. 1: A first integral of the system of ODEs,

$$\frac{dx_i}{dt} = X_i(t,x_1, \ldots ,x_n) \quad i=1, \ldots ,n, \tag{2.2}$$

is any nonconstant globally differentiable function $\Phi(t,x_1, \ldots ,x_n)$ *that retains a constant value on any integral curve of the system.*

This means its derivative with respect to t vanishes on the solution curves,

$$\frac{d\Phi}{dt} = 0 \Rightarrow \sum_i \frac{\partial \Phi}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial \Phi}{\partial t} = 0 \Rightarrow \tilde{D}[\Phi] = 0, \tag{2.3}$$

where $\tilde{D} := \Sigma_i X_i \partial/\partial x_i + \partial/\partial t$ is called the material derivative. For autonomous systems this reduces to

$$\sum_i X_i \frac{\partial \Phi}{\partial x_i} = 0, \tag{2.4}$$

where $D=\Sigma_i X_i \partial/\partial x_i$ is just the vector field associated with the given autonomous system. In many cases, the determination of a first integral is simplified considerably, by the existence of what are known as *second integrals*.

Definition 2.2: A second integral of a vector field D is a C^1 *function,* $f=f(x_1,\cdot,x_n):\mathbb{K}^n\to\mathbb{K}$ *such that* $D[f]=\lambda f$, *where* $\lambda=\lambda(x_1,\ldots,x_n):\mathbb{K}^n\to\mathbb{K}$.

Here $\mathbb{K}$ is a field of characteristic zero, and for our purposes may either be $\mathbb{R}$ or $\mathbb{C}$. Furthermore, it is appropriate here to introduce the notion of Darboux polynomials, as the determination of elementary first integrals is intimately connected to their existence.

Definition 2.3: The polynomial second integrals for polynomial vector fields are called Darboux polynomials (monic irreducible polynomials).

Finally one should mention what we mean by elementary first integrals. These are first integrals involving elementary functions only, which for the present purpose may be roughly stated as follows.

Definition 2.4: A function $F(x_1,\ldots,x_n)$, $\in\mathbb{C}^n$ is said to be elementary if it belongs to the set S, which in turn is obtained from rational functions on $\mathbb{C}^k, k=0,1,\ldots$, using a finite series of the following operations: (a) algebraic *operations such as addition, subtraction, multiplication, and division, (b) solution of algebraic equations, (c) derivations, and (d) exponential and logarithmic operations.*

Note that if in addition we include the operation of integration, then S becomes the set of Liouvillian function.

A. The Darboux integrability method

Let us consider planar polynomial differential systems,

$$\dot{x} = Q(x,y) \quad \text{and} \quad \dot{y} = P(x,y), \tag{2.5}$$

where $P(x,y)=\Sigma_{i=0}^m P_i(x,y), Q(x,y)=\Sigma_{i=0}^m Q_i(x,y)$ are coprime polynomials in $\mathbb{C}$ such that $\max\{\deg P, \deg Q\}=m$ and $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous components of degree i.

The planar differential system (2.5) may alternatively be described by the following vector field:

$$D = Q(x,y)\frac{\partial}{\partial x} + P(x,y)\frac{\partial}{\partial y}, \tag{2.6}$$

or a differential form

$$\omega = Pdx - Qdy.$$

The corresponding phase-flow being given by the solutions of first-order ODE,

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}. \tag{2.7}$$

The tangents to the trajectories of a planar polynomial differential system are defined everywhere. If $f(x,y)=0$ is the equation of an invariant curve, its tangent must coincide with the tangents of the trajectories. In other words, the gradient to f, $\nabla f=(\partial f/\partial x, \partial f/\partial y)$, and (Q,P) must be orthogonal over the curve $f(x,y)=0$,

$$\dot{f} = \left(Q\frac{\partial f}{\partial x} + P\frac{\partial f}{\partial y}\right)_{f=0} = 0.$$

An invariant curve $f(x,y)=0$ is called an algebraic curve of degree m when $f(x,y)$ is a polynomial of degree m.

Let D be the vector field associated with differential equation. A curve $f(x,y)=0$ is an *invariant* algebraic curve if $D[f]/f$ is a polynomial. The latter polynomial $\lambda_f=D[f]/f$ is usually called the *cofactor* of the invariant algebraic curve.

The system (2.5) is integrable on an open subset U of $\mathbb{K}^2$ ($\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$) if there exists a nonconstant function $I:U\to\mathbb{K}$, called a *first integral* of the system on U, which remains constant on all solutions curves $(x(t),y(t))$ of the system contained in U. The formal definition of a first integral of a vector field is as follows.

Definition 2.5: Let U be an open subset of $\mathbb{K}^2$. We say that a nonconstant function $I:U\to\mathbb{K}$ is a first integral of a vector field D on U, if and only if $D|_U(I)=0$.

It has been found that the existence of invariant algebraic curves (real or complex) forces the integrability of a differential system (2.5). This is the essential concept behind Darboux's theory of integrability, and arose in course of his analysis of differential equations in the complex projective plane.

Suppose the vector field admits s distinct invariant algebraic curves f_i $i=1,\ldots,s$.

(a) If there are $n_i\in\mathbb{C}$ not all zero, such that $\Sigma_{i=1}^{s}n_i\lambda_{f_i}=0$ then the function $\Pi_{i=1}^{s}f_i^{n_i}$ is a first integral of the vector field D.

(b) If there exists $n_i\in\mathbb{C}$ not all zero, such that $\Sigma_{i=1}^{s}n_i\lambda_{f_i}=-\text{div } D$, then $\Pi_{i=1}^{s}f_i^{n_i}$ is an integrating factor of D.

Improvements upon the Darboux theory have been attempted by many mathematicians. For instance, Juanolou studied the existence of rational first integrals for differential systems.[9] A rational first integral is more useful than a *Darbouxian* one because taking into account it and its inverse, there is a first integral defined at any point of the plane. Further improvement has been made by invoking exponential factors. If $f,g\in\mathbb{C}[x,y]$, we say that $e=\exp(g/f)$ is an exponential factor of the vector field D if $D(e)/e$ is a polynomial of degree at most $d-1$. The extension to such cases is given in Ref. 15.

In view of the above, one may identify two clearly distinguishable classes of first integrals, namely, the rational ones and those which are Darbouxian, that is, having the following essential structure $f^{\nu}(\exp(h/g))^{\mu}$, where, in general, $\nu,\mu\in\mathbb{C}$. In the Prelle–Singer method, it is shown that whenever a vector field D has an elementary first integral, the latter can be computed using only the invariant algebraic curves. Clearly, these first integrals may be found using the Darboux approach.

A major step toward the construction of an algorithm for solving first-order ODEs was put forward by Prelle and Singer.[13] In its original form, this method is a semialgorithmic procedure for solving nonlinear first-order ODEs of the form (2.7), when $P(x,y)$ and $Q(x,y)$ are polynomials, with coefficients defined on the field of complex numbers. The Prelle and Singer procedure can, not only determine polynomial first integrals, but more importantly may be applied to systems admitting even rational first integrals. On the other hand Singer in 1994 showed that for Liouvillian first integrals, their integrating factors are given by Darbouxian functions.[16]

Finally, we note that if an integrating factor is known then we can compute by quadrature a first integral of the system up to a constant.

Definition 2.6: We say that a nonzero function $R:U\to\mathbb{K}$ *is an integrating factor of a vector field D* on U if and only if

$$D(R)=-\,\text{div}(D)\cdot R$$

on U.

If $R(x,y)$ be an integrating factor of Eq. (2.7), then clearly

$$RPdx-RQdy=0 \quad \text{and} \quad (RP)_y=-(RQ)_x, \tag{2.8}$$

where the subscripts denote partial derivatives. The latter may be written as

$$D[R] = (Q\partial_x + P\partial_y)[R] = -\,\mathrm{div}(Q,P)R = -(Q_x + P_y)R. \tag{2.9}$$

Prelle–Singer then show that if the ODE has an elementary first integral then it may be written in the following form:

$$R = \prod_i f_i^{n_i}, \tag{2.10}$$

where f_i are Darboux polynomials and n_i are rational numbers.

If we can identify a sufficient number of Darboux polynomials f_i satisfying

$$D[f_i] = \lambda_i f_i, \tag{2.11}$$

where λ_i are suitable polynomials, then

$$\frac{D[R]}{R} = \sum_i n_i \frac{D[f_i]}{f_i} = -(Q_x + P_y). \tag{2.12}$$

Clearly Q_x, P_y are polynomials since Q, P are themselves polynomials; and therefore it is necessary that f_i divides $D[f_i]$. If we manage to find such Darboux polynomials, then all that remains is to determine the numbers n_i such that (2.12) is satisfied. This can be achieved by equating terms of various orders $x^\alpha y^\beta$ on either side and finding a consistent set of values of the n_i's. The problem lies in determining the f_i's, and the Prelle–Singer method provides a semialgorithm for determining these, whenever there exists a first integral which is an *elementary function*. This involves establishing bounds of different orders on the f_i's. For example, we start with $N=1$ and assume $f=\alpha x+\beta y+\gamma$; next we check for what values of α,β and γ, f divides $D[f]$. If we fail to find such an f, we go to the next level and set $N=2$, try $f=\alpha x^2+2\beta xy+\gamma y^2+\delta x+\epsilon y+\mu$ and find a particular combination which divides $D[f]$, and so on. It is clear that the process is semialgorithmic by its very nature.

One should point out that in the event

$$D[R] = 0, \tag{2.13}$$

then it is obvious that R is itself a first integral, since the equation is then exact.

III. THE GENERALIZED RAYCHAUDHURI EQUATION IN A TWO DIMENSIONAL DEFORMABLE MEDIA AND ITS FIRST INTEGRALS

Having explained the general procedure, we present the main result of this communication. In the study of spatially homogenous perfect fluid models in general relativity the relevant equations usually appear as a system of coupled ODEs. When expressed in terms of *expansion-normalized* variables these equations admit a symmetry which allows the equation for time evolution of expansion θ to decouple leading to the Raychaudhuri equation.[17]

In Ref. 18 the authors have considered geodesic flows on the surface of a deformable media and have deduced how the expansion, shear, and rotation of such flows evolve with time. The deformations of the media (at least locally) may be characterized in terms of time evolution of a deformation vector (θ,σ,w), where θ, σ, and w represent the expansion (E), shear (S), and rotation (R), respectively. The kinematics can be quantified in terms of these (ESR) variables and leads to the generalized Raychaudhuri equation for a two dimensional curved surface of constant curvature. When the exact solutions of the geodesic equations are used in them, one is led to the following system, after suitable relabeling of the variables involved [see Eqs. (2.20)–(2.23) of Ref. 18]:

$$\dot{x} + \tfrac{1}{2}x^2 + \alpha x + 2(y^2 + z^2 - t^2) + 2\beta = 0, \tag{3.1}$$

$$\dot{y} + (\alpha + x)y + \gamma = 0, \tag{3.2}$$

$$\dot{z} + (\alpha + x)z + \delta = 0, \tag{3.3}$$

$$\dot{t} + (\alpha + x)t = 0. \tag{3.4}$$

One must not interpret t here as the time, it is simply at par with variables x,y,z. However $\dot{x},\dot{y}$, etc., stand for the derivative of these variables with respect to the appropriate "temporal variable" relevant to the model. Thus from a mathematical point of view the above equations form a nonplanar dynamical system. Note that α, β, γ, and δ are suitable parameters of the model.

The vector field D is given by

$$-D = \left(\frac{1}{2}x^2 + \alpha x + 2(y^2 + z^2 - t^2) + 2\beta\right)\frac{\partial}{\partial x} + ((\alpha + x)y + \gamma)\frac{\partial}{\partial y} + ((\alpha + x)z + \delta)\frac{\partial}{\partial z} + ((\alpha + x)t)\frac{\partial}{\partial t}.$$

It can be easily verified that with $f_1 = -(\delta/\gamma)y + z$ we have

$$D[f_1] = D\left[-\frac{\delta}{\gamma}y + z\right] = -(\alpha + x)f_1, \quad \text{so that} \ \lambda_1 = -(\alpha + x). \tag{3.5}$$

Similarly we find $f_2 = t$ to be another Darboux polynomial whose associated eigenpolynomial is again $\lambda_2 = -(\alpha + x) = \lambda_1$. Consequently the exactness condition $D[R]=0$ which implies $\Sigma_i^2 n_i\lambda_i = 0$ leads to $(\alpha + x)(n_1 + n_2) = 0$ or $n_2 = -n_1$. Making the choice $n_1 = -1$ we obtain a first integral given by

$$I_1(x,y,z,t) = \frac{t}{\left(-\frac{\delta}{\gamma}y + z\right)}. \tag{3.6}$$

A. Additional new first integrals

On the other hand for the following specific choice of the parameters $\gamma = \delta = 0$ one finds the following Darboux polynomials:

$$D[g_i] = -(\alpha + x)g_i (i = 1,2,3) \quad \text{with} \ g_1 = y, \quad g_2 = z, \quad g_3 = t, \tag{3.7}$$

and

$$D[g_4] := D[z^2 + t^2 + zt] = -2(\alpha + x)(z^2 + t^2 + zt). \tag{3.8}$$

Hence, the exactness condition $D[R]=0$ implies

$$\sum_i n_i\lambda_i = 0 \Rightarrow (n_1 + n_2 + n_3 + 2n_4)(\alpha + x) = 0. \tag{3.9}$$

Choosing $n_1 = n_2 = 1$ and $n_3 = -1, n_4 = -\frac{1}{2}$ we get another first integral of the form

$$I = \frac{yz}{t(z^2 + t^2 + zt)^{1/2}}. \tag{3.10}$$

It will be noticed that all the above first integrals are independent of the variable x. To get a first integral explicitly dependent on x,y,z,t, we notice that when all the parameters $\alpha = \beta = \gamma = \delta = 0$ then the following Darboux polynomial depending on x is obtainable

$$-D[g_1] := -D\left[y^2 + z^2 - t^2 - \tfrac{1}{4}x^2\right] = x\left(y^2 + z^2 - t^2 - \tfrac{1}{4}x^2\right), \tag{3.11}$$

with associated eigenpolynomial given by $\lambda_1 = -x$. In addition the following are Darboux polynomials of degree of 2:

$$-D[g_2] := -D[zt] = 2xg_2 \Rightarrow \lambda_2 = -2x,$$

$$-D[g_3] := -D[yz] = 2xg_3 \Rightarrow \lambda_3 = -2x,$$

$$-D[g_4] := -D[yt] = 2xg_4 \Rightarrow \lambda_4 = -2x. \tag{3.12}$$

The exactness condition,

$$\sum_i n_i\lambda_i = 0 \ \text{ implies } \ -x[n_1 + 2(n_2+n_3+n_4)] = 0,$$

which may then be satisfied by the following choice $n_2=n_3=n_4=-\frac{1}{2}$ and $n_1=3$, leading to the rational first integral,

$$I(x,y,z,t) = \frac{\left(y^2+z^2-t^2-\frac{1}{4}x^2\right)^3}{yzt}. \tag{3.13}$$

B. Derivation of first integrals of two dimensional variant of generalized Raychaudhuri's equation: The traditional approach

Recently Grammaticos (we are grateful to Basil Grammaticos for communicating this to us) gave a simple method to compute the first integral of the generalized Raychaudhuri's equation by direct manipulations of the generalized system of Eqs. (3.1)–(3.4). First of all we translate x so as to put α to zero (this only changes the value of β). Next a scaling of y and z brings γ and δ to 1. Of course this changes the equation for $\dot{x}$ which is not so significant. Therefore, one arrives at

$$\dot{y} + xy + 1 = 0, \quad \dot{z} + xz + 1 = 0. \tag{3.14}$$

It is clear that we must introduce the sum and difference $u=(y+z)/2$ and $w=y-z$ to obtain

$$\dot{u} + xu + 1 = 0 \quad \text{and} \quad \dot{w} + xw = 0. \tag{3.15}$$

Since we also have

$$\dot{t} + xt = 0, \tag{3.16}$$

it is obvious that the difference of logarithmic derivatives of w and t is zero and thus the invariant is

$$I = t/w.$$

In fact, it is interesting to note that the invariants (3.10) and (3.13) may also be obtained in a similar fashion. For instance, when $\alpha=\gamma=\delta=0$, then $y/z, y/t$ are invariants and the first integral given in Eq. (3.10) is nothing but a combination of these two invariants, as can be easily verified. But when in addition $\beta=0$, setting $z=\lambda y$ and $t=\mu y$, where λ,μ are constants, the system of Eqs. (3.1)–(3.4) reduces to the following:

$$\dot{x} + \tfrac{1}{2}x^2 + cy^2 = 0, \tag{3.17}$$

$$\dot{y} + xy = 0. \tag{3.18}$$

Here the constants λ,μ have been lumped into an overall constant c. One can use (3.18) to eliminate the variable x from (3.17) and get

$$-\frac{\ddot{y}}{y} + \frac{3}{2}\left(\frac{\dot{y}}{y}\right)^2 + cy^2 = 0, \tag{3.19}$$

which by the transformation $w=1/y$ and the choice $c=\frac{1}{2}$, may be written in the canonical form,

$$\ddot{w} - \frac{\dot{w}^2}{2w} + \frac{1}{2w} = 0. \tag{3.20}$$

This is Eq. (32) in Ince[20] and the invariant is $K=\dot{w}^2-1/w^2$. Going back to the original variables we find $K=x^2/y^2-y^2$. The invariant given by (3.13) is just a combination of K and the two previous invariants *viz* $y/z, y/t$.

IV. CONCLUSION

In this paper we have illustrated how the Darboux theory of integrability may be used to determine the first integrals of the generalized Raychaudhuri equation. This is a powerful method and is likely to be of use to a wider audience of physicists and other researchers especially for those working with systems of ODEs, regardless of any specific fields.

ACKNOWLEDGMENTS

Authors are extremely grateful to Basil Grammaticos for careful reading of the manuscript and valuable suggestions. P.G. wishes to thank Pepin Cariñena, Jarmo Hietarinta, Peter Leach, and Sayan Kar for enlightening discussions. In addition A.G.C. wishes to acknowledge the support provided by the S. N. Bose National Centre for Basic Sciences, Kolkata in the form of an Associateship. Finally, A.G.C. and P.G. wish to thank IMI-IISc program on Nonlinear Dynamics for giving them an wonderful opportunity to start working on this project.

[1] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra* (Academic, New York, 1974).
[2] J. Hietarinta, Phys. Rep. **147**, 87 (1987).
[3] H. J. Giacomini, C. E. Repetto, and O. P. Zandron, J. Phys. A **24**, 4567 (1991).
[4] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, A. Strelcyn, and S. Wojciechowski, Physica A **163**, 683 (1990).
[5] B. Dorizzi, B. Grammaticos, J. Hietarinta, A. Ramani, F. Schwartz, Phys. Lett. A **116**, 432 (1986).
[6] H. J. Giacomini and S. Neukirch, Phys. Lett. A **227**, 309 (1997); Phys. Rev. E **56**, 3809 (1997).
[7] G. Darboux, Bull. Sci. Math. **2**, 60 (1878); **2**, 123 (1878).
[8] J. Chavarriga and M. Grau, Scientia, Ser. A: Math. Sci. **9**, 1 (2003).
[9] J. P. Jouanolou, *Équations de Pfaff Algébriques*, Lecture Notes in Mathematics Vol. 708 (Springer, Berlin, 1979) (in French).
[10] J. Giné and J. Llibre, *Differential Equations with Symbolic Computation*, Trends in Mathematics (Birkhauser, Basel, 2005), pp. 55–65.
[11] C. Christopher and C. Li, *Limit Cycles of Differential Equations*, Advanced Courses in Mathematics (Birkhauser Verlag, Basel, 2007).
[12] D. Schlomiuk, Expo. Math. **11**, 433 (1993).
[13] M. J. Prelle and M. F. Singer, Trans. Am. Math. Soc. **279**, 215 (1983).
[14] H. Stephani, *Differential Equations, Their Solution Using Symmetries*, edited by M. MacCallum (Cambridge University Press, Cambridge, 1989).
[15] C. Christofer and J. Llibre, Ann. of Diff. Eqns. **16**, 5 (2000).
[16] M. F. Singer, Trans. Am. Math. Soc. **333**, 673 (1992).
[17] S. Kar and S. Sengupta, Pramana, J. Phys., **69**, 49 (2007); e-print arXiv:gr-qc/0611123.
[18] A. Dasgupta, H. Nandan, and S. Kar, Int. J. Geom. Methods Mod. Phys. **6**, 645 (2009).e-print arXiv:0804.4089.
[19] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems*, Advanced Series in Nonlinear Dynamics Vol 19 (World Scientific, River Edge, NJ, 2001).
[20] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 115206 (13pp) doi:10.1088/1751-8113/42/11/115206

On adjoint symmetry equations, integrating factors and solutions of nonlinear ODEs

Partha Guha[1,2], A Ghose Choudhury[3] and Barun Khanra[4]

[1] Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany
[2] S N Bose National Centre for Basic Sciences JD Block, Sector III, Salt Lake Kolkata-700 098, India
[3] Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta-700 009, India
[4] Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Calcutta-700 004, India

E-mail: partha.guha@mis.mpg.de, a_ghosechoudhury@rediffmail.com and barunkhanra@rediffmail.com

Received 15 December 2008, in final form 3 February 2009
Published 20 February 2009
Online at stacks.iop.org/JPhysA/42/115206

Abstract
We consider the role of the adjoint equation in determining explicit integrating factors and first integrals of nonlinear ODEs. In Chandrasekar *et al* (2006 *J. Math. Phys.* **47** 023508), the authors have used an extended version of the Prelle–Singer method for a class of nonlinear ODEs of the oscillator type. In particular, we show that their method actually involves finding a solution of the adjoint symmetry equation. Next, we consider a coupled second-order nonlinear ODE system and derive the corresponding coupled adjoint equations. We illustrate how the coupled adjoint equations can be solved to arrive at a first integral.

PACS numbers: 02.40.Yy, 02.30.Hq
Mathematics Subject Classification: 58F05, 70H35

1. Introduction

The study of nonlinear ordinary differential equations (ODEs) has been an ongoing endeavor for well over two centuries now, with significant contributions from many of the greatest mathematicians of all times such as Euler, Lie, Painlevé, Poincaré to mention just a few. Their contributions have ranged from finding explicit solutions of ODEs, to developing general methods of classifications, to a qualitative analysis of their solutions etc. These in turn have often led to the opening up of entirely new branches of study in algebra, topology, geometry and have shed new light on several physical phenomena.

Over the years many techniques have been developed to obtain exact solutions of various kinds of ODEs. However, there does not exist any single common method for obtaining their solutions. Nevertheless, the apparently different techniques share one common feature: they somehow tend to exploit the symmetries of ODEs. Consequently, symmetry analysis of ODEs has become one of the most powerful tools for analyzing them.The foundations of this method are contained in the works of Sophus Lie [1, 2].

It is also well known that the existence of a sufficient number of first integrals greatly simplifies the process of solving any ODE. Having said this, it is not always quite obvious what these first integrals are. Indeed, their determination is, in general, a non-trivial task. In the case of conservative mechanical systems, one often has just a single first integral—the energy. In this context, the semi-algorithmic procedure developed by Prelle and Singer deserves mention [4]. In its original version it applied to first-order ODEs involving rational functions with coefficients belonging to the field of complex numbers $\mathbb{C}$. Subsequently their method, which involved the use of Darboux polynomials, was extended by Singer to include Liouvillian first integrals [14], by Duarte *et al* [5, 6] and also by Man and MacCullum [13]. Chandrasekhar *et al* have also extended the analysis in a series of papers [7–9].

Even though systematic techniques for solving nonlinear ODEs can be traced to the seminal works of Lie, certain aspects of the subject appear to have lain dormant for over a century. Notable among these is the notion of their linearization. Of late it has received renewed attention and notable progress has been made in this regard. In fact, Chandrasekar *et al* have recently proposed an extended Prelle–Singer method, based on generalized transformations, to linearize a class of equations that cannot be linearized by invertible point transformations [7].

In this paper we show how the extended Prelle–Singer method as proposed by Chandrasekar *et al* may be incorporated into the existing adjoint symmetry equation method. Essentially, as their method deals with a pair of first-order equations, in the variables R and S (to be called the RS-pair), these can be combined to obtain the corresponding second-order adjoint symmetry equation.

It is natural to enquire if similar analogs/correspondences may be identified between the adjoint equation method and the RS-pair method for coupled second-order systems. The answer is affirmative. In fact, by using a coupled version of the adjoint symmetry equation, we derive the first integral for a relatively new system [12], which has appeared in connection with stellar dynamics.

This paper is organized as follows. In section 2 we recall certain standard results concerning the solution of ODEs by using first integrals, and introduce the linearized symmetry equation, for determining the Lie point symmetry generators. Section 3 reviews the extended Prelle–Singer method as outlined in [8, 9] and contains a derivation of the adjoint symmetry equation, based on this approach. We illustrate the relative advantages of these methods with a few simple examples. Section 4 is dedicated to coupled second-order ODEs.

2. Preliminaries

Consider an nth-order ODE in the normal form

$$y^{(n)} = w(x, y, y', \ldots, y^{(n-1)}), \qquad \text{where} \quad y^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k}. \tag{2.1}$$

Corresponding to this ODE, there exists an equivalent first-order partial differential equation (PDE) in $(n + 1)$ variables [3, 10, 11],

$$\widetilde{D} f = (\partial_x + y'\partial_y + y''\partial_{y'} + \cdots + w\partial_{y^{(n-1)}}) f = 0, \tag{2.2}$$

in which the quantities $y', y''\ldots$ are treated as independent variables at par with x, y.

Their equivalence is provided by the first integrals of (2.1). By definition a first integral is a global function $I = I(x, y, y', \ldots, y^{(n-1)})$ that is constant along the solutions of (2.1), i.e.,

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \widetilde{D}I = I_x + y'I_y + y''I_{y'} + \cdots + wI_{y^{(n-1)}} = 0. \tag{2.3}$$

Having determined a first integral, say $I = I(x, y, y', \ldots, y^{(n-1)}) = I_0$, one can invert it to obtain

$$y^{(n-1)} = w_1(x, y, y', \ldots, y^{(n-2)}; I_0)$$

provided $I_{y^{(n-1)}} \neq 0$. This shows that the existence of a first integral allows for the reduction in the order of the differential equation by 1. Furthermore, it is evident that every first integral is a solution of the linear PDE (2.2) and conversely.

Let us assume ϕ^α ($\alpha = 1, \ldots, n$) denote a set of n functionally independent solutions of (2.1)/(2.2). Since each ϕ^α is a first integral, one has

$$\phi^\alpha(x, y, y', \ldots, y^{(n-1)}) = I_0^\alpha, \qquad \alpha = 1, 2, \ldots, n. \tag{2.4}$$

Consequently, by eliminating all derivatives from (2.4) one arrives at the general solution of (2.1) in the form

$$y = y\left(x; I_0^1, \ldots, I_0^n\right),$$

the I_0^α's being essentially constants of integration.

As mentioned earlier, the determination of even a single first integral is in most cases a non-trivial task; hence while in principle the above procedure is fine, its practical application is often a daunting task, to say the least.

It is also well known that symmetries play a crucial role in the solutions of differential equations. In fact much of the existing literature on symmetries of ODEs is restricted to what are known as Lie point symmetries. The differential equation (2.1)/(2.2) is said to admit a Lie point symmetry with generator

$$\mathbf{X} = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(k)}\partial_{y^{(k)}}, \qquad \text{where} \quad \eta^{(i)} = \frac{\mathrm{d}\eta^{(i-1)}}{\mathrm{d}x} - y^{(i)}\frac{\mathrm{d}\xi}{\mathrm{d}x},$$

if

$$[\mathbf{X}, \widetilde{D}] = g\widetilde{D} \tag{2.5}$$

holds. Here, $g = g(x, y, y', \ldots, y^{(n-1)})$ is some function and $\eta^{(i)}$'s denote the prolongations of the vector field (infinitesimal generators) $\mathbf{X}^{(0)} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$. For an nth-order ODE (2.1) the infinitesimal symmetry generators, when they exist, are determined from the linearized symmetry condition,

$$\eta^{(n)} = \xi w_x + \eta w_y + \eta^{(1)}w_{y'} + \cdots + \eta^{(n-1)}w_{y^{(n-1)}}, \tag{2.6}$$

when (2.1) holds [11]. In terms of the characteristic, $Q := \eta - y'\xi$, this condition may be written as

$$\widetilde{D}^n Q - w_{y^{(n-1)}}\widetilde{D}^{(n-1)}Q - \cdots - w_{y'}\widetilde{D}Q - w_y Q = 0. \tag{2.7}$$

For example when $y'' = w(x, y, y')$, the linearized symmetry condition is a second-order linear PDE

$$\widetilde{D}^2 Q - w_{y'}\widetilde{D}Q - w_y Q = 0 \tag{2.8}$$

with vector field

$$\widetilde{D} = \partial_x + y'\partial_y + w(x, y, y')\partial_{y'}.$$

3. Adjoint symmetries and integrating factors

The following equation is known as the adjoint of the linearized symmetry condition (2.7), and its solutions are usually called the adjoint symmetries

$$\widetilde{D}^n \Lambda + \widetilde{D}^{n-1}(w_{y^{(n-1)}}\Lambda) - \widetilde{D}^{n-2}(w_{y^{(n-2)}}\Lambda) + \cdots + (-1)^{n-1} w_y \Lambda = 0. \tag{3.1}$$

It must be stressed however that these solutions are neither symmetries nor generators of symmetries, and it is more appropriate to call a solution a *cocharacteristic* [11]. A systematic procedure for finding the solutions of (3.1) is to use an ansatz for Λ, for example, to assume that they are independent of $y^{(n-1)}$ or to even assume a suitable rational structure.

3.1. Review of the extended Prelle–Singer method

Let us consider once again the equation

$$y^{(n)} = w(x, y, y', \ldots, y^{(n-1)}), \tag{3.2}$$

together with the base one-forms $\mathrm{d}x, (\mathrm{d}y - y'\,\mathrm{d}x), \ldots, (\mathrm{d}y^{(n-1)} - w\mathrm{d}x)$. The null form obtained by multiplying all but the first one-form by functions $S_i(x, y, y', \ldots, y^{(n-1)})$ where $i = 0, \ldots, n-1$ and demanding that after addition the resultant one-form be exact is

$$\begin{aligned} -(S_0 y' + S_1 y'' + \cdots + S_{n-2} y^{(n-1)} + S_{n-1} w)\,\mathrm{d}x \\ + (S_0\,\mathrm{d}y + S_1\,\mathrm{d}y' + \cdots + S_{n-2}\,\mathrm{d}y^{(n-2)} + S_{n-1}\,\mathrm{d}y^{(n-1)}) \\ = \mathrm{d}I(x, y, y', \ldots, y^{(n-1)}) = 0. \end{aligned} \tag{3.3}$$

This implies

$$I_x = -(S_0 y' + S_1 y'' + \cdots + S_{n-2} y^{(n-2)} + w S_{n-1}) \tag{3.4}$$

$$I_y = S_0, \qquad I_{y'} = S_1, \ldots, I_{y^{(n-1)}} = S_{n-1}. \tag{3.5}$$

Clearly I is a first integral of the equation (3.2), provided it satisfies the integrability criteria

$$I_{xy^{(j)}} = I_{y^{(j)}x}, \qquad j = 0, \ldots, n-1, \tag{3.6}$$

$$I_{y^{(j)}y^{(k)}} = I_{y^{(k)}y^{(j)}}, \qquad 0 \leqslant j < k \leqslant n-1. \tag{3.7}$$

The vector field associated with (3.2) is

$$\widetilde{D} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \cdots + w \frac{\partial}{\partial y^{(n-1)}}, \tag{3.8}$$

in terms of which the integrability conditions (3.6) may be expressed as follows:

$$-\widetilde{D}[S_{n-1}] = (w_{y^{(n-1)}} S_{n-1} + S_{n-2}) \tag{3.9}$$

$$-\widetilde{D}[S_{n-2}] = (w_{y^{(n-2)}} S_{n-1} + S_{n-3}) \tag{3.10}$$

$$\vdots$$

$$-\widetilde{D}[S_2] = (w_{y'} S_{n-1} + S_0) \tag{3.11}$$

$$-\widetilde{D}[S_0] = w_y S_{n-1}. \tag{3.12}$$

The remaining integrability conditions (3.7) are all satisfied if

$$\frac{\partial S_{n-1}}{\partial y^{(j)}} = \frac{\partial S_j}{\partial y^{(n-1)}}, \qquad 0 \leqslant j \leqslant n-2. \tag{3.13}$$

Our primary interest is to know S_{n-1}, since the remaining ones can be determined algebraically from (3.9)–(3.12) in a recursive manner. Eliminating the S_i's by successively applying the vector field $\widetilde{D}$ to (3.9) and using the remaining ones, we obtain finally

$$\widetilde{D}^n[S_{n-1}] + \widetilde{D}^{n-1}[w_{y^{(n-1)}}S_{n-1}] - \widetilde{D}^{n-2}[w_{y^{(n-2)}}S_{n-1}] + \cdots + (-1)^{n-1}w_y S_{n-1} = 0. \tag{3.14}$$

But this is precisely the adjoint equation corresponding to the linearized symmetry equation (3.1), [11]. Thus the integrating factors of (2.1) are just the solutions of (3.14), which fulfil the integrability criteria stated in (3.13). Consequently, determination of the integrating factor S_{n-1} of (3.2) is basically equivalent to finding a solution of this equation. (The connection with the notation used in [9] is established by the following substitutions: $S_j \longrightarrow RS_{j+1}, \forall j = 0, \ldots, n-3$ and $S_{n-1} \longrightarrow R$.). The usual procedure to tackle such PDEs is to make an ansatz for S_{n-1}, for example assuming it to be a polynomial in $y^{(n-1)}$ of some suitable degree, and then obtaining its coefficients in a recursive manner. In their works, Chandrasekar *et al* have made a very interesting ansatz, in which they assumed a rational form for S_{n-1}. As a consequence, instead of solving the adjoint equation directly, they solved the set (3.9)–(3.12) of first-order equations by making appropriate ansätze for the S_i's. Suppose Λ^i be the solution(s) of the adjoint equation. Setting $S_{n-1} = \Lambda^i$ one can calculate the remaining S_j's in a recursive manner and check if (3.13) holds. In the event such an integrating factor exists and satisfies the integrability condition, its associated first integral may be obtained from the relation

$$I^i = \int S_0^i(\mathrm{d}y - y'\,\mathrm{d}x) + S_1^i(\mathrm{d}y' - y''\,\mathrm{d}x) + \cdots S_{n-1}^i(\mathrm{d}y^{(n-1)} - w\,\mathrm{d}x). \tag{3.15}$$

Essentially, therefore, one can choose to either solve the adjoint equation directly and obtain S_{n-1} through some suitable ansätze or make suitable ansätze for the S_k's and solve a set of n first-order PDEs. In general the former involves solving a single higher order equation, while the latter involves solving a system of first-order linear PDEs. It appears from the works [7–9] that the latter is much easier to implement, as far as practical computations are concerned. In the following, we illustrate these points with examples of second-order equations.

3.2. Some illustrative examples

Example 1. $y'' = w(x, y, y') = \frac{3y'^2}{y} + \frac{y'}{x}$.

Here the system of coupled first-order PDEs for the unknown functions S_0, S_1 is:

$$\widetilde{D}S_1 = -(w_{y'}S_1 + S_0) \tag{3.16}$$

$$\widetilde{D}S_0 = -w_y S_1, \tag{3.17}$$

where $\widetilde{D} = \partial_x + y'\partial_y + w\partial_{y'}$; the integrability condition is simply

$$S_{1y} = S_{0y'}. \tag{3.18}$$

The adjoint equation is

$$\widetilde{D}^2 S_1 + \widetilde{D}(w_{y'}S_1) - w_y S_1 = 0. \tag{3.19}$$

Assuming $\Lambda = S_1$ to be a solution of (3.19) independent of y', we have upon equating the coefficients of different powers of y' the following set of equations:

$$15\Lambda + 9y\Lambda_y + y^2\Lambda_{yy} = 0$$
$$3\Lambda + 3x\Lambda_x + y\Lambda_y + xy\Lambda_{xy} = 0$$
$$-\Lambda + x\Lambda_x + x^2\Lambda_{xx} = 0.$$

Their structure suggests an ansatz of the form $\Lambda = x^\alpha y^\beta$. One can verify that this leads to three solutions, namely,

$$\Lambda^1(x, y) = \frac{x}{y^3}, \qquad \Lambda^2(x, y) = \frac{1}{xy^3} \qquad \text{and} \qquad \Lambda^3(x, y) = \frac{1}{xy^5}.$$

However, only Λ^1 and Λ^2 are acceptable, as the other does not satisfy the integrability criterion. The results are summarized below along with the respective first integrals:

$$(i) \quad \Lambda^1 = S_1^1 = \frac{x}{y^3}, \qquad S_0^1 = -\frac{x}{y^3}\left(\frac{2}{x} + \frac{3y'}{y}\right), \qquad \text{with} \quad I^1(x, y, y') = \frac{xy' + y}{y^3}$$

$$(ii) \quad \Lambda^2 = S_1^2 = \frac{1}{xy^3}, \qquad S_0^2 = -\frac{3y'}{xy^4}, \qquad \text{with} \quad I^2(x, y, y') = \frac{y'}{xy^3}.$$

The first integral I^2 was obtained by Duarte *et al* in [5]. But for some reason the other one was not mentioned.

Example 2. In this example we study the equation

$$y'' = w(x, y, y') = -(kyy' + \lambda y),$$

where k, λ are constants, which represents a damped harmonic oscillator. As before one has to solve the adjoint symmetry equation (3.1) for $n = 2$, namely,

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + w_{y'}\Lambda_x + (w + y'w_{y'})\Lambda_y + (w_x + 2ww_{y'} + y'w_y)\Lambda_{y'}$$
$$+ \Lambda_{xx} + 2y'\Lambda_{xy} + y'^2\Lambda_{yy} + 2w\Lambda_{xy'} + 2wy'\Lambda_{yy'} + w^2\Lambda_{y'y'} = 0.$$

Solving this PDE is a rather daunting task even when $w(x, y, y')$ is fairly simple. It is therefore natural to make certain simplifying assumptions regarding the functional dependence of Λ. For instance one can begin by assuming Λ to be independent of a particular variable, say x, and see if that leads to a more manageable form of the adjoint equation. Alternatively, one may at the very outset assume that Λ depends on any one of the three variables x, y or y'. The choice of procedure to be adopted is one of sheer convenience. We illustrate this by first making the simplifying assumption $\Lambda_x = 0$, which leads to

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + (w + y'w_{y'})\Lambda_y + (w_x + 2ww_{y'} + y'w_y)\Lambda_{y'}$$
$$+ y'^2\Lambda_{yy} + 2wy'\Lambda_{yy'} + w^2\Lambda_{y'y'} = 0.$$

This is a linear parabolic PDE. Since $w = -(kyy' + \lambda y)$ we have

$$w_x = w_{y'y'} = 0, \qquad w_{y'} = -ky, \qquad w_y = -(ky' + \lambda) \qquad \text{and} \qquad w_{yy'} = -k.$$

As solving this PDE is still rather formidable, let us further assume $\Lambda_y = 0$. In other words Λ is just a function of y' and our equation simplifies further to

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + (w_x + 2ww_{y'} + y'w_{y'})\Lambda_{y'} + w^2\Lambda_{y'y'} = 0.$$

Plugging in the expressions for partial derivatives of w and equating the coefficients of different powers of y then leads to the following set of equations:

$$(ky' + \lambda)y'\Lambda_{y'} = \lambda\Lambda$$
$$2k\Lambda_{y'} + (ky' + \lambda)\Lambda_{y'y'} = 0.$$

These equations admit the particular solution $\Lambda^1(y') = \frac{y'}{(ky'+\lambda)}$ and one finds with $S_1^1 = \Lambda^1 = \frac{y'}{(ky'+\lambda)}$ that $S_0^1 = y$. The integrability condition $S^1_{1y} = S^1_{0y'}$ is trivially satisfied and the corresponding first integral is

$$I^1(x, y, y') = y' + \frac{1}{2}ky^2 - \frac{\lambda}{k}\log(ky' + \lambda).$$

Note that this first integral is independent of x by construction. For such first integrals, the method devised by Chandrasekar *et al* allows us to determine the form of S_0 *a priori*. We dwell on this aspect in the following section.

3.3. First integrals independent of a particular coordinate

In this subsection, we shall discuss the issue of first integrals independent of a particular coordinate. This usually leads to a reduction of the order of the equation, as will be explained below. The general ideas contained here will be illustrated with a specific example of a generic second-order ODE of the Liénard type.

An interesting feature occurs when the first integral is independent of a particular variable, say x, i.e., $I_x = 0$. Then, in general, (3.4) implies

$$S_0 = -\frac{1}{y'}(y''S_1 + \cdots + S_{n-2}y^{(n-1)} + S_{n-1}w),$$

which enables us to eliminate S_0, and causes a reduction in the order of the equations for determining the integrating factor. For instance in the case of a second-order ODE, we have $S_0y' + wS_1 = 0$, leading to $S_0 = -\frac{w}{y'}S_1$. As a result, one is left with a first-order PDE for determining S_1, namely,

$$\widetilde{D}(S_1) = -\left(w_{y'} - \frac{w}{y}\right)S_1. \tag{3.20}$$

On the other hand, for a third-order equation, we have

$$S_0 = -\frac{y''S_1 + wS_2}{y'}.$$

Elimination of S_0 from the system of equations (3.9)–(3.12) with $n = 3$ then requires us to solve for S_1 and S_2 from the coupled system:

$$\widetilde{D}[S_2] = -(w_{y''}S_2 + S_1)$$

$$\widetilde{D}[S_1] = -\left(\left(w_{y'} - \frac{w}{y'}\right)S_2 - \frac{y''}{y'}S_1\right).$$

This in turn leads to the following second-order equation for the integrating factor S_2:

$$\widetilde{D}^2S_2 + \widetilde{D}(w_{y''}S_2) - \frac{y''}{y'}\widetilde{D}S_2 - \left\{\left(w_{y'} - \frac{w}{y'}\right) + \frac{y''}{y'}w_{y''}\right\}S_2 = 0. \tag{3.21}$$

Thus the absence of one 'coordinate' in a first integral causes only marginal simplification, namely a reduction, by one, in the order of the equation to be solved for the integrating factor. Nevertheless this is extremely useful for second-order equations $y'' = w(x, y, y')$, since one is then required to solve a *single* first-order linear PDE for the integrating factor S_1. This fact was exploited in [7, 8]. Although in general for $n \geqslant 3$, the existence of an x independent first integral may not always lead to a substantial reduction of computational labor; nevertheless it is instructive to look into the *RS* method more carefully, as it has proved to be immensely successful in determining first integrals of many highly nonlinear oscillator-type systems. Generally, for equations of the generic form $y'' = -f_1(y)y' - f_0(y)$, (3.20) reduces to

$$\widetilde{D}S_1 = -\frac{f_0(y)}{y'}S_1.$$

The solution S_1^1 of example 2 suggests the ansatz $S_1 = \frac{y'}{h(y,y')}$ with the consequence

$$\widetilde{D}S_1 = \frac{\widetilde{D}(y')}{h} - \frac{y'}{h}\widetilde{D}h = -\frac{f_0(y)}{h}.$$

Therefore, the problem now reduces to a determination of the function $h(y, y')$ from the following relation (since $\widetilde{D}(y') = w$):

$$\begin{aligned} y'\widetilde{D}(h) &= (w + f_0)h = -f_1(y)y'h \\ \widetilde{D}(h) &= -f_1(y)h. \end{aligned} \tag{3.22}$$

The resulting PDE for h is explicitly given by

$$y'h_y + (-f_1y' - f_0)h_{y'} = -f_1y'h.$$

For $f_1 = ky$ and $f_0 = \lambda y$, assuming furthermore that h is independent of y, we obtain $h(y') = C(ky' + \lambda)$. Thus once again we get the solutions, setting constant $C = 1$,

$$S_1 = \frac{y'}{(ky' + \lambda)} \qquad \text{and} \qquad S_0 = y,$$

which satisfy the integrability criterion.

As pointed out in [9], it is often more convenient to modify the ansatz for S_1 to $S_1 = \frac{y'}{h(y,y')^r}$ to handle more complicated situations.

For generic equations of the form (Liénard type)

$$y'' = -f_1(y)y' - f_0(y)$$

with this ansatz for S_1, (3.22) is modified to

$$r\widetilde{D}(h) = -f_1(y)h. \tag{3.23}$$

Assuming $h(y, y') = A(y) + B(y)y' + C(y)y'^2$, substitution into (3.23) leads to the following set of equations for determining the unknown functions A, B, C upon equating coefficients of different powers of y':

$$C_y = 0, \qquad rB_y = (2rf_0 - f_1)C, \qquad rA_y = (rf_0 - f_1)B - 2rCf_1 \qquad \text{and} \qquad rf_0B = f_1A. \tag{3.24}$$

Suppose

$$f_0(y) = \lambda y^{\xi} \qquad \text{and} \qquad f_1(y) = \mu y^{\eta},$$

where λ, μ are parameters and ξ, η are constants. We obtain the following solutions for C, B and A:

$$C(y) = \gamma, \qquad rB(y) = \mu\gamma\frac{(2r-1)}{\eta+1}y^{\eta+1} + \beta$$

$$rA(y) = \frac{2\lambda r\gamma}{\xi+1}y^{\xi+1} + \mu(r-1)\left[\frac{(2r-1)\mu\gamma}{2r(\eta+1)^2}y^{2(\eta+1)} + \frac{\beta}{r(\eta+1)}y^{\eta+1}\right] + \alpha.$$

Here α, β and γ are constants of integration. From the last condition in (3.24), i.e., $rf_0B = f_1A$, it follows, assuming $\xi \neq \eta$, that $\alpha = \beta = 0$ and leads to the following relation:

$$\lambda r\left[\frac{(2r-1)}{(\eta+1)} - \frac{2}{(\xi+1)}\right]y^{\xi+\eta+1} = \frac{\mu^2(r-1)(2r-1)}{2r(\eta+1)^2}y^{3\eta+2}. \tag{3.25}$$

One can then identify two possible cases.

(a) When $r = 1$ we have $\xi = 2\eta + 1$ and $A(y) = \frac{\lambda\gamma}{(\eta+1)} y^{2(\eta+1)}$ and $B(y) = \frac{\mu\gamma}{(\eta+1)} y^{(\eta+1)}$. The corresponding integrating factor is

$$S_1^a = \frac{y'}{\left[\frac{\lambda\gamma}{(\eta+1)} y^{2(\eta+1)} + \frac{\mu\gamma}{(\eta+1)} y^{(\eta+1)} y' + \gamma y'^2\right]} \qquad \text{and} \qquad S_0^a = \frac{\mu y^\eta y' + \lambda y^{2\eta+1}}{y'} S_1.$$

(b) For $r \neq 1$, assuming the exponents of y in (3.25) to be equal, we find once again $\xi = 2\eta + 1$. Upon equating their coefficients, we obtain a quadratic equation for the exponent r, occurring in the denominator of the integrating factor, with solution $r = \frac{\mu^2}{4\lambda(\eta+1)}\left[1 \pm \sqrt{1 - \frac{4\lambda}{\mu^2}(\eta+1)}\right]$. Therefore, in this case $S_1^b = \frac{y'}{h^r}$ where

$$h(y, y') = \frac{\gamma}{(\eta+1)}\left[\lambda + \mu^2 \frac{(r-1)(2r-1)}{2r^2(\eta+1)}\right] y^{2(\eta+1)} + \frac{\gamma\mu(2r-1)}{r(\eta+1)} y^{\eta+1} y' + \gamma y'^2.$$

4. Coupled second-order equations

In this section, we consider a system of second-order ODEs to illustrate an application of the coupled version of the adjoint equation.

Let us consider the system of coupled second-order equations:

$$\ddot{x} = \phi_1(x, y) \qquad \text{and} \qquad \ddot{y} = \phi_2(x, y). \tag{4.1}$$

As before, consider the following base one forms $(\mathrm{d}x - \dot{x}\,\mathrm{d}t)$, $(\mathrm{d}y - \dot{y}\,\mathrm{d}t)$, $(\mathrm{d}\dot{x} - \phi_1\,\mathrm{d}t)$, $(\mathrm{d}\dot{y} - \phi_2\,\mathrm{d}t)$. Let S_1, S_2 and R_1, R_2 be functions such that

$$S_1(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + S_2(\mathrm{d}y - \dot{y}\,\mathrm{d}t) + R_1(\mathrm{d}\dot{x} - \phi_1\,\mathrm{d}t) + R_2(\mathrm{d}\dot{y} - \phi_2\,\mathrm{d}t) = \mathrm{d}I(t, x, y, \dot{x}, \dot{y}) = 0. \tag{4.2}$$

Hence

$$I_t = -(S_1\dot{x} + S_2\dot{y} + R_1\phi_1 + R_2\phi_2) \tag{4.3}$$

$$I_x = S_1, \quad I_y = S_2, \quad I_{\dot{x}} = R_1, \quad I_{\dot{y}} = R_2. \tag{4.4}$$

The functions R_1, R_2 are the integrating factors. Compatibility of the set of (4.3) and (4.4), namely,

$$\begin{aligned} &I_{tx} = I_{xt}, \quad I_{ty} = I_{yt}, \quad I_{t\dot{x}} = I_{\dot{x}t}, \quad I_{t\dot{y}} = I_{\dot{y}t} \\ &I_{xy} = I_{yx}, \quad I_{x\dot{x}} = I_{\dot{x}x}, \quad I_{x\dot{y}} = I_{\dot{y}x}, \quad I_{y\dot{x}} = I_{\dot{x}y}, \quad I_{y\dot{y}} = I_{\dot{y}y}, \end{aligned} \tag{4.5}$$

requires that the following hold:

$$D[R_1] = -(S_1 + R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}}) \tag{4.6}$$

$$D[R_2] = -(S_2 + R_1\phi_{1\dot{y}} + R_2\phi_{2\dot{y}}) \tag{4.7}$$

$$D[S_1] = -(R_1\phi_{1x} + R_2\phi_{2x}) \tag{4.8}$$

$$D[S_2] = -(R_1\phi_{1y} + R_2\phi_{2y}), \tag{4.9}$$

where $D = \partial_t + \dot{x}\partial_x + \dot{y}\partial_y + \phi_1\partial_{\dot{x}} + \phi_2\partial_{\dot{y}}$. It is evident that once R_1, R_2 are known the remaining S_1, S_2 can be determined algebraically from (4.6) and (4.7). Since our basic aim is to determine the integrating factors, we can eliminate, say, S_1 by differentiating (4.6) and using (4.8) to obtain

$$D^2[R_1] + D[R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}}] - (R_1\phi_{1x} + R_2\phi_{2x}) = 0. \tag{4.10}$$

Similarly eliminating S_2 yields

$$D^2[R_2] + D[R_1\phi_{1\dot{y}} + R_2\phi_{2\dot{y}}] - (R_1\phi_{1y} + R_2\phi_{2y}) = 0. \tag{4.11}$$

Equations (4.10)–(4.11) constitute the coupled version of the adjoint equation (3.1) when $n = 2$.

One needs to check, of course, that the solutions of the coupled adjoint equations indeed satisfy the compatibility conditions (4.5). In general one employs an ansatz for R_1, R_2 in order to solve the system of PDEs (4.10)–(4.11). From a knowledge of R_1, R_2 and S_1, S_2 it is straightforward to obtain the first integral from

$$I = \int S_1(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + S_2(\mathrm{d}y - \dot{y}\,\mathrm{d}t) + R_1(\mathrm{d}\dot{x} - \phi_1\,\mathrm{d}t) + R_2(\mathrm{d}\dot{y} - \phi_2\,\mathrm{d}t). \tag{4.12}$$

Example 3. Consider the following system of second-order equations:

$$\begin{aligned} &\ddot{x} + \frac{\alpha}{x^2}g(u) - \frac{\lambda}{x^3} = 0 \\ &\ddot{y} + \frac{\beta}{x^2}f(u) - \frac{\mu}{y^3} = 0, \qquad u = \frac{y}{x}. \end{aligned} \tag{4.13}$$

Here α, β, λ and μ are parameters and f and g are arbitrary functions. Writing these equations in the form $\ddot{x} = \phi_1(x, y)$ and $\ddot{y} = \phi_2(x, y)$, we identify

$$\phi_1(x, y) = -\frac{\alpha}{x^2}g(u) + \frac{\lambda}{x^3} \qquad \text{and} \qquad \phi_2(x, y) = -\frac{\beta}{x^2}f(u) + \frac{\mu}{y^3}.$$

Note here ϕ_1 and ϕ_2 are velocity independent and for a time-independent first integral $I_t = 0$, we may take $D = \dot{x}\partial_x + \dot{y}\partial_y + \phi_1\partial_{\dot{x}} + \phi_2\partial_{\dot{y}}$. In that event, with the following ansatz for R_1 and R_2, namely,

$$R_1 = a_1(x, y)\dot{x} + a_2(x, y)\dot{y} \qquad \text{and} \qquad R_2 = b_1(x, y)\dot{x} + b_2(x, y)\dot{y}, \tag{4.14}$$

(4.10) and (4.11) yield the following equations:

$$\begin{aligned} &\dot{x}^3 a_{1xx} + \dot{x}^2\dot{y}(a_{2xx} + 2a_{1xy}) + \dot{x}\dot{y}^2(2a_{2xy} + a_{1yy}) + a_{2yy}\dot{y}^3 \\ &\qquad + \dot{x}\{(\phi_1 a_1 + \phi_2 a_2)_x + 2a_{1x}\phi_1 + (a_{2x} + a_{1y})\phi_2\} \\ &\qquad + \dot{y}\{(\phi_1 a_1 + \phi_2 a_2)_y + 2a_{2y}\phi_2 + (a_{2x} + a_{1y})\phi_1\} \\ &\quad = \dot{x}(\phi_{1x}a_1 + \phi_{2x}b_1) + \dot{y}(\phi_{1x}a_2 + \phi_{2x}b_2), \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\dot{x}^3 b_{1xx} + \dot{x}^2\dot{y}(b_{2xx} + 2b_{1xy}) + \dot{x}\dot{y}^2(2b_{2xy} + b_{1yy}) + b_{2yy}\dot{y}^3 \\ &\qquad + \dot{x}\{(\phi_1 b_1 + \phi_2 b_2)_x + 2b_{1x}\phi_1 + (b_{2x} + b_{1y})\phi_2\} \\ &\qquad + \dot{y}\{(\phi_1 b_1 + \phi_2 b_2)_y + 2b_{2y}\phi_2 + (b_{2x} + b_{1y})\phi_1\} \\ &\quad = \dot{x}(\phi_{1y}a_1 + \phi_{2y}b_1) + \dot{y}(\phi_{1y}a_2 + \phi_{2y}b_2). \end{aligned} \tag{4.16}$$

Equating coefficients of different powers of the velocities we obtain the following system of equations:

$$a_{1xx} = 0, \qquad a_{2xx} + 2a_{1xy} = 0, \qquad a_{1yy} + 2a_{2xy} = 0, \qquad a_{2yy} = 0, \tag{4.17}$$

$$(\phi_1 a_1 + \phi_2 a_2)_x + 2a_{1x}\phi_1 + (a_{2x} + a_{1y})\phi_2 = (\phi_{1x}a_1 + \phi_{2x}b_1), \tag{4.18}$$

$$(\phi_1 a_1 + \phi_2 a_2)_y + 2a_{2y}\phi_2 + (a_{2x} + a_{1y})\phi_1 = (\phi_{1x}a_2 + \phi_{2x}b_2) \tag{4.19}$$

$$b_{1xx} = 0, \qquad b_{2xx} + 2b_{1xy} = 0, \qquad b_{1yy} + 2b_{2xy} = 0, \qquad a_{2yy} = 0, \tag{4.20}$$

$$(\phi_1 b_1 + \phi_2 b_2)_x + 2b_{1x}\phi_1 + (b_{2x} + b_{1y})\phi_2 = (\phi_{1y}a_1 + \phi_{2y}b_1), \tag{4.21}$$

$$(\phi_1 b_1 + \phi_2 b_2)_y + 2b_{2y}\phi_2 + (b_{2x} + b_{1y})\phi_1 = (\phi_{1y}a_2 + \phi_{2y}b_2). \tag{4.22}$$

Observe that the choice $a_k =$ constant and $b_k =$ constant $(k = 1, 2)$ satisfies (4.17) and (4.20), while the remaining equations then simplify to

$$\phi_{2x}(b_1 - a_2) = 0, \qquad \phi_{1y}(a_2 - b_1) = 0$$

$$(\phi_{1x} - \phi_{2y})a_2 - \phi_{1y}a_1 + \phi_{2x}b_2 = 0$$

$$\phi_{1y}a_1 + (\phi_{2y} - \phi_{1x})b_1 - \phi_{2x}b_2 = 0.$$

The first two equations imply $a_2 = b_1$, which renders the second and the third equations identical, namely,

$$(\phi_{1x} - \phi_{2y})a_2 - \phi_{1y}a_1 + \phi_{2x}b_2 = 0.$$

If equations (4.13) are derivable from a potential then it is necessary that $\phi_{1y} = \phi_{2x}$. With this in mind the above equation can be satisfied by making the choice $a_2 = b_1 = 0$ whilst a_1 and b_2 are arbitrary. Therefore, the choice $a_1 = b_2 = 1$ and $a_1 = b_1 = 0$ leads to the following solution:

$$R_1 = \dot{x} \qquad R_2 = \dot{y}. \tag{4.23}$$

In this case the solutions of S_1 and S_2 from (4.6) and (4.7) are found to be

$$S_1 = -\phi_1 = \frac{\alpha}{x^2}g(u) - \frac{\lambda}{x^3}$$

$$S_2 = -\phi_2 = \frac{\beta}{x^2}f(u) - \frac{\mu}{y^3}, \qquad u = \frac{y}{x}.$$

Using the above values of R_i and S_i $(i = 1, 2)$ we obtain from (4.12) the first integral as

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} + N(x, y),$$

where

$$N(x, y) = \int \frac{\alpha}{x^2}g(u)\,\mathrm{d}x + \int \frac{\beta}{x^2}f(u)\,\mathrm{d}y.$$

On the other hand the condition $\phi_{1y} = \phi_{2x}$ translates to

$$\alpha g'(u) + 2\beta f(u) + \beta u f'(u) = 0. \tag{4.24}$$

Using this condition $N(x, y)$ may be evaluated and we find that

$$N(x, y) = -\frac{\beta}{x}\left(\frac{\alpha}{\beta}g(u) + uf(u)\right).$$

Hence a first integral for the system of second-order equations is

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} - \frac{\beta}{x}\left(\frac{\alpha}{\beta}g(u) + uf(u)\right). \tag{4.25}$$

Let us now look for another solution set of the coupled adjoint equations for R_1 and R_2. It is easily verified that

$$a_1(x, y) = y^2, \qquad a_2(x, y) = -xy = b_1(x, y) \qquad \text{and} \qquad b_2(x, y) = x^2 \tag{4.26}$$

satisfy (4.17) and (4.20) while (4.18) and (4.22) are identically satisfied. The remaining equations (4.19) and (4.21) become identical and reduce to the following equation:

$$3(y\phi_1 - x\phi_2) = (\phi_{2y} - \phi_{1x})xy - \phi_{1y}y^2 + \phi_{2x}x^2. \tag{4.27}$$

Substituting the values of ϕ_i $(i = 1, 2)$ and their derivatives leads to the following condition on the functions f and g, namely:

$$\alpha u g(u) - \beta f(u) = 0, \qquad u = \frac{y}{x}. \tag{4.28}$$

From (4.26) we derive the following solution for R_i $(i = 1, 2)$:

$$R_1 = -y(x\dot{y} - y\dot{x}) \qquad \text{and} \qquad R_2 = x(x\dot{y} - y\dot{x}). \tag{4.29}$$

The corresponding values of S_i $(i = 1, 2)$ are now

$$S_1 = (x\dot{y} - y\dot{x})\dot{y} - \lambda\frac{y^2}{x^3} + \mu\frac{x}{y^2} \qquad \text{and} \qquad S_2 = -(x\dot{y} - y\dot{x})\dot{x} + \lambda\frac{y}{x^3} - \mu\frac{x^2}{y^3}, \tag{4.30}$$

where use has been made of the condition (4.28). Hence from (4.12) we obtain another first integral given by

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(y\dot{x} - x\dot{y})^2 + \frac{\lambda}{2}\left(\frac{y}{x}\right)^2 + \frac{\mu}{2}\left(\frac{x}{y}\right)^2. \tag{4.31}$$

The two first integrals given by (4.25) and (4.31) will be valid simultaneously provided we can find functions f and g which satisfy (4.24) and (4.28). It is easily verified that these require the functions f and g to be given by

$$g(u) = \frac{1}{(1 + u^2)^{3/2}} \qquad \text{and} \qquad f(u) = \frac{\alpha}{\beta}\frac{u}{(1 + u^2)^{3/2}},$$

respectively. Under the circumstances the system of second-order equations reduces to the following well-known system

$$\ddot{x} + \frac{\alpha x}{(x^2 + y^2)^{3/2}} - \frac{\lambda}{x^3} = 0 \qquad \ddot{y} + \frac{\alpha y}{(x^2 + y^2)^{3/2}} - \frac{\mu}{y^3} = 0,$$

with the first integrals

$$I_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} - \frac{\alpha}{\sqrt{x^2 + y^2}}$$

$$I_2 = \frac{1}{2}(y\dot{x} - x\dot{y})^2 + \frac{\lambda}{2}\left(\frac{y}{x}\right)^2 + \frac{\mu}{2}\left(\frac{x}{y}\right)^2.$$

A more interesting situation from the physical point of view arises when the functions f and g satisfy condition (4.24) but *not* condition (4.28). In that event the system of equations (4.13) admits just one first integral given by (4.25), with f and g satisfying (4.24). In [12] the authors obtained a system of equations similar in structure to (4.13), in the context of the dynamics of stellar systems, with

$$f(u) = 2(1 - ug(u)).$$

Condition (4.24) then leads to the following differential equation determining $g(u)$:

$$(1 - 2u^2)g'(u) = 2(3ug(u) - 2)$$

and the first integral assumes the form (setting all the parameters equal to unity)

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(y\dot{x} - x\dot{y})^2 + \frac{1}{2x^2} + \frac{1}{2y^2} - \frac{1}{x}(2u + (1 - 2u^2)g(u)), \qquad u = \frac{y}{x}.$$

In fact this first integral serves as the Hamiltonian.

5. Outlook

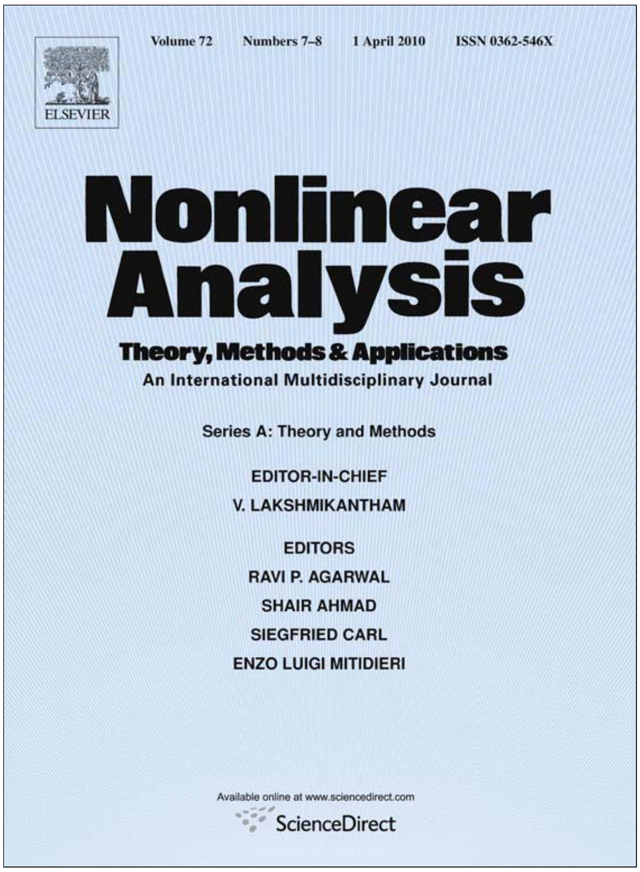
In this paper we have studied the *RS*-pair method, for determination of first integrals of ODEs, as proposed by Chandrasekar *et al* and have shown how their procedure may be brought within the general ambit of the adjoint equation method. In a similar spirit we have derived the coupled adjoint equations for analysis of coupled systems of second-order ODEs. Its use has been illustrated for a system occurring in the context of stellar dynamics. It is obvious that the procedure can easily be extended to systems of higher order equations. Lastly, it may be mentioned that one can apply this method to the equations of the Painlevé–Gambier classification and that this is currently being pursued.

Acknowledgment

The authors wish to thank the referees for their detailed comments. They wish to thank Pepin Cariñena, Basil Grammaticos and Manolo Rañada for enlightening discussions. In addition AGC wishes to acknowledge the support provided by the S N Bose National Centre for Basic Sciences, Kolkata in the form of an associateship.

References

[1] Lie S 1874 Verhandlungen der Gesellschaft der Wissenschaften zu Christinia
[2] Lie S 1888 Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen *x*,*y*, die eine Gruppe von Transformationen gestatten *Math. Ann.* **32** 213–81
[3] Bluman G and Anco S C 2002 *Symmetry and Integration Methods for Differential Equations (Applied Mathematical Sciences* vol 154) (New York: Springer)
[4] Prelle M J and Singer M F 1983 Elementary first integrals of differential equations *Trans. Amer. Math. Soc.* **279** 215–29
[5] Duarte L G S, Duarte S E S, da Mota L A C and Skea J E F 2001 Solving second-order ordinary differential equations by extending the Prelle–Singer method *J. Phys. A: Math. Gen.* **34** 3015–24
[6] Duarte L G S, Duarte S E S and da Mota L A C 2002 Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution *J. Phys. A: Math. Gen.* **35** 1001–6
[7] Chandrasekar V K, Senthilvelan M and Lakshmanan M 2006 A unification in the theory of linearization of second-order nonlinear ordinary differential equations *J. Phys. A: Math. Gen.* **39** L69–76
[8] Chandrasekar V K, Senthilvelan M and Lakshmanan M 2005 Extended Prelle–Singer method and integrability/solvability of a class of nonlinear *n*th order ordinary differential equations *J. Nonlinear Math. Phys.* **12** 184–201
[9] Chandrasekar V K, Senthilvelan M and Lakshmanan M 2006 A simple and unified approach to identify integrable nonlinear oscillators and systems *J. Math. Phys.* **47** 023508
[10] Stephani H 1989 *Differential Equations—Their Solution Using Symmetries* ed M MacCullum (Cambridge: Cambridge University Press)
[11] Hydon P E 1999 *Symmetry Methods for Differential Equations* (Cambridge: Cambridge University Press)
[12] Sridhar S and Nityananda R 1989 Undamped oscillations of collisionless systems: spheres, spheroids and discs *J. Astrophys. Astron.* **10** 279–93
[13] Man Y K and MacCallum M 1997 A rational approach to the Prelle–Singer algorithm *J. Symb. Comput.* **24** 31–43
[14] Singer M 1992 Liouvillian first integrals of differential equations *Trans. Am. Math. Soc.* **333** 673–88


Volume 72 Numbers 7–8 1 April 2010 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
EDITOR-IN-CHIEF
V. LAKSHMIKANTHAM
EDITORS
RAVI P. AGARWAL
SHAIR AHMAD
SIEGFRIED CARL
ENZO LUIGI MITIDIERI
Available online at www.sciencedirect.com
ScienceDirect

Nonlinear Analysis 72 (2010) 3247–3257

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On generalized Sundman transformation method, first integrals, symmetries and solutions of equations of Painlevé–Gambier type

Partha Guha [a,b,*], Barun Khanra [c], A. Ghose Choudhury [d]

[a] *Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103, Leipzig, Germany*
[b] *S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India*
[c] *Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Kolkata-700004, India*
[d] *Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Kolkata-700009, India*

A R T I C L E   I N F O

Article history:
Received 1 October 2009
Accepted 2 December 2009

MSC:
34C14
34C20

Keywords:
Sundman transformation
Sundman symmetries
Painlevé–Gambier equations
First integrals
Jacobi equation

A B S T R A C T

We employ the generalized Sundman transformation method to obtain certain new first integrals of autonomous second-order ordinary differential equations belonging to the Painlevé–Gambier classification scheme. This method not only yields systematically the known first integrals of a large number of the Painlevé–Gambier equations but also some time dependent ones, which greatly simplify the computation of their corresponding solution. In addition we also compute the Sundman symmetries of these equations.

1. Introduction

The problem of constructing solutions of a given differential equation forms the cornerstone of their analysis. Not unrelated to this problem is the issue of determining first integrals of the differential equation under consideration. This is because the existence of a sufficient number of first integrals often enables us to construct a solution by mere elimination of the derivatives of the dependent variable.

Although there are a number of well-defined methods for the solution of linear ordinary differential equations (ODEs) the same, however, cannot be said for nonlinear ODEs. It was only through the efforts of Lie towards the end of the nineteenth century that many *ad hoc* methods for the solution of nonlinear ODEs were gradually systemized. Besides it is generally acknowledged that, whenever a differential equation is amenable to a solution, it is because of some sort of underlying symmetry of the equation [1,2]. Much of Lie's work was concerned with point transformations of the form

$$(t, x) \mapsto (T, X) \quad \text{where } T = G(t, x),\ X = F(t, x)$$

with the transformation often involving one or more continuous real parameters.

Furthermore towards the very end of the nineteenth century the fact that a given differential equation could be transformed to a linear equation, that is, it could be *linearized* came to light [3]. This provided a mechanism to work out the solutions of many nonlinear differential equations by systematically transforming them to linear equations. In fact Lie

* Corresponding author at: Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103, Leipzig, Germany.
E-mail addresses: partha@bose.res.in (P. Guha), barunkhanra@rediffmail.com (B. Khanra), a_ghosechoudhury@rediffmail.com (A.G. Choudhury).

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.12.004

himself [4] solved the linearization problem for second-order ordinary differential equations in the sense that he found the general form of all second-order ODEs that could be reduced to a linear equation by changing the independent and dependent variables [4].

Subsequently attempts were made to look beyond point transformations when dealing with second and higher-order ODEs, with some degree of success. For instance, one may look for nonlocal transformations under which a given ordinary differential equation is linearizable. This problem was studied by Duarté et al. [5] and considers transformations of the form

$$X(T) = F(t, x), \qquad \mathrm{d}T = G(t, x)\mathrm{d}t. \tag{1.1}$$

Here F and G are arbitrary smooth functions and it is assumed that the Jacobian $J \equiv \frac{\partial(T,X)}{\partial(t,x)} \neq 0$. If one knows the functional form of $x(t)$, then the latter transformation ceases to be nonlocal, but knowledge of $x(t)$ is what we are interested in, in the first place. Consequently (1.1) constitutes a particular type of nonlocal transformation. It must be pointed out that term nonlocal is very general in nature and it is therefore better to refer to such a transformation as a generalized Sundman transformation (GST) [6–8], in view of its similarity with the original transformation used in Sundman's analysis [9]. Other authors have also used such transformations, but have preferred to call them non point transformations [10,11].

In [5] the authors derived the most general condition under which a second-order ordinary differential equation is transformable to the linear equation

$$X''(T) = 0,$$

(here $X' = \frac{\mathrm{d}X}{\mathrm{d}T}$) under a generalized Sundman transformation. Euler and Euler, studied the case of the general anharmonic oscillator in [7], wherein they investigated the *Sundman symmetries* of second-order and third-order nonlinear ODEs. These symmetries, which are in general nonlocal transformations can be calculated systematically and can be used to find first integrals of the equations. Euler et al. used the generalized Sundman transformation to obtain a relation between a generalized Emden–Fowler equation and the first Painlevé transcendent [6].

1.1. *Result and plan*

In this paper we concentrate on generalized Sundman transformations and Sundman symmetries of second-order ordinary differential equations of the Painlevé–Gambier classification [12,13,18]. We compute new first integrals of some of the autonomous Painlevé–Gambier equations, which are not mentioned in the classic text by Ince [14]. The method used for this purpose is the generalized Sundman transformation. Barring the six Painlevé equations, it is known that, the remaining 44 equations of this classification scheme admit solutions in terms of known special functions. Therefore knowledge of additional (time dependent) first integrals is not essential, as far as construction of their solutions is concerned. But the deduction of time dependent first integrals is interesting from a broader perspective because of the recent interest in non autonomous ODEs. Secondly, we also compute the associated Sundman symmetries of these equations. It is true that the first integrals of the Painlevé–Gambier (PG) equations are known from other methods. In this paper we demonstrate that the first integrals of PG equations can be computed in a simple manner using Sundman's method. As a bonus we obtain the Sundman symmetries of these class of equations which are not stated elsewhere.

The *organization* of the paper is as follows. In Section 2 we introduce the notion of a generalized Sundman transformation and define the associated Sundman symmetry. Section 3 begins with a discussion of the generalized Sundman transformation for the Jacobi equation and proceeds to outline the format for its explicit evaluation. It then examines, as a special case of the Jacobi equation, particular equations of the Painlevé–Gambier classification, notably the equations numbered 11, 17, 37, 41 and 43 of Ince's book, from the viewpoints of the generalized Sundman transformation, the associated Sundman symmetry including also their solution. In Section 4, four more equations of the Painlevé–Gambier classification (namely the equations numbered 18, 19, 21 and 22) which also arise as special cases of the Jacobi equation are analyzed and their parametric solutions are constructed by exploiting the Sundman transformation.

2. Generalized Sundman transformation and symmetry

Consider an nth-order ordinary differential equation given by

$$x^{(n)} = w(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}) \tag{2.1}$$

where $x = x(t)$ and $x^{(k)} = \mathrm{d}^k x/\mathrm{d}t^k$. Such an ODE may always be written as a system of first-order ODEs

$$\dot{x^i} = w^i(\mathbf{x}) \quad i = 1, \ldots, n.$$

By a first integral we mean the following.

Definition 2.1. Let $I(\mathbf{x}, t)$ be a C^1 function on an open set $U \in \mathbb{R}^n$. Then $I(\mathbf{x}, t)$ is a first integral of the vector field $\mathbf{w} \cdot \partial_{\mathbf{x}}$ corresponding to the system of ODEs $\dot{\mathbf{x}} = \mathbf{w}(\mathbf{x})$ if and only if it is constant along any solution of the equation.

This means that given a time interval T, $I(\mathbf{x}(t), t)$ is independent of t for all $t \in T$. Formally we define a generalized Sundman transformation for (2.1) as follows.

Definition 2.2 (*Sundman Transformation*). A coordinate transformation of the form

$$X(T) = F(t, x), \qquad dT = G(t, x)dt, \qquad \frac{\partial F}{\partial x} \neq 0, \quad G \neq 0 \tag{2.2}$$

is said to be a generalized Sundman transformation of Eq. (2.1) if differentiable functions F and G are determined such that (2.1) is transformed to the autonomous equation

$$X^{(n)} = w_0(X, X', \ldots, X^{(n-1)}), \tag{2.3}$$

where $X' = dX/dT$ etc.

This notion of generalized Sundman transformation, as a kind of nonlocal extension of invertible point transformation was made by Duarte et al. Its nonlocal character is apparent from the fact that $T = \int G(t, x(t))\, dt$. If (2.3) happens to be a linear ordinary differential equation, then we say that the original ordinary differential equation, (2.1), is linearizable. In the event $w_0 = 0$ one says that (2.1) has been mapped to the free particle equation.

Closely related to the concept of a generalized Sundman transformation is the notion of an associated Sundman symmetry. This is similar in spirit to the existence of a Lie symmetry under point transformations.
Suppose that we have a generalized Sundman transformation (GST)

$$X(T) = F(\tilde{t}, \tilde{x}), \qquad dT = G(\tilde{t}, \tilde{x})d\tilde{t}$$

which maps the equation

$$\tilde{x}^{(n)} = w(\tilde{t}, \tilde{x}, \dot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}) \longmapsto X^{(n)} = w_0(X, X', \ldots, X^{(n-1)}).$$

If there exists a transformation of the differentiable functions $F(\tilde{t}, \tilde{x})$ and $G(\tilde{t}, \tilde{x})$, considered as functions of $F(t, x)$ and $G(t, x)$, such that our original differential equation (2.1) remains invariant under the transformation, then the transformation defines a Sundman symmetry. Formally it may be defined as follows.

Definition 2.3 (*Sundman Symmetry*). A Sundman symmetry [7] for Eq. (2.1) is a transformation of the form

$$F(\tilde{t}, \tilde{x}) = M(F(t, x), G(x, t)), \qquad G(\tilde{t}, \tilde{x})d\tilde{t} = N(F(t, x), G(t, x))dt, \tag{2.4}$$

where M and N are some differentiable functions such that the transformation keeps (2.1) invariant. In other words (2.1) is transformed to

$$\tilde{x}^{(n)} = w(t; \tilde{x}, \dot{\tilde{x}}, \ddot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}). \tag{2.5}$$

If $M(F, G) = F$ and $N(F, G) = G$, then of course, the symmetry is trivial. The set of conditions on the differentiable functions F and G when the differential equation (2.1) is mapped to the autonomous differential equation (2.3) are referred to as *the Sundman determining equations*. This is illustrated below.

A Sundman symmetry (2.4) is obtained by choosing M and N in such a way that the Sundman determining equations remain invariant. If

$$X = F(\tilde{t}, \tilde{x}), \qquad dT = G(\tilde{t}, \tilde{x})d\tilde{t}$$

transforms (2.5) to (2.3) and

$$X = M(F(t, x), G(t, x)), \qquad dT = N(F(t, x), G(t, x))dt$$

also transforms (2.1) to (2.3), then the composition of these two GSTs leads to the Sundman symmetry (2.4) for (2.1).

3. GST and Sundman symmetry for Jacobi's equation

We begin this section by considering the well-known Jacobi equation since many of the second-order equations of the Painlevé–Gambier classification may then be regarded as special cases of this rather general equation. The Jacobi equation is given by [15,16]

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x) = 0, \tag{3.1}$$

and may be transformed to $X'' = 0$ under the transformation (2.4) when its coefficients satisfy the following Sundman determining equations:

$$\frac{1}{2}\phi_x(F, G; t, x) = \frac{F_{xx}}{F_x} - \frac{G_x}{G} \tag{3.2}$$

$$\phi_t(F, G; t, x) = \frac{2F_{xt}}{F_x} - \frac{F_t}{F_x}\frac{G_x}{G} - \frac{G_t}{G} \tag{3.3}$$

$$B(F, G; t, x) = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x}. \tag{3.4}$$

Further it admits a Sundman symmetry of the form (2.4) if and only if M and N are given by

$$M(F,G) = M(F(t,x)) \quad \text{and} \quad N(F,G) = G(t,x)\psi(F). \tag{3.5}$$

The Sundman symmetry of (3.1) is of the form

$$F(\tilde{x},\tilde{t}) = M(F(x,t)), \tag{3.6}$$

$$G(\tilde{t},\tilde{x}) = G(t,x)\frac{\mathrm{d}M(F(t,x))}{\mathrm{d}F}\mathrm{d}t \tag{3.7}$$

with no further condition on the differentiable function M.
This follows from the following observation. Suppose for the sake of notational convenience we denote

$$F(\tilde{t},\tilde{x}) = \hat{F} \quad \text{and} \quad G(\tilde{t},\tilde{x}) = \hat{G}.$$

The invariance of the Sundman determining equations requires each expression occurring in (3.2)–(3.4) to be invariant. From (3.4) we observe, making use of (3.5)

$$\frac{\hat{F}_{tt}}{\hat{F}_x} - \frac{\hat{G}_t}{\hat{G}}\frac{\hat{F}_t}{\hat{F}_x} = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + \left(\frac{M''(F)}{M'(F)} - \frac{\psi'(F)}{\psi(F)}\right)\frac{F_t^2}{F_x}.$$

The left hand side is clearly an invariant, provided

$$\left(\frac{M''(F)}{M'(F)} - \frac{\psi'(F)}{\psi(F)}\right) = 0 \tag{3.8}$$

which in turn implies

$$\psi(F) = \frac{\mathrm{d}M}{\mathrm{d}F}, \tag{3.9}$$

where we have chosen the integration constant to be unity. It may be verified that (3.2) and (3.3) are also invariant under (3.5) provided condition (3.9) holds, i.e.,

$$\frac{\hat{F}_{xx}}{\hat{F}_x} - \frac{\hat{G}_x}{\hat{G}} = \frac{F_{xx}}{F_x} - \frac{G_x}{G},$$

$$\frac{2\hat{F}_{xt}}{\hat{F}_x} - \frac{\hat{F}_t}{\hat{F}_x}\frac{\hat{G}_x}{\hat{G}} - \frac{\hat{G}_t}{\hat{G}} = \frac{2F_{xt}}{F_x} - \frac{F_t}{F_x}\frac{G_x}{G} - \frac{G_t}{G}.$$

3.1. *Case I: When $\phi_t = 0$ and $B(x,t) = 0$*

In this subsection we examine the following special case of the Jacobi equation

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 = 0,$$

and explicitly derive the forms of the functions F and G. We present this method algorithmically.

Step I: *Writing G in term of F*

Since $B(t,x) = 0$, from (3.4) we can set

$$G = a(x)F_t, \tag{3.10}$$

where a is an arbitrary function of x.

Step II: *Expressing F and its derivatives in terms of coefficients*

Again, since $\phi_t = 0$, from (3.3) and using (3.10) we have

$$\frac{F_{xt}}{F_x} - \frac{a_x(x)}{a(x)}\frac{F_t}{F_x} - \frac{F_{tt}}{F_t} = 0$$

i.e.,

$$\frac{\partial}{\partial t}\left(\frac{F_x}{F_t}\right) = \frac{a_x(x)}{a(x)}.$$

Integrating this with respect to t we have

$$\frac{F_x}{F_t} = \frac{a_x}{a}t + b(x), \tag{3.11}$$

where b is an arbitrary function of x. Finally from (3.2) we get

$$\frac{F_x}{G} = c(t)\mathrm{e}^{\frac{\phi}{2}} = c(t)K(x), \tag{3.12}$$

where c is an arbitrary function of t and

$$\mathrm{e}^{\phi/2} = K(x). \tag{3.13}$$

Since $\phi_t = 0$, the r.h.s. is independent of t.

Step III: *Equations and solutions of coefficients*

Using (3.10) and (3.11) one can show that (3.12) can be reduced to

$$\frac{a_x}{a^2}t + \frac{b(x)}{a} = c(t)K(x). \tag{3.14}$$

There are two possibilities (a) $c(t) = c_0$ (constant), in this case a is also constant; (b) $c(t) = t$. The second case is more interesting. Equation of the coefficient of t from (3.14) leads to

$$\frac{a_x}{a^2} = K(x), \tag{3.15}$$

which implies

$$a(x) = -\frac{1}{K_1(x) + f} \tag{3.16}$$

where

$$K_1(x) = \int K(x)\mathrm{d}x \tag{3.17}$$

and f is an arbitrary constant. Assuming $f = 0$ one finds

$$a(x) = -\frac{1}{K_1(x)}. \tag{3.18}$$

Step IV: *Finding F and G using solutions of coefficients*

Using (3.18) in (3.11) and with $b(x) = 0$ we find that

$$\frac{F_x}{F_t} = -\frac{K(x)}{K_1(x)}t$$

or

$$\frac{K_1(x)}{K(x)}F_x + tF_t = 0. \tag{3.19}$$

By using the method of characteristics we obtain the general solution of $F(t, x)$ in the form

$$F(t, x) = J\left(\frac{K_1(x)}{t}\right), \tag{3.20}$$

where $J(\lambda)$ is any arbitrary function of the characteristic coordinate $\lambda = K_1(x)/t$. Hence from (3.10) using (3.18) and F as given by (3.20) we easily find that

$$G(t, x) = \frac{1}{t^2}J'(\lambda). \tag{3.21}$$

It is interesting to note that, when $J(\lambda) = \lambda$, the nonlocal character of the transformation vanishes for we have

$$X = F(t, x) = \frac{K_1(x)}{t} \quad \text{and} \quad G(t, x) = \frac{1}{t^2} \text{ so that } \mathrm{d}T = \frac{1}{t^2}\mathrm{d}t \text{ leading to } T = -\frac{1}{t}. \tag{3.22}$$

Step V: *Finding first integrals from F and G*

As the standard first integrals of the linear ODE $X'' = 0$ are

$$I_1 = X' = \frac{\mathrm{d}X}{\mathrm{d}T} \quad \text{and} \quad I_2 = X - TX'$$

respectively, we obtain as a result of the GST these in the following form:

$$I_1 = \frac{F_x}{G}\dot{x} + \frac{F_t}{G} = tK(x)\dot{x} - K_1(x) \tag{3.23}$$

and

$$I_2 = X - TX' = F(t, x) - (tK(x)\dot{x} - K_1(x)) \int G(t, x)\mathrm{d}t. \tag{3.24}$$

In particular, when F and G are given by (3.22), I_2 assumes the following simple form

$$I_2 = \dot{x}K(x). \tag{3.25}$$

It is important to note that in the following examples we repeatedly use this expression in order to compare the results of our calculations with the known time independent first integrals given in Ince's book [14].

Secondly, in view of the fact that we have at our disposal two first integrals, it is a straightforward matter to obtain the general solution by eliminating $\dot{x}$ from these expression.

3.1.1. Examples from the Painlevé–Gambier class of equations

Apart from the six Painlevé equations, the remaining 44 second-order ODEs of the Painlevé–Gambier classification scheme possess solutions that can be expressed in terms of elementary functions [17]. These solutions fall into two classes—(a) solutions which are rational in the independent variable and (b) solutions which are expressed in terms of the classical special functions. Since the latter are the solutions of linear equations, this second kind of solutions is referred to as the 'linearizable' case, obviously these exist only for special values of the parameters.

In this subsection we focus on some of these equations. In particular, using the generalized Sundman transformations we obtain certain new first integrals for the equations 11, 17, 37, 41 and 43 of the Painlevé–Gambier classification, as given in Ince's classic text [14]. The results are presented below.

Example 1 (*Painlevé–Gambier Equation XI*). The first system we examine is equation number 11 in the Painlevé–Gambier classification:

$$\ddot{x} - \frac{1}{x}\dot{x}^2 = 0. \tag{3.26}$$

Comparison with the Jacobi equation (3.1) reveals that

$$\frac{1}{2}\phi_x = -\frac{1}{x}. \tag{3.27}$$

Hence from (3.13) we have

$$K(x) = \mathrm{e}^{\frac{\phi}{2}} = \frac{1}{x} \tag{3.28}$$

and from (3.17)

$$K_1(x) = \ln x. \tag{3.29}$$

Therefore making use of (3.22) we have

$$F = \left(\frac{\ln x}{t}\right)^2 \quad \text{and} \quad G(t, x) = \frac{2\ln x}{t^3} \tag{3.30}$$

while from (3.23) and (3.24) the first integrals for this equation are

$$I_1 = \frac{t}{x}\dot{x} - \ln x \tag{3.31}$$

and

$$I_2 = \frac{\dot{x}}{x}. \tag{3.32}$$

Notice that whereas the time independent first integral I_2 is mentioned in [14] the remaining first integral I_1 is time dependent and is not stated therein. This is a trivial example in the sense that one could have deduced these results even otherwise. Moreover $G(t, x)$ being a function of t only actually produces a point transformation. But the Sundman symmetry of this simple example is quite interesting.

Table 1
Summary of results of Sundman transformations and symmetries.

Painlevé–Gambier equation no.	Sundman transformation	Sundman symmetry
XI. $\ddot{x} - \frac{1}{x}\dot{x}^2 = 0$	$F(x,t) = \left(\frac{\log x}{t}\right)^2$	$\tilde{t} = -\left[c + \int \frac{\log x}{t^3\sqrt{M(F)}} \frac{\mathrm{d}M(F)}{\mathrm{d}F}\mathrm{d}t\right]^{-1}$
	$G(x,t) = \frac{2\ln x}{t^3}$	$\tilde{x} = \exp\left(-\frac{\sqrt{M(F)}}{c+\int \frac{\log x}{t^3\sqrt{M(F)}} \frac{\mathrm{d}M(F)}{\mathrm{d}F}\mathrm{d}t}\right)$
XVII. $\ddot{x} - \frac{m-1}{mx}\dot{x}^2 = 0$	$F(x,t) = \left(\frac{mx^{1/m}}{t}\right)^2$	$\tilde{t} = c - \left[m\int \frac{x^{1/m}}{t^3\sqrt{M}} \frac{\mathrm{d}M}{\mathrm{d}F}\mathrm{d}t\right]^{-1}$
	$G(x,t) = 2\frac{mx^{1/m}}{t^3}$	$\tilde{x} = \left(\frac{\tilde{t}\sqrt{M(F)}}{m}\right)^m$
XXXVII. $\ddot{x} - \left\{\frac{1}{2x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0$	$F(x,t) = \left(\frac{1}{t}\log\frac{x^{1/2}-1}{x^{1/2}+1}\right)^2$	$\tilde{t} = c - \left[\int \log\left(\frac{x^{1/2}-1}{x^{1/2}+1}\right)\frac{1}{\sqrt{M}}\frac{\mathrm{d}M}{\mathrm{d}F}\mathrm{d}t\right]^{-1}$
	$G(x,t) = \frac{2}{t^3}\log\frac{x^{1/2}-1}{x^{1/2}+1}$	$\tilde{x} = \left(\frac{1+\mathrm{e}^{\tilde{t}\sqrt{M(F)}}}{1-\mathrm{e}^{\tilde{t}\sqrt{M(F)}}}\right)^2$
XLI. $\ddot{x} - \frac{2}{3}\left\{\frac{1}{x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0$	$F(x,t) = \frac{K_1^2(x)}{t^2}$	
	$G(x,t) = \frac{2K_1(x)}{t^3}$	
	$K_1(x) = -3(-x)^{1/3}{}_2F_1(1/3, 2/3; 4/3; x)$	
XLIII. $\ddot{x} - \frac{3}{4}\left\{\frac{1}{x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0$	$F(x,t) = \frac{K_1^2(x)}{t^2}$	
	$G(x,t) = \frac{2K_1(x)}{t^3}$	
	$K_1(x) = -4(-x)^{1/4}\,{}_2F_1(3/4, 1/4; 5/4; x)$	

3.1.2. The Sundman symmetry for $\ddot{x} - \frac{1}{x}\dot{x}^2 = 0$

To deduce the Sundman symmetry for this equation, it is convenient to assume that $J(\lambda) = \lambda^2$ in the rest of this subsection so that from (3.20) we have

$$F(t,x) = \left(\frac{K_1(x)}{t}\right)^2 = \left(\frac{\ln x}{t}\right)^2. \tag{3.33}$$

Now the Sundman symmetry of (3.26) being of the form (3.6), we assume that

$$\hat{F} = F(\tilde{t},\tilde{x}) = M(F(t,x)).$$

Consequently with F given as in (3.33) one finds that

$$\tilde{x} = \exp\left(\tilde{t}\sqrt{M(F)}\right). \tag{3.34}$$

On the other hand from (3.7), using (3.21) to calculate G which now is given by $G(t,x) = 2\ln x/t^3$, we have

$$\hat{G}\mathrm{d}\tilde{t} = G\frac{\mathrm{d}M(F)}{\mathrm{d}F}\mathrm{d}t \Rightarrow \frac{\ln\tilde{x}}{\tilde{t}^3}\mathrm{d}\tilde{t} = \frac{\ln x}{t^3}\frac{\mathrm{d}M}{\mathrm{d}F}\mathrm{d}t.$$

Upon using (3.34) to eliminate $\tilde{x}$ from the l.h.s of the above expression, we obtain the following transformation for the time variable:

$$\tilde{t} = -\left[c + \int \frac{\ln x}{t^3\sqrt{M(F)}}\frac{\mathrm{d}M(F)}{\mathrm{d}F}\mathrm{d}t\right]^{-1}. \tag{3.35}$$

Here c is a constant of integration. Substituting this expression into (3.34) we get the transformation for the spatial variable, *viz*

$$\tilde{x} = \exp\left(-\frac{\sqrt{M(F)}}{c + \int \frac{\ln x}{t^3\sqrt{M(F)}}\frac{\mathrm{d}M(F)}{\mathrm{d}F}\mathrm{d}t}\right). \tag{3.36}$$

Here $M(F)$ is an arbitrary function of F and c is a constant of integration. Eqs. (3.35) and (3.36) constitute a Sundman symmetry for the Painlevé–Gambier XI equation.

The above procedure for finding Sundman symmetries may easily be applied to some of the other equations of the Painlevé–Gambier classification. The results for this and some of the other equations of the Painlevé–Gambier classification are summarized in Table 1.

In the above table ${}_2F_1(a,b,c;x)$ is the hypergeometric series which converges for $-1 < x < 1$. For Eqs. 41 and 43 it is difficult to obtain explicit expressions for the corresponding symmetries and we do not display them here.

In Table 2 we summarize the results for the time independent and time dependent first integrals of the above equations.

Table 2
Summary of first integrals.

Painlevé–Gambier equation no.	Time dependent F.I	Time independent F.I
XVII. $\ddot{x} - \frac{m-1}{mx}\dot{x}^2 = 0$	$tx^{\frac{1-m}{m}}\dot{x} - mx^{\frac{1}{m}}$	$x^{\frac{1-m}{m}}\dot{x}$
XXXVII. $\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t}{x^{1/2}(x-1)}\dot{x} - \log\frac{x^{1/2}-1}{x^{1/2}+1}$	$-\frac{1}{x^{1/2}(x-1)}\dot{x}$
XLI. $\ddot{x} - \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t\dot{x}}{x^{\frac{2}{3}}(x-1)^{\frac{2}{3}}} + 3(-x)^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x)$	$\frac{\dot{x}}{x^{2/3}(x-1)^{2/3}}$
XLIII. $\ddot{x} - \frac{3}{4}\left(\frac{1}{x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t\dot{x}}{x^{\frac{3}{4}}(x-1)^{\frac{3}{4}}} + 4(-x)^{1/4}\ {}_2F_1(3/4, 1/4; 5/4; x)$	$\frac{\dot{x}}{x^{3/4}(x-1)^{3/4}}$

3.2. *Case B: When $\phi_t = 0 = B_t$*

The prototype equation for this section has the generic form

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 + B(x) = 0. \tag{3.37}$$

Once again there are a number of equations of the Painlevé–Gambier classification which belong to this category.

3.2.1. *Generalized Sundman transformation*

For such equations our first objective is to construct a generalized Sundman transformation (2.2) (GST) such that (3.37) is mapped to the following equation

$$X'' + a_0(X) = 0, \tag{3.38}$$

where $X' = \mathrm{d}X/\mathrm{d}T$. The exact form of $a_0(X)$ is specified below. This is true if the following conditions (i.e. the Sundman determining equations) on the coefficients of (3.37) hold good:

$$\frac{1}{2}\phi_x = \frac{F_{xx}}{F_x} - \frac{G_x}{G} \tag{3.39}$$

$$0 = 2\frac{F_{xt}}{F_x} - \frac{G_x}{G}\frac{F_t}{F_x} - \frac{G_t}{G} \tag{3.40}$$

$$B(x) = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + a_0(F)\frac{G^2}{F_x}. \tag{3.41}$$

From (3.39) we have

$$\ln F_x - \ln G = \int \frac{1}{2}\phi_x \mathrm{d}x - \ln b(t).$$

Here $b(t)$ is an arbitrary constant of integration. It follows that

$$G(t, x) = b(t)\mathrm{e}^{-\phi/2}F_x. \tag{3.42}$$

Substituting G from (3.42) to (3.41) we have

$$\frac{F_{tt}}{F_x} - \frac{F_{xt}F_t}{F_x^2} - \frac{\dot{b(t)}}{b(t)}\frac{F_t}{F_x} + a_0(F)b(t)^2\mathrm{e}^{-\phi}F_x = B(x). \tag{3.43}$$

If we set $b(t) = \beta$, i.e., a constant independent of t and assume that

$$\frac{\partial}{\partial t}\left(\frac{F_t}{F_x}\right) = 0, \tag{3.44}$$

then (3.43) implies

$$a_0(F)\beta^2\mathrm{e}^{-\phi}F_x = B(x). \tag{3.45}$$

Instead of trying to determine the form of F first, it is more convenient to stipulate $a_0(F)$ and see whether we can satisfy the remaining equations with such a choice of $a_0(F)$. To this end we suppose

$$a_0(F) = \pm F. \tag{3.46}$$

Then (3.45) yields

$$F^2 = \pm\frac{2}{\beta^2}\int B(x)\mathrm{e}^{\phi}\mathrm{d}x. \tag{3.47}$$

Thus F is a function of x only and as a result it is obvious that (3.44) is trivially satisfied. It remains to verify whether such an expression for F is consistent with (3.40). Since $b(t) = \beta$ is a constant, we have from (3.42),

$$G(t,x) = \beta e^{-\phi/2} F_x = \frac{B(x)e^{\phi/2}}{\left(\pm 2\int B(x)e^{\phi}dx\right)^{1/2}}, \tag{3.48}$$

which is clearly independent of t and hence $G_t = 0$. Consequently, since F and G are only functions of x, it follows that (3.40) is clearly satisfied. In summary we therefore have the following form of the GST mapping (3.37) to the equation $X'' \pm X = 0$, *viz*

$$X = F(x) = \left(\pm\frac{2}{\beta^2}\int B(x)e^{\phi(x)}dx\right)^{1/2}, \qquad dT = \frac{B(x)e^{\phi/2}}{\left(\pm 2\int B(x)e^{\phi}dx\right)^{1/2}}dt. \tag{3.49}$$

The latter is obviously a nonlocal transformation.

3.2.2. *The Sundman symmetry*

The Sundman symmetry associated with (3.37) is not difficult to deduce. As above, for notational convenience we denote

$$\hat{F} = F(\tilde{t},\tilde{x}) \quad \text{and} \quad \hat{G} = G(\tilde{t},\tilde{x}).$$

To ensure invariance of the Sundman determining equations, namely (3.39)–(3.41), we assume that

$$\hat{F} = M(F) \quad \text{and} \quad \hat{G} = G(t,x)\psi(F). \tag{3.50}$$

The functional forms of M and ψ are determined by demanding invariance of the Sundman determining equations. Invariance of (3.39) leads to

$$\psi(F) = K\frac{dM(F)}{dF},$$

where K is a constant of integration, which may be set to unity, so that

$$\psi(F) = M'(F). \tag{3.51}$$

Invariance of (3.41) then leads to the equation

$$\frac{dM}{dF} = \frac{a_0(F)}{a_0(M)}$$

whence it follows, with $a_0(F) = \pm F$, that

$$M = \pm\sqrt{F^2 + c}, \tag{3.52}$$

where c is a constant of integration. Note that, if $c = 0$, then we get a trivial symmetry. The functional form of ψ is therefore given by

$$\psi(F) = \pm\frac{F}{\sqrt{F^2 + c}}. \tag{3.53}$$

With M and ψ given by (3.52) and (3.53) respectively, one can easily verify that the final Sundman determining equation, namely (3.40), is identically satisfied. Thus in summary we have the following Sundman symmetry for (3.37)

$$F(\tilde{t},\tilde{x}) = \pm\sqrt{F^2(t,x) + c} \quad \text{and} \quad G(\tilde{t},\tilde{x})d\tilde{t} = \pm G(t,x)\frac{F}{\sqrt{F^2 + c}}dt. \tag{3.54}$$

In the following we consider only the case in which the GST maps equations of the Painlevé–Gambier classification belonging to the class of (3.37) to a harmonic oscillator equation

$$X'' + X = 0. \tag{3.55}$$

Note that a first integral for (3.55) is obviously

$$X'^2 + X^2 = I_1. \tag{3.56}$$

Example (*Painlevé–Gambier Equation XXI*).

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0. \tag{3.57}$$

Table 3
Summary of Sundman symmetry.

	Painlevé–Gambier equation	Sundman symmetry
XVIII.	$\ddot{x} - \frac{1}{2x}\dot{x}^2 - 4x^2 = 0$	$\tilde{x} = \frac{1}{2\mathrm{i}}\sqrt{c\beta^2 - 4x^2}$ $\tilde{t} = A + \int \frac{(2\mathrm{i}x)^{3/2}}{(c\beta^2-4x^2)^{3/4}}\mathrm{d}t$
XXI.	$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0$	$\tilde{x} = \left(\frac{1}{2\mathrm{i}}\right)^{4/3}(c\beta^2 - 4x^{3/2})^{2/3}$ $\tilde{t} = A + \int \frac{(2\mathrm{i})^{5/3}x^{5/4}}{(c\beta^2-4x^{3/2})^{5/6}}\mathrm{d}t$
XXII.	$\ddot{x} - \frac{3}{4x}\dot{x}^2 + 1 = 0$	$\tilde{x} = \frac{16}{(c\beta^2-4x^{-1/2})^2}$ $\tilde{t} = A + \int \frac{8\mathrm{i}x^{-3/4}}{(c\beta^2-4x^{-1/2})^{3/2}}\mathrm{d}t$
XIX.	$\ddot{x} - \frac{1}{2x}\dot{x}^2 - (4x^2 + 2x) = 0$	$\tilde{x} = \frac{-1+\sqrt{4x^2+4x+1-c\beta^2}}{2}$ $\tilde{t} = A + \int \frac{\sqrt{2x}(2x+1)}{\sqrt{4x^2+4x+1-c\beta^2}\sqrt{\sqrt{4x^2+4x+1-c\beta^2}-1}}\mathrm{d}t$

Here $\frac{1}{2}\phi_x = -\frac{3}{4x}$ implying $\phi = \ln x^{-3/2}$ and $B(x) = -3x^2$. As a result from (3.49) taking the positive square root we find $F(x) = \frac{2\mathrm{i}}{\beta}x^{\frac{3}{4}}$ and it turns out that $G = \frac{3\mathrm{i}}{2}\sqrt{x}$. Hence the Sundman transformation has the explicit form

$$X = F(x) = \frac{2\mathrm{i}}{\beta}x^{\frac{3}{4}}, \qquad \mathrm{d}T = \frac{3\mathrm{i}}{2}\sqrt{x}\,\mathrm{d}t. \tag{3.58}$$

When the first integral (3.56) is evaluated in terms of the preceding transformation, it reproduces the result in [14].

Table 3 contains a summary of the Sundman symmetry for some of the Painlevé–Gambier equations falling under Case B.

4. Parametric solutions

As remarked earlier, when we have two first integrals for a second-order ODE, then its general solution may be obtained simply by eliminating the first derivatives from the two first integrals. However, the problem of finding a sufficient number of first integrals is itself a non trivial exercise. In most instances, one is lucky if there exists even a single first integral. In such cases a parametric solution of the ODE can often be constructed by integrating the first integral in terms of a parameter. Using Euler et al. [7,8] scheme we present parametric solutions of some Painlevé–Gambier equations.
Consider a first-order equation of the form

$$F\left(x(t), \frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0. \tag{4.1}$$

Let $x(t) = f(\tau)$, $\frac{\mathrm{d}x}{\mathrm{d}t} = g(\tau)$ and $\tau = \tau(t)$ where f and g satisfy the relation $F(f(\tau), g(\tau)) = 0$ with τ being a parameter. Since $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}$ so $g(\tau) = \frac{\mathrm{d}f}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}$, i.e.,

$$\int \mathrm{d}t = \int \frac{\mathrm{d}f}{\mathrm{d}\tau}\frac{1}{g(\tau)}\mathrm{d}\tau + C, \tag{4.2}$$

where C is an integrating constant. The general solution (parametric) of (4.1) is then given by

$$x(\tau) = f(\tau), \tag{4.3}$$

$$t(\tau) = \int \frac{\mathrm{d}f}{\mathrm{d}\tau}\frac{1}{g(\tau)}\mathrm{d}\tau + C, \tag{4.4}$$

$$F(f(\tau), g(\tau)) = 0. \tag{4.5}$$

Using this method we can integrate (3.56) with respect to the parameter τ to obtain the general solution of (3.55) in the form

$$X(\tau) = \sqrt{I_1 - \tau^2} \tag{4.6}$$

$$T(\tau) = C_1 - \arcsin\left(\frac{\tau}{\sqrt{I_1}}\right), \tag{4.7}$$

with I_1 and C_1 being the arbitrary constants of integration.

Table 4
Summary of parametric solutions.

Painlevé–Gambier equation no.	Sundman transformation	Parametric solution
	$F(x,t) = \frac{2i}{\beta}(x^2+x)^{1/2}$	$x(\tau) = \frac{1}{2}\left(-1+\sqrt{\beta^2\tau^2+1-\beta^2 I_1}\right)$
XIX. $\ddot{x} - \frac{1}{2x}\dot{x}^2 - (4x^2+2x) = 0$	$dT = \frac{\beta x^{1/2}(2x+1)}{2\sqrt{x^2+x}}dt$	$t(\tau) = \frac{\beta}{\sqrt{2}}\int \frac{d\tau}{\left((\beta^2\tau^2+1-\beta^2 I_1)\sqrt{\beta^2\tau^2+1-\beta^2 I_1}-1\right)^{1/2}} + C_2$
	$F(x,t) = \frac{2i}{\beta}x$	$x(\tau) = \frac{\beta}{2i}\sqrt{I_1-\tau^2}$
XVIII. $\ddot{x} - \frac{1}{2x}\dot{x}^2 - 4x^2 = 0$	$dT = 2i\sqrt{x}dt$	$t(\tau) = -\frac{1}{\sqrt{2i\beta}}\int \frac{d\tau}{(I_1-\tau^2)^{3/4}} + C_2$
	$F(x,t) = \frac{2i}{\beta}x^{-1/4}$	$x(\tau) = \frac{16}{\beta^4}\frac{1}{\sqrt{I_1-\tau^2}}$
XXII. $\ddot{x} - \frac{3}{4x}\dot{x}^2 + 1 = 0$	$dT = -\frac{i}{2}x^{-1/2}dt$	$t(\tau) = c_2 + \frac{8i}{\beta}\int \frac{d\tau}{(I_1-\tau^2)^{3/2}}$

The general solution of (3.57) is then obtained by using the transformation (3.58) together with the parametric solutions (4.6) and (4.7) and is given by

$$x(\tau) = \left(\frac{\beta}{2i}\right)^{4/3}(I_1-\tau^2)^{2/3}, \tag{4.8}$$

$$t(\tau) = -\frac{4}{3\beta}\left(\frac{\beta}{2i}\right)^{1/3}\int \frac{d\tau}{(I_1-\tau^2)^{5/6}} + C_2 \tag{4.9}$$

where I_1 and C_2 are arbitrary constants.
In Table 4 we present the parametric solutions for some of the other equations of the Painlevé–Gambier classification scheme, obtained by using the above method.

5. Conclusion

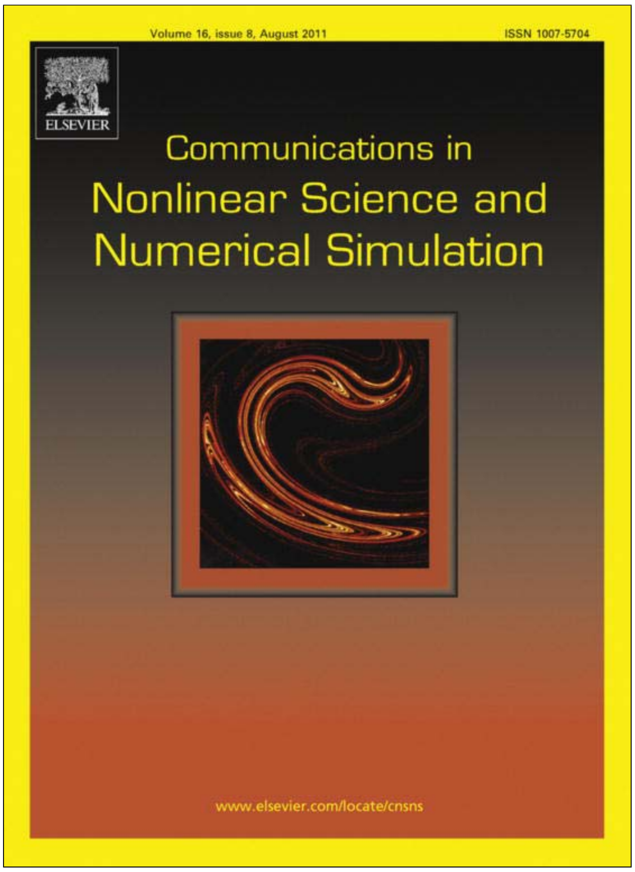
In this paper we have computed the first integrals of the Painlevé–Gambier class of equations using Sundman transformation. It turns out that the Sundman method simplifies our job and also yields the Sundman symmetries of these equations. The calculation of the Sundman symmetries, appears to be new to the existing literature on the Painlevé–Gambier equations. Hence it would be interesting and tempting to apply this scheme to compute the first integrals of the Chazy systems and their corresponding (Sundman) symmetries.

Acknowledgements

We wish to thank Basil Grammaticos, Norbert Euler, M. Lakshmanan and Peter Leach for enlightening discussions. In particular we are grateful to Basil Grammaticos for suggesting this problem. In addition AGC wishes to acknowledge the support provided by the S.N. Bose National Centre for Basic Sciences, Kolkata, in the form of an Associateship.

References

[1] G. Bluman, S. Kumei, Symmetries and Differential Equations, Springer, New York, 1996.
[2] W.-H. Steeb, Invertible Point Transformation and Nonlinear Differential Equations, World Scientific, Singapore, ISBN: 981-02-1355-7, 1993.
[3] A.M. Tresse, Détermination des Invariants Ponctules de l'Équation Différentielle ordinaire du seconde ordre $y'' = w(x, y, y')$, Leipzig, Hirzel, 1896.
[4] S. Lie, Arch. Math. Nat. 8 (1883) 371–427. Reprinted in Lie's Gessammelte Abhanlundgen, 1924, 5, paper XIY, pp. 362–427.
[5] L.G.S. Duarte, I.C. Moreira, F.C. Santos, Linearization under non-point transformations, J. Phys. A 27 (1994) L739–L743.
[6] M. Euler, N. Euler, A. Strömberg, E. Ström, Transformation between a generalized Emden–Fowler equation and the first Painlevé transcendent, Math. Methods Appl. Sci. 30 (16) (2007) 2121–2124.
[7] N. Euler, M. Euler, Sundman symmetries of nonlinear second-order and third-order ordinary differential equations, J. Nonlinear Math. Phys. 11 (3) (2004) 399–421.
[8] N. Euler, T. Wolf, P.G.L. Leach, M. Euler, Linearisable third-order ordinary differential equations and generalised Sundman transformations: The case $X''' = 0$, Acta Appl. Math. 76 (1) (2003) 89–115.
[9] K.F. Sundman, Mémoire sur le problém des trois corps, Acta Math. 36 (1912–1913) 105–179.
[10] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, On the complete integrability and linearization of nonlinear ordinary differential equations—Part II: Third order equations. arXiv:nlin/0510036v1.
[11] V. Szebehely, V. Bond, Transformations of the perturbed two-body problem to unperturbed harmonic oscillators, Celestial Mech. Dynam. Astronom. 30 (1983) 59–69.
[12] P. Painlevé, Memoire sur les quations différentielles dont l'intégrale générale est uniforme, Bull. Soc. Math. France 28 (1900) 201–261.
[13] P. Painlevé, Sur les quations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, Acta Math. 25 (1902) 1–85.
[14] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956, reprinted.
[15] C.G.J. Jacobi, Theoria novi multiplicatoris systemati quationum differentialium vulgarium applicandi, J. Reine Angew. Math. 29 (1845) 213–279. 333–376.
[16] M.C. Nucci, K.M. Tamizhmani, Using old method of Jacobi to derive Lagrangians, 2008. arXiv:0807.2796v1[nlin.SI].
[17] V.A. Gromak, N.A. Lukashevich, The Analytic Solutions of the Painlev Equations, Universitetskoye, Minsk, 1990 (in Russian).
[18] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intgrale générale est á points critique fixés, Acta Math. 33 (1910) 1–55.


Volume 16, issue 8, August 2011
ISSN 1007-5704
ELSEVIER
Communications in
Nonlinear Science and
Numerical Simulation
www.elsevier.com/locate/cnsns

Commun Nonlinear Sci Numer Simulat 16 (2011) 3062–3070

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

First integrals for time-dependent higher-order Riccati equations by nonholonomic transformation

Partha Guha [a,*], A. Ghose Choudhury [b], Barun Khanra [c]

[a] S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata 700098, India
[b] Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Kolkata 700009, India
[c] Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Kolkata 700004, India

ARTICLE INFO

Article history:
Received 29 October 2010
Accepted 10 December 2010
Available online 17 December 2010

Keywords:
Nonholonomic transformation
First integrals
Second-order Riccati equation
Gambier equation

ABSTRACT

We exploit the notion of nonholonomic transformations to deduce a time-dependent first integral for a (generalized) second-order nonautonomous Riccati differential equation. It is further shown that the method can also be used to compute the first integrals of a particular class of third-order time-dependent ordinary differential equations and is therefore quite robust.

1. Introduction

The time-independent second-order Riccati equation (SORE) plays an important role in dynamical systems. It is also sometimes called the Painlevé–Ince equation. The SORE was studied in [6] from a geometric perspective and shown to admit two alternative Lagrangian formulations, with both Lagrangians belonging to a nonnatural class (neither potential nor kinetic term). The Lie point symmetries of the SORE are known to have an algebra identical to that of the eight-parameter group SL $(3,\mathbf{R})$ [20]. Since the free particle also possesses a similar symmetry algebra, it is not surprising that under an appropriate transformation, the SORE may be transformed into that of the free particle.

The aim of the present work is to study a time-dependent generalization of the second-order Riccati equation, namely

$$\ddot{x} + 3kx\dot{x} + k^2x^3 = 0. \tag{1.1}$$

In [5] the authors have studied the second-, third- and fourth-order cases of the hierarchy of Riccati equations and have shown the existence of Darboux functions and generators of t-dependent constants of motion.

The time-dependent second-order Riccati equation (TDSORE) is given by

$$\ddot{x} + 3h(t)x\dot{x} + h^2x^3 + \dot{h}(t)x^2 = 0. \tag{1.2}$$

This is an interesting equation which differs from the SORE in many respects. For instance, unlike the SORE its time-dependent counterpart is not a bi-Lagrangian system. Furthermore it turns out that this equation is actually a truncated version of the Gambier equation [10]. Indeed exactly a hundred years ago Gambier in course of his classification of integrable second-order differential equations solved the following equation, which is listed as Equation number XXVII of the Painlevé–

* Corresponding author.
E-mail addresses: partha@bose.res.in (P. Guha), a_ghosechoudhury@rediffmail.com (A.G. Choudhury), barunkhanra@rediffmail.com (B. Khanra).

1007-5704/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cnsns.2010.12.012

Gambier series in Ince's book [15] and occurs as Eq. 15 in Gambier's minimal list of 24 s-order equations with the Painlevé property. The Gambier equation (see [13] for a relatively recent update) is given by

$$\ddot{x} = \frac{n-1}{n}\frac{\dot{x}^2}{x} + a\frac{n+2}{n}x\dot{x} + b\dot{x} - \frac{n-2}{n}\frac{\dot{x}}{x}\sigma - \frac{a^2}{n}x^3 + (\dot{a} - ab)x^2 + (cn - \frac{2a}{n}\sigma)x - b\sigma - \frac{\sigma^2}{nx}, \tag{1.3}$$

where a, b and c are functions of the independent variable, σ is a constant which can be scaled to 1 unless it happens to be 0 and n is an integer. If we set $b = c = 0$ and assume that $n = 1$ and $\sigma = 0$, then (G.1) reduces to a time-dependent second-order Riccati equation which can be mapped to (G14) of the minimal list of Gambier.

It is also interesting to note that the TDSORE also arises in the context of Calogero–Leyvraz's construction of isochronous dynamical systems [11]. Indeed such systems are not as rare as previously thought. Furthermore it is in fact possible to construct isochronous Hamiltonian systems for the many-particle case with a translation invariant potential by making a novel use of suitable canonical variables [1,2].

The TDSORE arises from a Riccati sequence which may be introduced as follows. Let $h(t)$ be an arbitrary differentiable function and $\mathbb{D}_R$ denote the following differential operator

$$\mathbb{D}_R := \frac{d}{dt} + h(t)x,$$

which will be called the 'Riccati differential operator'. Now consider the sequence obtained by applying such a differential operator to the function x in an iterative way. For example when

$$\begin{aligned}
n = 1, \quad & \mathbb{D}_R x = \left(\frac{d}{dt} + h(t)x\right)x = \dot{x} + h(t)x^2, \\
n = 2, \quad & \mathbb{D}_R^2 x = \left(\frac{d}{dt} + h(t)x\right)^2 x = \ddot{x} + 3h(t)x\dot{x} + h^2(t)x^3 + \dot{h}(t)x^2, \\
n = 3, \quad & \mathbb{D}_R^3 x = \left(\frac{d}{dt} + h(t)x\right)^3 x = \dddot{x} + 4h(t)x\ddot{x} + 6h^2(t)x^2\dot{x} + 3h(t)\dot{x}^2 + h^3(t)x^4 + 5\dot{h}(t)x\dot{x} + 3h(t)\dot{h}(t)x^3 + \ddot{h}(t)x^2,
\end{aligned} \tag{1.4}$$

and analogous expressions for higher values of n which turn out to be quite lengthy.

The equation $R^{(k)}(x, \ldots, x^{(k)}) = \mathbb{D}_R^k x = 0,\ k = 0, 1, \ldots$ defines a Riccati equation with variable coefficients of order k. Note that $R^{(0)}(x) = x$, and the particular Riccati equation $R^{(1)}(x, \dot{x}) = 0$ obtained for $k = 1$ is the standard Riccati equation but with a variable coefficient $h(t)$,

$$\dot{x} + h(t)x^2 = 0. \tag{1.5}$$

The time-dependent second-order Riccati equation actually belongs to the class of equations of Liénard type but with time-dependent coefficients namely,

$$\ddot{x} + f(x,t)\dot{x} + g(x,t) = 0. \tag{1.6}$$

Recently Gladwin Pradeep et al. [12] studied a system of N-coupled Liénard-type nonlinear oscillators which is completely integrable and possesses N time-independent and N time-dependent explicit integrals. There are various methods applied to study first integrals of time-dependent systems, for example, Sarlet [22] used a direct ad hoc procedure for the construction of first integrals for one-dimensional particle motion in a non-linear, time-dependent potential field. In general method varies from problem to problem.

1.1. Motivation, result and plan

The canonical list of second-order equations with the Painlevé property is still an unsettled question and as such any new development in this area is always interesting. The *objective of this paper* is to study the first integrals of the time-dependent higher-order equations of Riccati type. We will firstly be concerned with Eq. (1.2) and derive a first integral for it. Thereafter we consider a generalization of the time-dependent Riccati equation and examine its relation with the Sugai equation. It is shown how nonholonomic transformations [16–19] provide us with an effective tool for the determination of a first integral in several particular cases. Finally we consider applications of the method to third-order ODEs, which mimic in a sense Eq. (1.6) above.

In [4,20] the authors, by considering a nonlocal transformation, have derived certain nonlinear ODEs from well-known simple linear ODEs such as the linear harmonic oscillator and its damped counterpart. While the TDSORE can be shown to belong to a category of equations derived by them, its formal solution as presented by the authors still retains a nonlocal character. Our motivation is not to derive classes of nonlinear ODEs from linear ODEs using a nonlocal transformation; rather it is to linearize a given nonlinear ODE by means of a nonlocal differential transformation and to obtain an explicit expression for the first integral, when the latter exists.

The *organization* of the paper is as follows. In Section 2 we introduce the nonholonomic transformation method and use it to compute the first integrals of the time dependent second-order Riccati equation. In Section 3 we apply such a transformation to compute first integrals of a class of third-order ODEs. We illustrate the method with several examples and complete the paper with a modest outlook.

2. Nonholonomic transformations and first integrals of time dependent second-order Riccati equation

Of late it has been found that Sundman transformations are often quite useful for the determination of first integrals of second- and higher-order ordinary differential equations (ODEs), [8,9,7]. On the other hand for nonholonomic transformations, which may be regarded as the generalization of Sundman transformations [24], one assumes that both the new variables X and T are given by nonlocal transformations. In this sense they are the complete opposite of point transformations. Given a second-order ordinary differential equation

$$\ddot{x} = w(t, x, \dot{x}), \tag{2.1}$$

where $w(t,x,\dot{x})$ is linear in $\dot{x}$, suppose we seek a nonlocal transformation of $(t,x) \mapsto (T,X)$ of the form

$$dX = A(x,t)dx + B(x,t)dt, \tag{2.2}$$

$$dT = C(x,t)dx + D(x,t)dt \tag{2.3}$$

such that the ODE (2.1) is transformed to the autonomous linear equation [21]

$$\frac{d^2X}{dT^2} = 0. \tag{2.4}$$

Our first objective is therefore to determine the differentiable functions A, B, C and D which enable such a linearization to be made for the particular case of (1.2). The nonholonomic nature of the above transformation may be ensured by demanding that

$$A_t \neq B_x, \quad C_t \neq D_x. \tag{2.5}$$

It is obvious that, if such a transformation exists, then in terms of the new variables we immediately obtain a first integral given by

$$\frac{dX}{dT} = \text{constant}. \tag{2.6}$$

However, from (2.6) it follows that such a first integral is clearly a time-dependent one when expressed in terms of the original variables, x and t, because

$$I(t,x,\dot{x}) = \frac{A(x,t)\dot{x} + B(x,t)}{C(x,t)\dot{x} + D(x,t)} \tag{2.7}$$

clearly defines a time-dependent first integral of (1.6).

The crucial question is whether one can derive a nonholonomic transformation which enables such a first integral to be identified for a given second-order equation. In answer to this question we note that, if (2.7) is indeed a first integral of (1.6), then we must have $dI/dt = 0$. This in turn leads to the following condition:

$$\Delta(x,t)\ddot{x} + (CA_x - AC_x)\dot{x}^3 + (C(A_t + B_x) - A(C_t + D_x) + DA_x - BC_x)\dot{x}^2 + (CB_t - BC_t + DA_t - AD_t + DB_x - BD_x)\dot{x} + (DB_t - BD_t) = 0, \tag{2.8}$$

where

$$\Delta(x,t) = A(x,t)D(x,t) - B(x,t)C(x,t).$$

Comparison with (1.6) shows that firstly we must have

$$CA_x = AC_x, \quad \text{implying} \quad C(x,t) = a(t)A(x,t), \tag{2.9}$$

since there is no term proportional to $\dot{x}^3$ while the vanishing of the coefficient of $\dot{x}^2$ implies

$$C(A_t + B_x) - A(C_t + D_x) + DA_x - BC_x = 0. \tag{2.10}$$

In view of (2.9) we rewrite $\Delta(x,t)$ as

$$\Delta(x,t) = A(x,t)(D(x,t) - a(t)B(x,t)).$$

We now make a simplifying assumption, *viz*,

$$D(x,t) := a(t)B(x,t) + b(x,t)A(x,t) \quad \text{so that} \quad \Delta(x,t) = A^2(x,t)b(x,t). \tag{2.11}$$

Here $a(t)$ and $b(x,t)$ are functions to be determined below. Under these circumstances we may express (2.2) and (2.3) as

$$dX = A(x,t)[dx + S(x,t)dt], \tag{2.12}$$

$$dT = a(t)A(x,t)\left[dx + \left(S(x,t) + \frac{b}{a}\right)dt\right], \tag{2.13}$$

where $S(x,t) := B(x,t)/A(x,t)$. In view of (2.9) and (2.11) condition (2.10) simplifies to

$$\dot{a} + b_x = 0. \tag{2.14}$$

Furthermore returning to (2.8) we require the coefficient of $\dot{x}$ to satisfy

$$f(x,t) = \frac{1}{\Delta}[CB_t - BC_t + DA_t - AD_t + DB_x - BD_x] \tag{2.15}$$

and

$$g(x,t) = \frac{1}{\Delta}[DB_t - BD_t]. \tag{2.16}$$

Using (2.9) and (2.11) one can rewrite (2.15) as

$$f(x,t) = S_x - \left(\frac{2\dot{a} + b_x}{b}\right)S - \frac{b_t}{b} \tag{2.17}$$

while (2.16) becomes

$$g(x,t) = S_t - \left(\frac{\dot{a}S + b_t}{b}\right)S. \tag{2.18}$$

Thus, when there exists a function $S(x,t)$ such that the above two equations are satisfied, it follows from (2.12) and (2.13) that there exists a first integral of the form

$$I(t,x,\dot{x}) = \frac{\dot{x} + S}{a(\dot{x} + S) + b}. \tag{2.19}$$

Before presenting our main result, we quickly see whether the above procedure works for the following equation of modified Emden type [6,3].

Example. Notice that, if we choose $S(x,t) = \beta x^2$, $b(x,t) = -kx$ and $a(t) = kt$, then substitution into (2.15) and (2.16) leads to the equation

$$\ddot{x} + 3\beta x\dot{x} + \beta^2 x^3 = 0.$$

Its associated first integral by the formula stated in (2.19) is

$$I(t,x,\dot{x}) = \frac{\dot{x} + \beta x^2}{kt(\dot{x} + \beta x^2 - x/t)}.$$

2.1. Generalized time-dependent Riccati equation

We consider the Gambier Eq. (1.3) with minor changes of notation

$$\ddot{x} = \frac{n-1}{n}\frac{\dot{x}^2}{x} + \alpha\frac{n+2}{n}x\dot{x} + \beta\dot{x} - \frac{n-2}{n}\frac{\dot{x}}{x}\sigma - \frac{\alpha^2}{n}x^3 + (\dot{\alpha} - \alpha\beta)x^2 + (\gamma n - \frac{2\alpha}{n}\sigma)x - \beta\sigma - \frac{\sigma^2}{nx}, \tag{2.20}$$

where $\gamma = \frac{\dot{\beta}}{2} - \frac{\beta^2}{4}$.

If we assume $\sigma = 0$ and $n = 1$ then the above system reduces to the generalized time-dependent Riccati equation or Sugai equation [23]. It is evident that the Sugai equation includes as special cases the TDSORE for the particular choice $\beta = 0$ while the Gambier V equation, namely

$$\ddot{x} = -3x\dot{x} + \beta(t)x - x^3 + \beta(t)x^2, \tag{2.21}$$

corresponds to the specific choice $n = 1$, $\sigma = 0$, $\alpha(t) = -1$ and $\gamma = 0$. One of the main results of this paper is the following proposition.

Proposition 2.1. *A time-dependent first integral of the variable coefficient second-order equation*

$$\ddot{x} - [3h(t)x + r(t)]\dot{x} + h^2(t)x^3 - [\dot{h}(t) - h(t)r(t)]x^2 + \lambda(t)x + \left(\frac{r^2(t)}{4} - \frac{\dot{r}(t)}{2}\right)x = 0 \tag{2.22}$$

is given by the function

$$I(t,x,\dot{x}) = \frac{\dot{x}+S}{a(t)(\dot{x}+S)-\dot{a}x} \tag{2.23}$$

where

$$S(x,t) = \left(\frac{\ddot{a}}{\dot{a}} - r(t)\right)\frac{x}{2} - h(t)x^2$$

and $\lambda(t)$ is given by the Schwarzian derivative

$$\lambda(t) = \frac{1}{2}\left[\frac{\dddot{a}}{\dot{a}} - \frac{3}{2}\frac{\ddot{a}^2}{\dot{a}^2}\right].$$

Proof. The proof essentially revolves around finding the function $S(x,t)$. Using (2.14) to simplify (2.17) it follows that S must satisfy the following:

$$f(x,t) = -(3h(t)x + r(t)) = S_x - \left(\frac{\dot{a}}{b}\right)S - \frac{b_t}{b}, \tag{2.24}$$

$$g(x,t) = S_t - \left(\frac{\dot{a}S + b_t}{b}\right)S, \tag{2.25}$$

where

$$g(x,t) = h^2(t)x^3 - [\dot{h}(t) - h(t)r(t)]x^2 + \lambda(t)x + \left(\frac{r^2(t)}{4} - \frac{\dot{r}(t)}{2}\right)x.$$

A particular solution of (2.14) is clearly given by

$$b(x,t) = -\dot{a}x, \tag{2.26}$$

where we have set the constant of integration to zero. Next we make the following ansatz for $S(x,t)$, *viz.*

$$S(x,t) = S_2(t)x^2 + S_1(t)x + S_0(t).$$

Upon substitution of this into the right side of (2.24) and after equation of the coefficients of the different powers of x we get

$$S_0(t) = 0, \quad S_1(t) = \frac{1}{2}\left(\frac{\ddot{a}}{\dot{a}} - r(t)\right) \quad \text{and} \quad S_2(t) = -h(t),$$

so that

$$S(x,t) = \left(\frac{\ddot{a}}{\dot{a}} - r(t)\right)\frac{x}{2} - h(t)x^2. \tag{2.27}$$

It is easy to verify that this expression for S gives the required form of the function $g(x,t)$ when substituted in (2.25). □

Corollary 2.1. *A time-dependent first integral of the variable coefficient second-order equation*

$$\ddot{x} + 3h(t)x\dot{x} + h^2(t)x^3 + \lambda(t)x + \dot{h}(t)x^2 = 0, \tag{2.28}$$

is given by the function

$$I(t,x,\dot{x}) = \frac{\dot{x}+S}{a(t)(\dot{x}+S)-\dot{a}x}, \tag{2.29}$$

where $S(x,t) = h(t)x^2 + \ddot{a}x/2\dot{a}$ and $\lambda(t)$ is given by the Schwarzian derivative

$$\lambda(t) = \frac{1}{2}\left[\frac{\dddot{a}}{\dot{a}} - \frac{3}{2}\frac{\ddot{a}^2}{\dot{a}^2}\right].$$

Proof. The proof follows from the previous proposition by setting $r(t) = 0$ and replacing $h(t) \to -h(t)$. □

Proposition 2.2. *If $a(t)$ be such that the function $R(t) := \ddot{a}/\dot{a}$ satisfies a first-order Riccati equation, then the* Eq. (2.28) *can be mapped to a standard variable coefficient second-order Riccati equation.*

Proof. The second-order Riccati equation of variable coefficients is given by

$$\ddot{x} + 3h(t)x\dot{x} + h^2(t)x^3 + \dot{h}(t)x^2 = 0.$$

It is easy to see that, when $\lambda(t) = 0$, (2.28) reduces to a second-order Riccati equation with variable coefficients. The vanishing of $\lambda(t)$ leads to the first-order Riccati equation, namely

$$R_t - \frac{1}{2}R^2 = 0. \tag{2.30}$$

In a similar manner it can be shown that a first integral for the Gambier V Eq. (2.21) is given by

$$I(x, \dot{x}, t) = \frac{\dot{x} + x^2}{\beta(t)(\dot{x} + x^2) - x\beta(t)^2/2},$$

when $\beta(t)$ is a solution of the first-order Riccati equation (2.30).

It is evident that once a first integral is obtained one can easily read off the nonholonomic transformation from its numerator and denominator respectively in view of (2.6). □

3. Nonholonomic transformations for third-order differential equations

The linearization problem in case of third-order ODEs has previously been studied from the perspective of point and contact transformations [14]. However, continuing in the same spirit as above, we consider in this section a general third-order differential equation (TODE) of the form

$$\dddot{x} + a_0(x,t)\ddot{x} + g_2(x,t)\dot{x}^2 + g_1(x,t)\dot{x} + g_0(x,t) = 0 \tag{3.1}$$

and search for a nonholonomic transformation such that it is mapped to the following equation

$$X'''(T) = 0, \tag{3.2}$$

(here $X' = dX/dT$), by the nonholonomic transformation

$$dX = A(x,t)dx + B(x,t)dt, \quad dT = H(x,t)dt. \tag{3.3}$$

It is a matter of straightforward computation to show that the TODE (3.1) is mapped to (3.2) provided its coefficients satisfy the following conditions:

$$2\frac{A_t}{A} + \frac{B_x}{A} - \frac{B}{A}\frac{H_x}{H} - 3\frac{H_t}{H} = a_0(x,t), \tag{3.4}$$

$$3\frac{A_x}{A} - 4\frac{H_x}{H} = 0, \tag{3.5}$$

$$\frac{A_{xx}}{A} - \frac{H_{xx}}{H} - 3\frac{H_x}{H}\frac{A_x}{A} + 3\left(\frac{H_x}{H}\right)^2 = 0, \tag{3.6}$$

$$2\frac{A_{xt}}{A} - 2\frac{H_{xt}}{H} + \frac{B_{xx}}{A} - \frac{H_{xx}}{H}\frac{B}{A} - 3\frac{H_t}{H}\frac{A_x}{A} + 6\frac{H_x}{H}\frac{H_t}{H} - 3\frac{H_x}{H}\frac{A_t}{A} - 3\frac{B_x}{A}\frac{H_x}{H} + 3\frac{B}{A}\left(\frac{H_x}{H}\right)^2 = g_2(x,t), \tag{3.7}$$

$$\frac{A_{tt}}{A} - \frac{H_{tt}}{H} + 2\frac{B_{xt}}{A} - 2\frac{B}{A}\frac{H_{xt}}{H} - 3\frac{H_t}{H}\frac{A_t}{A} + 3\left(\frac{H_t}{H}\right)^2 - 3\frac{H_t}{H}\frac{B_x}{A} + 6\frac{B}{A}\frac{H_x}{H}\frac{H_t}{H} - 3\frac{H_x}{H}\frac{B_t}{A} = g_1(x,t), \tag{3.8}$$

$$\frac{B_{tt}}{A} - \frac{H_{tt}}{H}\frac{B}{A} - 3\frac{H_t}{H}\frac{B_t}{A} + 3\frac{B}{A}\left(\frac{H_t}{H}\right)^2 = g_0(x,t). \tag{3.9}$$

Thus, given a TODE so that the explicit form of the coefficients $a_0(x,t)$ and g_i, $i = 0,\ldots,2$, are known, the set of Eqs. (3.4)–(3.9) constitutes an over-determined set for the three unknown functions A, B and H. Therefore, if upon solving the above set of Eqs. (3.4)–(3.9) one can deduce the functions A, B and H, then the linearizing transformation can be determined and consequently equations of the form (3.1) may be linearized to (3.2).

It is obvious that a first integral of (3.2) is given by

$$I_1(t, x, \dot{x}, \ddot{x}) = X'' = \text{constant}. \tag{3.10}$$

The explicit form of the first integral can be immediately calculated from the transformation (3.3) and has the following appearance,

$$X'' = \frac{1}{H^3}\left[HA\ddot{x} + (HA_x - AH_x)\dot{x}^2 + ((HA_t - AH_t) + (HB_x - BH_x))\dot{x} + (HB_t - BH_t)\right],$$

which may be written as

$$X'' = \frac{1}{H}\left[\frac{A}{H}\ddot{x} + \left(\frac{A}{H}\right)_x \dot{x}^2 + \left(\left(\frac{A}{H}\right)_t + \left(\frac{B}{H}\right)_x\right)\dot{x} + \left(\frac{B}{H}\right)_t\right] = \text{constant}. \tag{3.11}$$

Having explained the general idea behind the construction of a linearizing transformation for an equation of the form (3.1), we proceed to the determination of the unknown functions A, B and H. From (3.5) we have

$$H(x,t) = \alpha(t)A^{3/4}, \tag{3.12}$$

where $\alpha(t)$ is an arbitrary function of t. Elimination of H from (3.6) leads to the following equation determining the function $A(x,t)$, namely

$$\frac{A_{xx}}{A} - \frac{3}{2}\left(\frac{A_x}{A}\right)^2 = 0, \tag{3.13}$$

which admits the solution

$$A(x,t) = \frac{\gamma(t)}{(2 - x\beta(t))^2}. \tag{3.14}$$

Here β and γ are arbitrary functions of t. Next eliminating H from (3.4) we have

$$\left(\frac{B}{A}\right)_x + \frac{1}{4}\frac{A_x}{A}\left(\frac{B}{A}\right) = \frac{1}{4}\frac{A_t}{A} + 3\frac{\dot{\alpha}}{\alpha} + a_0$$

which has the formal solution,

$$B(x,t) = A^{3/4}(x,t)\left[\int A^{1/4}\left(\frac{A_t}{4A} + 3\frac{\dot{\alpha}}{\alpha} + a_0\right)dx + \delta(t)\right]. \tag{3.15}$$

Note that, since α, β, γ and δ are arbitrary functions of t, we can choose them to be constants in order to simplify the calculations. Secondly, having completed the determination of the unknown functions A, B and H involved in our nonholonomic transformation, it remains to examine their compatibility with Eqs. (3.7)–(3.9). In the following we consider the case when the functions α, β, γ and δ assume the following values.

Case (i) $\alpha = \beta = \gamma = 1$ and $\delta = 0$

With these values one finds that

$$A(x,t) = \frac{1}{(2-x)^2}, \quad H(x,t) = \frac{1}{(2-x)^{3/2}}, \quad \frac{B}{A} = \frac{1}{A^{1/4}}\int^x A^{1/4}a_0(s,t)ds. \tag{3.16}$$

Consequently from (3.7) to (3.9) we arrive at the following relations:

$$g_2(x,t) = a_{0x} - \frac{1}{2}a_0\left(\frac{A_x}{A}\right) - \frac{3}{4}\left(\frac{A_x}{A}\right)_x\left[\frac{1}{A^{1/4}}\int^x A^{1/4}a_0(s,t)ds\right], \tag{3.17}$$

$$g_1(x,t) = 2a_{0t} - \frac{3}{4}\left(\frac{A_x}{A}\right)\left[\frac{1}{A^{1/4}}\int^x A^{1/4}a_{0t}(s,t)ds\right], \tag{3.18}$$

$$g_0(x,t) = \left[\frac{1}{A^{1/4}}\int^x A^{1/4}a_{0tt}(s,t)ds\right]. \tag{3.19}$$

Instead of requiring the right hand sides of the above equations to match the given values of $g_i(x,t)$, $i = 0,\ldots,2$, we could choose to define the g_i by these very relations. Furthermore we assume

$$a_0(x,t) = h(t)f(x)$$

and define a function,

$$F(x) := \frac{1}{A^{1/4}}\int^x A^{1/4}f(s)ds,$$

so that

$$\frac{B}{A} = h(t)F(x).$$

Then the expressions for g_i become

$$g_0(x,t) = \ddot{h}(t)F(x), \tag{3.20}$$

$$g_1(x,t) = \dot{h}(t)\left[2f(x) - \frac{3F(x)}{2(2-x)}\right], \tag{3.21}$$

$$g_2(x,t) = h(t)\left[f'(x) - \frac{f(x)}{(2-x)} + \frac{3F(x)}{2(2-x)^2}\right]. \tag{3.22}$$

Note that the explicit form of $F(x)$ is given by

$$F(x) = (2-x)^{1/2}\int^x \frac{f(s)}{(2-s)^{1/2}}ds.$$

Therefore a third-order equation of the form

$$\dddot{x} + h(t)f(x)\ddot{x} + [g_2(x,t)\dot{x}^2 + g_1(x,t)\dot{x} + g_0(x,t)] = 0$$

with g_0, g_1 and g_2 given by the Eqs. (3.20)–(3.22) may be linearized to $X''' = 0$ by the nonholonomic transformation

$$dX = \frac{1}{(2-x)^2}[dx + h(t)F(x)dt], \quad dT = \frac{1}{(2-x)^{3/2}}dt. \tag{3.23}$$

Its associated first integral may be obtained from (3.11) and is given by the following expression

$$I(x,t,\dot{x},\ddot{x}) = (2-x)\ddot{x} + \frac{1}{2}\dot{x}^2 + h(t)\left((2-x)F'(x) + \frac{F(x)}{2}\right)\dot{x} + (2-x)F(x)\dot{h} = \text{constant}. \tag{3.24}$$

Case (ii) $\alpha = 1, \gamma = 4$ and $\beta = \delta = 0$

This case is considerably simply because, when $\beta = 0$ and $\gamma = 4$, it follows that $A(x,t) = H(x,t) = 1$ while from (3.15) we have $B = \int^x a_0(s,t)ds = h(t)\int^x f(s)ds$ (assuming $a_0 = h(t)f(x)$). Moreover the expressions for g_i, as stated above, reduce to the following:

$$g_2(x,t) = a_{0x} = h(t)f'(x), \quad g_1(x,t) = 2\dot{h}f(x), \quad g_0(x,t) = \ddot{h}\int^x f(s)ds := \ddot{h}F_1(x). \tag{3.25}$$

The explicit nature of the transformation in this case is interesting since it does not involve any change in the time coordinate,

$$dX = dx + h(t)F_1(x)dt, \quad dT = dt. \tag{3.26}$$

The corresponding first integral is now given by

$$I_1(x,t,\dot{x},\ddot{x}) = \left[\ddot{x} + h(t)F_1'(x)\dot{x} + \dot{h}(t)F_1(x)\right] = \text{constant}. \tag{3.27}$$

4. Conclusion

In this paper we have computed the first integrals of the time-dependent second-order Riccati equation and its generalization by using the method of nonholonomic transformations. It appears that unlike Sundman transformation this method leads to considerable computational simplification. In the latter half of the paper we have calculated the first integral in certain particular cases of a generic third-order nonlinear equation, which may be viewed as a kind of generalization of the second-order equation of Liénard type, $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Hence it would be interesting and tempting to apply this scheme to compute the first integrals of the Chazy systems.

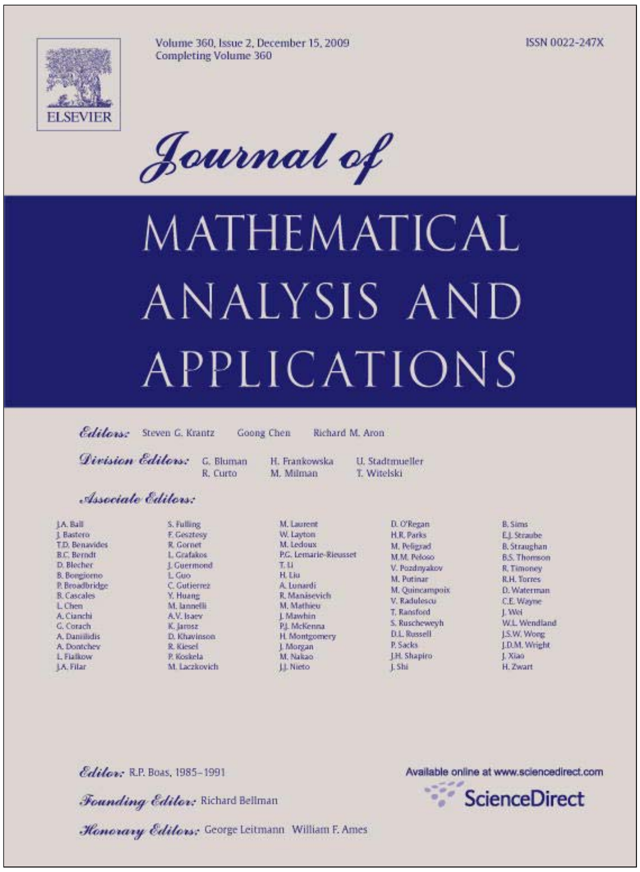
Acknowledgements

We are extremely grateful to Professor Basil Grammaticos for his comments, encouragement and careful reading of our manuscript. We wish to thank Pepin Cariñena for his valuable remarks. In addition AGC wishes to acknowledge the support provided by the S.N. Bose National Centre for Basic Sciences, Kolkata in the form of an Associateship.

References

[1] Calogero F, Leyvraz F. General technique to produce isochronous Hamiltonians. J Phys A Math Theor 2007;40:12931–44.
[2] Calogero F, Leyvraz F. Examples of isochronous Hamiltonians: classical and quantal treatments. J Phys A Math Theor 2008;41:175202. 11p.

[3] Chandrasekhar VK, Senthilvelan M, Lakshmanan M. On the general solution of the modified Emden-type equation $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$. J Phys A Math Gen 2007;40:4717–27.
[4] Chandrasekhar VK, Senthilvelan M, Kundu A, Lakshmanan M. A nonlocal connection between certain linear and nonlinear ordinary differential equations/oscillators. J Phys A Math Gen 2006;39:9743–54.
[5] Cariñena JF, Guha P, Rañada MF. A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion. J Phys: Conf Ser 2009;175:012009.
[6] Cariñena JF, Rañada MF, Santander M. Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability. J Math Phys 2005;46:062703.
[7] Duarte LGS, Moreira IC, Santos FC. Linearization under nonpoint transformations. J Phys A 1994;27:L739–43.
[8] Euler N, Euler M. Sundman symmetries of nonlinear second-order and third-order ordinary differential equations. J Nonlinear Math Phys 2004;11(3):399–421.
[9] Euler N, Wolf T, Leach PGL, Euler M. Linearisable third-order ordinary differential equations and generalised Sundman transformations: the case $X''' = 0$. Acta Appl Math 2003;76(1):89–115.
[10] Gambier B. Sur les équations différentielles du second ordre et du premier degré dont l'intgrale générale est á points critique fixés. Acta Math 1910;33:1–55.
[11] Guha P, Ghose A, Choudhury. On planar and non-planar isochronous systems and poisson structures. To Appear in Int J Geom Methods Mod Phys.
[12] Gladwin Pradeep R, Chandrasekar VK, Senthilvelan M, Lakshmanan M. Dynamics of a completely integrable N-coupled Linard-type nonlinear oscillator. J Phys A Math Theor 2009;42:135206.
[13] Grammaticos B, Ramani A, Lafortune S. The Gambier mapping, revisited. Physica A 1998;253:260–70.
[14] Exton H. On nonlinear ordinary differential equations with fixed critical points. Rend de Mat 1973;6(6):419–62.
[15] Ince EL. Ordinary differential equations; 1956, New York: Dover, reprinted.
[16] Kennedy J. The hydrogen atom as an example of quantization with a constraint. Proc R Irish Acad A 1982;82:1–8.
[17] Kleinert H. Nonholonomic mapping principle for classical and quantum mechanics in spaces with curvature and torsion, arXiv:gr-qc/0203029v1.
[18] Kleinert H. Chapter 10 of path integrals in quantum mechanics, statistics, polymer physics, and financial markets. Singapore: World Scientific; 2009.
[19] Kustaanheimo P, Stiefel E. Perturbation theory of Kepler motion based on spinor regularization. J Reine Angew Math 1965;218:204–19.
[20] Mahomed FM, Leach PGL. The linear symmetries of a nonlinear differential equation. Quaest Math 1985;8(3):241–74.
[21] Pelster A, Kleinert H. Transformation properties of classical and quantum laws under some nonholonomic spacetime transformations, arXiv:quant-ph/9608037v1.
[22] Sarlet W. First integrals for one-dimensional particle motion in a non-linear, time-dependent potential field. Int J Non-Linear Mech 1983;18(4):259–68.
[23] Sugai I. Riccati's nonlinear differential equation. Am Math Monthly 1960;67:134–9.
[24] Sundman KF. Mémoire sur le problème des trois corps, Acta Math 36 (1912–1913), 105–179.


Volume 360, Issue 2, December 15, 2009
Completing Volume 360
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL ANALYSIS AND APPLICATIONS
Editors: Steven G. Krantz Goong Chen Richard M. Aron
Division Editors: G. Bluman H. Frankowska U. Stadtmueller R. Curto M. Milman T. Witelski
Associate Editors:
Editor: R.P. Boas, 1985–1991
Founding Editor: Richard Bellman
Honorary Editors: George Leitmann William F. Ames
Available online at www.sciencedirect.com
ScienceDirect

J. Math. Anal. Appl. 360 (2009) 651–664

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Jacobi Last Multiplier, integrating factors and the Lagrangian formulation of differential equations of the Painlevé–Gambier classification

A. Ghose Choudhury [a], Partha Guha [b,c,*], Barun Khanra [d]

[a] Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta-700009, India
[b] Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany
[c] S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India
[d] Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Calcutta-700004, India

A R T I C L E   I N F O

Article history:
Received 15 December 2008
Available online 23 June 2009
Submitted by P.G.L. Leach

Keywords:
Painlevé equations
First integral
Jacobi's Last Multiplier
Lagrangian

A B S T R A C T

We use a formula derived almost seventy years ago by Madhav Rao connecting the Jacobi Last Multiplier of a second-order ordinary differential equation and its Lagrangian and determine the Lagrangians of the Painlevé equations. Indeed this method yields the Lagrangians of many of the equations of the Painlevé–Gambier classification. Using the standard Legendre transformation we deduce the corresponding Hamiltonian functions. While such Hamiltonians are generally of non-standard form, they are found to be constants of motion. On the other hand for second-order equations of the Liénard class we employ a novel transformation to deduce their corresponding Lagrangians. We illustrate some particular cases and determine the conserved quantity (first integral) resulting from the associated Noetherian symmetry. Finally we consider a few systems of second-order ordinary differential equations and deduce their Lagrangians by exploiting again the relation between the Jacobi Last Multiplier and the Lagrangian.

1. Introduction

The study of nonlinear ordinary differential equations (ODEs) has been an ongoing endeavor for well over two centuries now with significant contributions by many of the greatest mathematicians of all times, such as Euler, Lie, Painlevé and Poincaré to mention just a few. Their contributions have ranged from finding explicit solutions of ODEs, to developing general methods of classifications, to the qualitative analysis of their solutions etc. These in turn have often led to the opening of entirely new branches of study in algebra, topology, geometry and have shed new light on several physical phenomena.

Over the years many techniques have been developed to obtain closed-form solutions of various kinds of ODEs. However, there does not exist any single common method for obtaining their solutions. Nevertheless the apparently different techniques share one common feature: they somehow tend to exploit the symmetries of ODEs. Consequently symmetry analysis of ODEs has become one of the most powerful tools for analyzing them. The foundations of this method are contained in the works of Sophus Lie [1,2].

* Corresponding author at: S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India.
E-mail addresses: a_ghosechoudhury@rediffmail.com (A. Ghose Choudhury), partha.guha@mis.mpg.de (P. Guha), barunkhanra@rediffmail.com (B. Khanra).

0022-247X/$ – see front matter © 2009 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2009.06.052

It is also well known that the existence of a sufficient number of first integrals greatly simplifies the process of solving any ODE. Having said so, it is not always quite obvious what these first integrals are. Indeed their determination is, in general, a nontrivial task. In the case of conservative mechanical systems, one often has just a single first integral – the energy. In this context the semialgorithmic procedure developed by Prelle and Singer deserves mention [6]. In its original version it applied to first-order ODEs involving rational functions with coefficients belonging to the field of complex numbers $\mathbb{C}$. Subsequently their method, which involved the use of Darboux polynomials, was extended by Duarte et al. [7,8] and Chandrasekhar et al. in a series of papers [3–5].

As is often the case in a field which has been so thoroughly investigated over the years, some aspects often tend to fade out only to resurface after many years when new results point to a mysterious link with those of the past. One such result, which has appeared in the recent literature on differential equations, concerns the Jacobi Last Multiplier. The credit for resurrecting this has to go to Nucci and Leach, who have shown how it may be used to determine the first integrals and also Lagrangians of a wide variety of nonlinear differential equations. It appears that the connection of the Jacobi Last Multiplier to the existence of Lagrangian functions was the subject of investigation by a few authors in the early 1900s. However, the precise nature of their interrelation was brought out by Rao in [9] in the 1940s. Thereafter this did not attract much attention among researchers in the field of differential equations.

Recently there has been a renewal of interest in this area and it appears that Jacobi's Last Multiplier can be incorporated in the formalism initiated by Lie for the study of differential equations.

1.1. Motivation and plan

It is clear that Rao's formula can be used to deduce the Lagrangian of a second-order ODE or even a system of such ODEs once the last multiplier is known. Unlike the Hamiltonian structure of the six Painlevé equations, which have received much attention [17], the Lagrangian formulation has not been sufficiently nurtured. In a recent paper Wolf and Brand [18] proposed a Lagrangian for Painlevé VI. In this paper we investigate the Lagrangians for the majority of the Painlevé equations, using Rao's formula and also indicate its applicability to other equations of the Painlevé–Gambier classification.

The organization of the paper is as follows. In Section 2 we introduce the basic ideas underlying the Jacobi Last Multiplier and state its defining equation for an nth-order ODE or an equivalent system of first-order ODEs. The connection between the last multiplier and symmetries is also mentioned. It also contains a discussion of certain geometrical aspects underlying Jacobi's Last Multiplier. Section 3 constitutes the main body of this paper and explains the relationship between the Jacobi Last Multiplier and the Lagrangian description of second-order ODEs. It includes a deduction of the Lagrangians for four of the six Painlevé equations and also their corresponding Hamiltonians. It also briefly outlines the procedure for other equations of the Painlevé–Gambier classification. In Section 4 we analyze in this context second-order equations of the Liénard type. It contains a specific example of a generic equation of nonlinear oscillator type. Finally in Section 5 we apply the technique to a coupled system of second-order ODEs, which has not been very extensively studied, and also summarize the results for a couple of other more well-known systems.

2. The Jacobi Last Multiplier

Consider the nth-order ODE in the normal form

$$y^{(n)} = w\big(x, y, y', \ldots, y^{(n-1)}\big). \tag{2.1}$$

Corresponding to this ODE there exists an equivalent first-order partial differential equation (PDE) in $(n+1)$ variables,

$$\widetilde{D} f = (\partial_x + y'\partial_y + y''\partial_{y'} + \cdots + w\partial_{y^{(n-1)}}) f = 0, \tag{2.2}$$

in which the quantities $y', y'' \ldots$ are treated as independent variables at par with x and y.

Their equivalence is provided by the first integrals of (2.1). By definition a first integral is a global function, $I = I(x, y, y', \ldots, y^{(n-1)})$, that is constant along the solutions of (2.1), i.e.,

$$\frac{dI}{dx} = \widetilde{D} I = I_x + y' I_y + y'' I_{y'} + \cdots + w I_{y^{(n-1)}} = 0. \tag{2.3}$$

Having determined a first integral, say $I = I(x, y, y', \ldots, y^{(n-1)}) = I_0$, one can invert it to obtain

$$y^{(n-1)} = w_1\big(x, y, y', \ldots, y^{(n-2)}; I_0\big)$$

provided $I_{y^{(n-1)}} \neq 0$. This shows that the existence of a first integral allows for the reduction in the order of the differential equation by one. Furthermore it is evident that every first integral is a solution of the linear PDE (2.2) and conversely.

Assume that ϕ^α, $\alpha = 1, \ldots, n$ denote a set of n functionally independent solutions of (2.2). As each ϕ^α is a first integral, one has

$$\phi^\alpha\big(x, y, y', \ldots, y^{(n-1)}\big) = I_0^\alpha, \quad \alpha = 1, 2, \ldots, n. \tag{2.4}$$

Consequently by eliminating all derivatives from (2.4) one arrives at the general solution of (2.1) in the form

$$y = y\big(x; I_0^1, \dots, I_0^n\big),$$

the I_0^α's being essentially constants of integration.

As we mentioned above, the determination of even a single first integral is in most cases a nontrivial task. Hence, while in principle the above procedure is fine, its practical application is often a daunting task, to say the least.

Nevertheless, assuming we have at our disposal $(n-1)$ solutions ϕ^α of the linear PDE $\widetilde{D}f = 0$, by means of the Jacobi Last Multiplier the nth solution can be obtained by a quadrature. The formal definition of the Jacobi Last Multiplier is as follows.

Definition 2.1. Given an nth-order ODE or its equivalent linear PDE in $(n+1)$ variables

$$\widetilde{D}f = (\partial_x + y'\partial_y + y''\partial_{y'} + \dots + w\partial_{y^{(n-1)}})f = 0,$$

the Jacobi Last Multiplier M is defined by

$$M\widetilde{D}f := \frac{\partial(f, \phi^1, \phi^2 \dots \phi^{n-1})}{\partial(x, y, y', \dots, y^{(n-1)})} = \det\begin{pmatrix} f_x & f_y & \dots & f_{y^{(n-1)}} \\ \phi^1_x & \phi^1_y & \dots & \phi^1_{y^{(n-1)}} \\ \vdots & \vdots & \dots & \vdots \\ \phi^{(n-1)}_x & \phi^{(n-1)}_y & \dots & \phi^{(n-1)}_{y^{(n-1)}} \end{pmatrix} = 0. \tag{2.5}$$

From the above definition it follows that the Jacobi Last Multiplier (JLM) can be varied by selecting a different set of $(n-1)$ independent solutions $\psi^1, \psi^2, \dots, \psi^{n-1}$ of (2.2). If the corresponding JLM be $\widetilde{M}$, then

$$\widetilde{M}\widetilde{D}f = \frac{\partial(f, \psi^1, \psi^2, \dots, \psi^{n-1})}{\partial(x, y, y', \dots, y^{(n-1)})} = \frac{\partial(f, \phi^1, \phi^2, \dots, \phi^{n-1})}{\partial(x, y, y', \dots, y^{(n-1)})}\frac{\partial(\psi^1, \psi^2, \dots, \psi^{n-1})}{\partial(\phi^1, \phi^2, \dots, \phi^{n-1})} = M\frac{\partial(\psi^1, \psi^2, \dots, \psi^{n-1})}{\partial(\phi^1, \phi^2, \dots, \phi^{n-1})}.$$

Indeed each JLM, as defined above, turns out to be a solution of the following linear PDE

$$\frac{\partial M}{\partial x} + \sum_{k=1}^{n} \frac{\partial\big(y^{(k)}M\big)}{\partial y^{(k-1)}} = 0 \quad \text{on } y^{(n)} = w\big(x, y, y', \dots, y^{(n-1)}\big) \tag{2.6}$$

or, if the ODE be expressed as a system of first-order ODEs of the form

$$\dot{x}_k = W_k(t, x_1, \dots, x_n), \quad k = 1, 2, \dots, n, \tag{2.7}$$

as a solution of the equation

$$\frac{d}{dt}\log M + \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0. \tag{2.8}$$

It is evident that the classical definition of the JLM is overly restrictive, requiring as it does almost complete knowledge of the system. However, being dependent on first integrals, it is natural to expect that it should bear some connection to the symmetries of the equation under investigation. This connection was unravelled by Lie and its formulation in terms of the generators of the Lie symmetry algebra is outlined below.

For the ODE in Eq. (2.1) or its equivalent PDE given by (2.2) let $X_i = \xi_i\partial_x + \eta_i\partial_y$ denote n Lie point symmetry generators of the equation. The vector field associated with $\widetilde{D}f = 0$ has $(n+1)$ components $(1, y', \dots, w)$ on $y^{(n)} = w(x, y, \dots, y^{(n-1)})$. Using standard methods for constructing the prolongations of these generators X_i up to the $(n-1)$th-order, *viz.*

$$X_i^{(n-1)} = \xi_i\partial_x + \eta_i\partial_y + \eta_i^{(1)}\partial_{y'} + \dots + \eta_i^{(n-1)}\partial_{y^{(n-1)}}, \quad i = 1, 2, \dots, n,$$

consider the determinant

$$\Delta = \det\begin{pmatrix} 1 & y' & y'' & \dots & f_{y^{(n-1)}} & w \\ \xi_1 & \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-2)} & \eta_1^{(n-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-2)} & \eta_n^{(n-1)} \end{pmatrix}. \tag{2.9}$$

If $\Delta \neq 0$, then the JLM is given by $M = \Delta^{-1}$.

Similarly for a system of n first-order ODEs given by (2.7) the associated vector field has components $(1, W_1, \ldots, W_n)$. If we can find n symmetry generators of the form $X_i = \sum_{j=0}^{n} a_{ij} \frac{\partial}{\partial x_j}$, $i = 1, \ldots, n$, with $x_0 = t$, then the last multiplier is given by

$$M^{-1} = \Delta = \det \begin{pmatrix} 1 & W_1 & \cdots & W_n \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n0} & a_{n1} & \ldots & a_{nn} \end{pmatrix}. \tag{2.10}$$

Since M satisfies (2.8), it follows that the ratio of two last multipliers is a first integral, i.e.,

$$\frac{d}{dt}\left(\frac{M_1}{M_2}\right) = 0.$$

In other words the ratios of the Δ_i's ($i \geqslant 2$) provide us first integrals for the system of equations (2.7).

2.1. *Geometric description of Jacobi's Last Multiplier*

Let $M = M(\mathbf{x})$ be a non-negative C^1 function non-identically vanishing on some open subset of $\mathbb{R}^n$. Then Jacobi's Last Multiplier is a solution of the linear partial differential equation

$$\sum_{i=1}^{n} \frac{\partial(MW_i)}{\partial x_i} = 0, \tag{2.11}$$

where $\mathbf{W} = \sum_{i=1}^{n} W_i \partial_{x_i}$ is the vector field of the system of first-order ODEs. Essentially, if a Jacobi multiplier is known together with $(n-2)$ first integrals, then we can reduce locally to a $2D$ vector field on the intersection of the $(n-2)$ level sets formed by first integrals.

Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be a volume form on $\mathbb{R}^n$. Define an inner product, $\langle\cdot,\cdot\rangle$, between 1-forms and $(n-1)$-forms on $\mathbb{R}^n$ as

$$\omega_1 \wedge \omega_2 = \langle \omega_1, \omega_2 \rangle \Omega.$$

So both the space of vectors and the space of $(n-1)$-forms are dual to the space of 1-forms. Hence there is a natural isomorphism between the space of vectors and the space of $(n-1)$-forms. Let $\mathbf{W}$ be a vector field. Then $\mathbf{W}$ corresponds to $(n-1)$ form $\omega_{\mathbf{W}}$ under isomorphism

$$i_{\mathbf{W}}\Omega = \omega_{\mathbf{W}} = \sum_{i=1}^{n} (-1)^{(i-1)} W_i \, dx^1 \wedge \cdots \wedge \hat{d}x_i \wedge \cdots \wedge dx_n. \tag{2.12}$$

Thus the condition for Jacobi's Last Multiplier can be manifested as [11,12]

$$0 = d(M\omega_{\mathbf{W}}) = \left(\sum_{i=1}^{n} \frac{\partial(MW_i)}{\partial x_i}\right)\Omega.$$

Therefore an element, M, is called a Jacobi's Last Multiplier for an ODE if

$$d(M\omega_{\mathbf{W}}) = dM \wedge \omega_{\mathbf{W}} + M d\omega_{\mathbf{W}} = 0. \tag{2.13}$$

Using

$$L_{\mathbf{W}}\omega_{\mathbf{W}} = (\mathrm{div}_\Omega \, \mathbf{W})\omega_{\mathbf{W}} \tag{2.14}$$

and $(dg) \wedge \omega_{\mathbf{W}} = (\mathbf{W}g)\Omega$ $(\forall g \in C^\infty(\mathbb{R}^n))$ we can show that the Jacobi Last Multiplier M satisfies

$$\mathbf{W}M + M \, \mathrm{div}_\Omega \, \mathbf{W} = 0. \tag{2.15}$$

This equation reveals that M is a last multiplier for the divergence-free vector field $\mathbf{W}$ if and only if M is a first integral of $\mathbf{W}$. In general the vector field $\mathbf{W}$ is not divergence-free and in this situation the theory of multipliers, namely, the ratio of two multipliers is a first integral etc., holds good. In fact the set of last multipliers measures how far away $\mathbf{W}$ is from the divergence-free condition.

The theory of the Jacobi Last Multiplier is also connected to another important method namely adjoint symmetry equation in determining explicit integrating factors and first integrals of nonlinear ODEs. One can define the action of the adjoint vector field $\mathbf{W}^*$ corresponding to $\mathbf{W}$ on functions [13] as

$$\mathbf{W}^*(M) = -\mathbf{W}(M) - M \, \mathrm{div}_\Omega \, \mathbf{W} = 0. \tag{2.16}$$

Thus solving the adjoint equation one can obtain Jacobi's Last Multiplier. This is the essential feature of adjoint method.

We state another geometrical idea related to the last multiplier. Characterization of Jacobi's Last Multiplier can be obtained in terms of the Marsden differential [14]. Let $m \in C^\infty(M)$. Then the Marsden differential,

$$d^m : \Lambda^*(M) \to \Lambda^{*+1}(M)$$

is given by

$$d^m(\eta) = \frac{1}{m} d(m\eta). \tag{2.17}$$

Thus M is a Jacobi Last Multiplier if and only if $\omega_{\mathbf{W}}$ is d^M-closed.

A vector field, **S**, is called a symmetry of an ODE given by a vector field, **W**, if

$$L_{\mathbf{S}}\mathbf{W} = [\mathbf{W}, \mathbf{S}] = \lambda \mathbf{W}, \quad \lambda \in C^\infty. \tag{2.18}$$

Let $\mathbf{S}_1, \ldots, \mathbf{S}_{n-1}$ be $(n-1)$ symmetries. Define

$$h = i_{S_{n-1}} \cdots i_2 i_1 \omega_{\mathbf{W}}. \tag{2.19}$$

Then $M = h^{-1}$ is a last multiplier for $\omega_{\mathbf{W}}$, i.e., $d(M\omega_{\mathbf{W}}) = 0$. This can be proved using the symmetry condition.

$$L_{\mathbf{W}} h = L_{\mathbf{W}} i_{S_{n-1}} \cdots i_2 i_1 \omega_{\mathbf{W}} = (i_{[\mathbf{W}, \mathbf{S_{n-1}}]} + i_{S_{n-1}} L_{\mathbf{W}}) i_{S_{n-2}} \cdots i_{S_2} i_{S_1} \mathbf{W}.$$

The first term in the expression above vanishes. Thus recursively one can prove that

$$L_{\mathbf{W}} h = h \operatorname{div}_\Omega \mathbf{W}, \tag{2.20}$$

where the function $M = h^{-1}$ is called an inverse multiplier.

At last we wish to outline a connection between last multiplier and Nambu mechanics. Consider a special case of (2.14), a divergence-free condition

$$\operatorname{div}_\Omega \mathbf{W} = \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0. \tag{2.21}$$

In this situation Eq. (2.7) can be mapped to Nambu dynamical systems, i.e. systems of time-autonomous ODEs of the form [15], with a special value of W_i

$$\dot{x}^i = W_i(x) = \epsilon_{j_1,\ldots,j_n} \delta^i_{j_1} \frac{\partial H_2}{\partial x^{j_2}} \cdots \frac{\partial H_n}{\partial x^{j_n}}. \tag{2.22}$$

In other words a system obeying Nambu mechanics automatically satisfies the Liouville condition. In fact by duality the vector field $\mathbf{W} = W_i(x)\partial/\partial x_i$ maps to an $(n-1)$-differential form given by

$$\begin{aligned}
\omega_{\mathbf{W}} &= \frac{1}{(n-1)!} \epsilon_{j_1,\ldots,j_n} W_{j_1}\, dx^{j_2} \wedge \cdots \wedge dx^{j_n} \\
&= \frac{1}{(n-1)!} \epsilon_{k_1,\ldots,k_n} \epsilon_{j_1,\ldots,j_n} \delta_{k_1 j_1} \left[\frac{\partial H_2}{\partial x_{j_2}} dx^{j_2}\right] \wedge \cdots \wedge \left[\frac{\partial H_n}{\partial x_{j_n}} dx^{j_n}\right] \\
&= dH_2 \wedge \cdots \wedge dH_n.
\end{aligned}$$

Thus $\omega_{\mathbf{W}}$ is a decomposable and closed $(n-1)$-form and the set of $(n-1)$ independent functions, $H_2, \ldots, H_n$, are such that every integral curve is given by an equation of the form $H_2(\mathbf{x}) = C_2, \ldots, H_n(\mathbf{x}) = C_n$.

3. Lagrangians and the last multiplier

In a series of recent papers Leach, Nucci and Tamizhmani (for example, [16,19–21] and references therein) have investigated the relation between integrating factors and the Hessian. It appears that this connection has a long history, which can be traced to Jacobi's attempts to obtain the last multiplier [23,24]. In 1874 Lie [1,2] showed that point symmetries could be used to determine Jacobi's Last Multiplier (JLM). The explicit nature of the relation between the JLM and Hessian was clarified by Rao in a article [9] and is also mentioned in Whittaker's book on analytical dynamics [10].

3.1. Second order equations

For a second-order ODE $y'' = w(x, y, y')$ which admits a Lagrangian function $L(x, y, y')$ the Jacobi Last Multiplier, M, is given by

$$M = \frac{\partial^2 L}{\partial y'^2}. \tag{3.1}$$

On the other hand, given a system of first order equations

$$y_k' = f_k(x, y), \qquad y = (y_1, y_2, \ldots, y_n),$$

the JLM is a solution of the equation

$$\frac{d \log M}{dx} + \sum_{k=1}^{n} \frac{\partial f_k}{\partial y_k} = 0.$$

It follows that, if a solution of this equation is obtained, then from a knowledge of the JLM one can construct the Lagrangian function as

$$L(x, y, y') = \int \left(\int M \, dy' \right) + f_1(x, y) y' + f_2(x, y). \tag{3.2}$$

3.2. Lagrangians for the Painlevé equations

A large number of second-order ODEs in the Painlevé–Gambier classification system belong to the following class of equations, namely

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x) = 0. \tag{3.3}$$

Writing this equation in the form

$$\ddot{x} = \mathcal{F}(t, x, \dot{x}) = -\left[\frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x)\right],$$

the Jacobi Last Multiplier M for (3.3) is given by the solution of

$$\frac{d}{dt} \log M = -\frac{\partial \mathcal{F}}{\partial \dot{x}}. \tag{3.4}$$

In the present case we have

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} = \exp\big[\phi(t, x)\big]. \tag{3.5}$$

By (3.2) we then obtain the Lagrangian as

$$L(t, x, \dot{x}) = \frac{1}{2} e^{\phi(t,x)} \dot{x}^2 + f_1(t, x)\dot{x} + f_2(t, x). \tag{3.6}$$

To determine the unknown functions, f_1 and f_2, we substitute this Lagrangian into the Euler–Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{3.7}$$

and use (3.3) to get

$$f_{1t} - f_{2x} = e^{\phi} B(t, x).$$

The making of a gauge transformation $f_1 = G_x$ and $f_2 = G_t + f_3(t, x)$ allows us to satisfy the last equation when

$$f_3(t, x) = -\int e^{\phi} B(t, x) \, dx. \tag{3.8}$$

Consequently the final Lagrangian for (3.3) becomes

$$L(t, x, \dot{x}) = e^{\phi(t,x)} \frac{\dot{x}^2}{2} - \int e^{\phi} B(t, x) \, dx + \frac{dG}{dt}. \tag{3.9}$$

The total derivative term obviously is of little consequence. Hence we may safely discard it.

The conjugate momentum may be defined by

$$p = \frac{\partial L}{\partial \dot{x}} = e^{\phi} \dot{x} \quad \text{which implies} \quad \dot{x} = e^{-\phi} p$$

and leads to the Hamiltonian

$$H = e^{-\phi}\frac{p^2}{2} + \int e^{\phi} B(t,x)\,dx$$

by the usual Legendre transformation. It is clear that the Lagrangian obtained in the above manner is a non-standard one.

One can attempt to bring it closer to the standard form by means of the transformation

$$\dot{y} = e^{\phi/2}\dot{x} \text{ or } y(t,x) = \int e^{\phi(t,x)/2}\,dx. \tag{3.10}$$

We illustrate this by a specific example in the sequel.

3.2.1. The Painlevé III equation

The P_{III} equation may be written as

$$\ddot{x} - \frac{1}{x}\dot{x}^2 + \frac{1}{t}\dot{x} + B(t,x) = 0, \tag{3.11}$$

where $B(t,x) = -[\frac{1}{t}(\alpha x^2+\beta) + \gamma x^3 + \frac{\delta}{x}]$. Comparison with (3.3) shows that $\phi_x = -2/x$ and $\phi_t = 1/t$ which yields for the last multiplier $M = \exp\phi = t/x^2$. Then from (3.9) we obtain

$$L_{III} = \frac{t}{x^2}\frac{\dot{x}^2}{2} + \alpha x - \frac{\beta}{x} + t\left(\frac{\gamma x^2}{2} - \frac{\delta}{2x^2}\right) \tag{3.12}$$

and the Hamiltonian as

$$H_{III} = \frac{x^2}{t}\frac{p^2}{2} + \left(\frac{\beta}{x^2} - \alpha x\right) + \frac{t}{2}\left(\frac{\delta}{x^2} - \gamma x^2\right). \tag{3.13}$$

3.2.2. The Painlevé V equation

The P_V equation may be written as

$$\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 + \frac{1}{t}\dot{x} + B(t,x) = 0, \tag{3.14}$$

where

$$B(t,x) = -\left[\frac{(x-1)^2}{t^2}\left(\alpha x + \frac{\beta}{x}\right) + \frac{\gamma x}{t} + \frac{\delta x(x+1)}{x-1}\right].$$

Following the same procedure as before we obtain for the Jacobi Last Multiplier

$$M = \frac{t}{x(x-1)^2}$$

and the Lagrangian

$$L_V = \frac{t}{x(x-1)^2}\frac{\dot{x}^2}{2} + \frac{1}{t}\left(\alpha x - \frac{\beta}{x}\right) - \frac{\gamma}{x-1} - \delta\frac{tx}{(x-1)^2}. \tag{3.15}$$

The corresponding Hamiltonian is

$$H_V = \frac{x(x-1)^2}{t}\frac{p^2}{2} - \frac{1}{t}\left(\alpha x - \frac{\beta}{x}\right) + \frac{\gamma}{x-1} + \delta\frac{tx}{(x-1)^2}. \tag{3.16}$$

3.2.3. The Painlevé IV equation

The P_{IV} equation may be written as

$$\ddot{x} - \frac{1}{2x}\dot{x}^2 + B(t,x) = 0, \tag{3.17}$$

where

$$B(t,x) = -\left[\frac{3}{2}x^3 + 4tx^2 + 2(t^2-\alpha)x + \frac{\beta}{x}\right].$$

Unlike the previous two Painlevé equations, here we have $\phi_t = 0$ so that the last multiplier is now time independent. Indeed for the P_{IV} equation we have $M = 1/x$ while the corresponding Lagrangian is

$$L_{IV} = \frac{1}{x}\frac{\dot{x}^2}{2} + \left[\beta\ln|x| + (t^2-\alpha)x^2 + \frac{4}{3}tx^3 + \frac{3}{8}x^4\right]. \tag{3.18}$$

The associated Hamiltonian is

$$H_{IV} = \frac{xp^2}{2} - \left[\beta \ln|x| + \left(t^2 - \alpha\right)x^2 + \frac{4}{3}tx^3 + \frac{3}{8}x^4\right]. \tag{3.19}$$

3.2.4. The Painlevé VI equation

The P_{VI} equation is perhaps one of the most well-studied equations of the Painlevé class. It may be written as

$$\ddot{x} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\dot{x}^2 + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\dot{x} + B(t,x) = 0, \tag{3.20}$$

where

$$-B(t,x) = \frac{(x-1)(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha + \frac{\beta t}{x^2} + \frac{\gamma(t-1)}{(x-1)^2} + \frac{\delta t(t-1)}{(x-t)^2}\right].$$

In this case we have

$$\phi_x = -\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right) \quad \text{and} \quad \phi_t = \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)$$

so that the last multiplier is given by

$$M = e^{\phi} = \frac{t(t-1)}{x(x-1)(x-t)}. \tag{3.21}$$

The Lagrangian for the P_{VI} equation is found to be

$$L_{VI}(t,x,\dot{x}) = \frac{t(t-1)}{x(x-1)(x-t)}\frac{\dot{x}^2}{2} + \int \frac{t(t-1)}{x(x-1)(x-t)}\left(-B(t,x)\right)dx + \frac{dG}{dt},$$

$$L_{VI}(t,x,\dot{x}) = \frac{t(t-1)}{x(x-1)(x-t)}\frac{\dot{x}^2}{2} + \frac{\alpha x}{t(t-1)} - \frac{\beta}{x(t-1)} - \frac{\gamma}{t(x-1)} - \frac{\delta}{x-t} + \frac{dG}{dt}. \tag{3.22}$$

Let p be the conjugate momentum. With

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{t(t-1)}{x(x-1)(x-t)}\dot{x}$$

the corresponding Hamiltonian is

$$H_{VI} = \frac{t(t-1)}{x(x-1)(x-t)}\frac{p^2}{2} - \frac{\alpha x}{t(t-1)} + \frac{\beta}{x(t-1)} + \frac{\gamma}{t(x-1)} + \frac{\delta}{x-t}. \tag{3.23}$$

Besides the Painlevé equations many other equations of the Painlevé–Gambier classification may also be treated in a similar manner. We illustrate this below.

3.2.5. The Painlevé–Gambier equations XXI

This equation is of the form

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0. \tag{3.24}$$

The Jacobi Last Multiplier is given by $M = x^{-3/2}$ and the corresponding Lagrangian is

$$L_{21} = x^{-3/2}\frac{\dot{x}^2}{2} + 2x^{3/2}. \tag{3.25}$$

The associated Hamiltonian H_{21} provides a first integral (i.e., $\frac{dH_{21}}{dt} = 0$), namely

$$H_{21} = x^{-3/2}\frac{\dot{x}^2}{2} - 2x^{3/2}. \tag{3.26}$$

It is interesting to note that L_{21} and H_{21} both have a 'wrong relative sign'. Consider the transformation

$$x \mapsto y = 4x^{1/4} \quad \text{so that} \quad \dot{y} = x^{-3/4}\dot{x}. \tag{3.27}$$

Under this transformation the Lagrangian L_{21} assumes the more familiar form

$$L_{21}(t,y,\dot{y}) = \left[\frac{1}{2}\dot{y}^2 + \left(2(y/4)^6\right)\right].$$

4. Equations of the Liénard type

In a series of interesting papers Chandrasekhar et al. have made a thorough study of many nonlinear equations of the oscillator type, using an extension of the Prelle–Singer method [3–5]. We investigate below one such generic equation of the Liénard type,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{4.1}$$

from the perspective of the Jacobi Last Multiplier.

4.1. Lagrangian for second-order Liénard type of equations

From (3.4) the last multiplier for Eq. (4.1) is given by $M = \exp(\int f(x)\,dt)$. Following [20] we introduce a new variable v by setting

$$\int f(x)\,dt = \log\left(v^{-\alpha^{-1}}\right) \tag{4.2}$$

which implies

$$\dot{v} + \alpha f(x) v = 0, \tag{4.3}$$

with α being a nonzero scalar to be determined. As a result we have

$$M = v^{-\alpha^{-1}}. \tag{4.4}$$

Indeed, if we can map the original equation, (4.1), to the first-order Eq. (4.3) in terms of the variable v, then a suitable Lagrangian can be easily deduced. It is obvious that v must be linear in $\dot{x}$. In fact it is shown in [20] that such a map exists and is given by

$$v = \dot{x} + \frac{g}{\alpha f} \tag{4.5}$$

provided f and g satisfy the condition

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \alpha(1-\alpha) f. \tag{4.6}$$

From (4.4), since $M = \partial^2 L/\partial \dot{x}^2$, we find that

$$L = \frac{1}{(2-\alpha^{-1})(1-\alpha^{-1})} v^{2-\alpha^{-1}} + f_1 v + f_2. \tag{4.7}$$

Now we substitute this into the Euler–Lagrange equation leads to the condition

$$f_{1t} - f_{2x} = \frac{d}{dx}\left(f_1 \frac{g}{\alpha f}\right),$$

which may be satisfied by setting $f_1 = G_x$ and $f_2 = G_t + f_3$ yielding

$$f_{3x} = -\frac{d}{dx}\left(G_x \frac{g}{\alpha f}\right) \quad \Rightarrow \quad f_3 = -G_x \frac{g}{\alpha f}.$$

The simple choice $G_x = 0$, i.e., $f_1 = 0$ gives, $f_3 = 0$ and $f_2 = dG/dt$. Thus

$$L = \frac{1}{(2-\alpha^{-1})(1-\alpha^{-1})}\left(\dot{x} + \frac{g}{\alpha f}\right)^{2-1/\alpha} + \frac{dG}{dt}, \quad \alpha \neq 0, \frac{1}{2}, 1. \tag{4.8}$$

We can rescale the Lagrangian to get rid of the inconsequential scalar factors and also drop the total time derivative to get it into the neater form

$$L = \left(\dot{x} + \frac{g}{\alpha f}\right)^{2-1/\alpha}. \tag{4.9}$$

This Lagrangian, being invariant under time translation, admits a Noether symmetry with corresponding conserved quantity or first integral (disregarding overall scalar factors)

$$I = \left(\dot{x} + \frac{g}{\alpha f}\right)^{1-1/\alpha} \frac{(\alpha-1) f \dot{x} - g}{f}. \tag{4.10}$$

4.2. Example: A generic equation of nonlinear oscillator type

Consider the following equation

$$\ddot{x} + \left(k_1 x^q + k_2\right)\dot{x} + \left(k_3 x^{2q+1} + k_4 x^{q+1} + k_5 x\right) = 0. \tag{4.11}$$

This is a generic equation of nonlinear oscillator type, which includes many subcases depending upon the choice of the k_i, which are parameters. The case $q = 0$ corresponds to a damped harmonic oscillator, while $q = 1$ corresponds to the force-free Helmholtz oscillator. Substituting f and g from (4.11) into the condition (4.6), we obtain the following equations from the different coefficients of x.

$$\alpha(1-\alpha) = (q+1)\frac{k_3}{k_1^2}, \tag{4.12}$$

$$\alpha(1-\alpha) = \frac{k_5}{k_2^2}, \tag{4.13}$$

$$k_1 k_4 + k_2 k_3(2q+1) = \alpha(1-\alpha) k_1^2 k_2, \tag{4.14}$$

$$k_1 k_5(1-q) + k_2 k_4(1+q) = 3\alpha(1-\alpha) k_1 k_2^2. \tag{4.15}$$

Equating (4.12) and (4.13) we find that

$$q + 1 = \frac{k_1^2 k_5}{k_2^2 k_3}. \tag{4.16}$$

Using this value of q in the remaining Eqs. (4.14) and (4.15) while eliminating α by means of (4.13), we get

$$k_5 = \frac{k_2}{k_1^2}(k_1 k_4 - k_2 k_3). \tag{4.17}$$

The constant α is determined from the quadratic equation (4.13) and is

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_5}{k_2^2}}\right), \tag{4.18}$$

where k_5 is given by (4.17). Given q there exists another relation between the k_i $(i = 1, \dots, 5)$ derivable from (4.16) and (4.17), *viz.*

$$\frac{k_1 k_4}{k_2 k_3} = q + 2. \tag{4.19}$$

Thus of the five parameters k_i $(i = 1, \dots, 5)$ only three are independent and to summarize we have the following relations:

$$k_4 = \frac{k_2 k_3}{k_1}(q+2),$$

$$k_5 = \frac{k_2^2 k_3}{k_1}(q+1),$$

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_3}{k_1^2}(q+1)}\right).$$

4.2.1. Special cases

When $q = 0$, we have $k_1 k_4 = 2k_2 k_3$ and $k_5 = k_2^2 k_3 / k_1^2$. Consequently $\alpha = \frac{1}{2}(1 \pm \sqrt{1 - \frac{4k_3}{k_1^2}})$ and the equation $\ddot{x} + (k_1 + k_2)\dot{x} + (k_3 + k_4 + k_5)x = 0$, which is simply the damped harmonic oscillator, has Lagrangian

$$L = \left(\dot{x} + \frac{(k_3 + k_4 + k_5)x}{\alpha(k_1 + k_2)}\right)^{2 - 1/\alpha}.$$

When $q = 1$, we have $k_1 k_4 = 3k_2 k_3$ and $k_5 = 2k_2^2 k_3 / k_1^2$ while $\alpha = \frac{1}{2}(1 \pm \sqrt{1 - 8k_3/k_1^2})$. The Lagrangian for the equation, $\ddot{x} + (k_1 x + k_2)\dot{x} + k_3(x^3 + 3k_2/k_1 x^2 + 2k_2^2/k_1^2 x) = 0$ is

$$L = \left\{\dot{x} + \frac{k_3}{\alpha k_1}\left(x^2 + 2k_2/k_1 x\right)\right\}^{2 - 1/\alpha}.$$

From this Lagrangian one can easily compute the conjugate momentum to obtain the corresponding Hamiltonian.

5. A system of second-order coupled equations

The extension of the above technique to a system of second-order ODEs is also possible under certain conditions. We describe below the formulation as presented in [22]. In the case of a system of n degrees of freedom the Lagrangian $L = L(t, q, \dot{q})$, where $q = \{q_1, \dots, q_n\}$ and $\dot{q} = \{\dot{q_1}, \dots, \dot{q_n}\}$ define the generalized coordinates and corresponding velocities, we may define the ijth Jacobi Last Multiplier by

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q_i} \partial \dot{q_j}}, \quad i, j = 1, \dots, n. \tag{5.1}$$

It is assumed that the equations of motion:

$$\ddot{q_k} = w_k(t, q, \dot{q}), \quad k = 1, \dots, n, \tag{5.2}$$

are derivable from the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \dots, n. \tag{5.3}$$

It is evident that the conjugate momenta are

$$p_j = \frac{\partial L}{\partial \dot{q_j}} = p_j(t, q, \dot{q}), \quad j = 1, \dots, n,$$

which implies

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial t} + \sum_{k=1}^{n}\left(\dot{q_k}\frac{\partial p_j}{\partial q_k} + w_k\frac{\partial p_j}{\partial \dot{q_k}}\right) = \frac{\partial L}{\partial q_j}.$$

This means

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q_j}}\right) + \sum_{k=1}^{n}\left(\dot{q_k}\frac{\partial^2 L}{\partial q_k \partial \dot{q_j}} + w_k\frac{\partial p_j}{\partial \dot{q_k}\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j}, \quad j = 1, \dots, n. \tag{5.4}$$

Differentiating (5.4) with respect to $\dot{q}_i$ and using the definition of the last multiplier given in (5.1) we find

$$\frac{\partial M_{ij}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q_k}M_{ij}) + \frac{\partial}{\partial \dot{q_k}}(w_k M_{ij})\right) + \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q_i}}M_{kj} - \frac{\partial w_k}{\partial \dot{q_k}}M_{ij}\right) + \frac{\partial^2 L}{\partial q_i \partial \dot{q_j}} - \frac{\partial^2 L}{\partial \dot{q_i} \partial q_j} = 0. \tag{5.5}$$

Interchanging i and j in (5.5) we get

$$\frac{\partial M_{ji}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q_k}M_{ji}) + \frac{\partial}{\partial \dot{q_k}}(w_k M_{ji})\right) + \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q_j}}M_{ki} - \frac{\partial w_k}{\partial \dot{q_k}}M_{ji}\right) + \frac{\partial^2 L}{\partial q_j \partial \dot{q_i}} - \frac{\partial^2 L}{\partial \dot{q_j} \partial q_i} = 0. \tag{5.6}$$

Adding (5.5) and (5.6) and making use of the fact that $M_{ij} = M_{ji}$ we have

$$\frac{\partial M_{ij}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q_k}M_{ij}) + \frac{\partial}{\partial \dot{q_k}}(w_k M_{ij})\right) + \sum_{k=1}^{n}\left(\frac{1}{2}\left(\frac{\partial w_k}{\partial \dot{q_i}}M_{kj} + \frac{\partial w_k}{\partial \dot{q_j}}M_{ki}\right) - \frac{\partial w_k}{\partial \dot{q_k}}M_{ij}\right) = 0. \tag{5.7}$$

It is evident that M_{ij} satisfies the defining relation (2.6) for the JLM whenever

$$\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q_i}}M_{kj} + \frac{\partial w_k}{\partial \dot{q_j}}M_{ki}\right) = 2\sum_{k=1}^{n}\frac{\partial w_k}{\partial \dot{q_k}}M_{ij} \quad \text{for each } k = 1, \dots, n. \tag{5.8}$$

A trivial way to ensure this condition is satisfied is to assume the w_k's to be velocity independent,

$$\frac{\partial w_k}{\partial \dot{q_l}} = 0 \quad \text{for all } k, l = 1, \dots, n.$$

On the other hand, when $i = j$, the last two terms in (5.5) cancel leaving

$$\frac{\partial M_{ii}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q_k}M_{ii}) + \frac{\partial}{\partial \dot{q_k}}(w_k M_{ii})\right) + \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q_i}}M_{ki} - \frac{\partial w_k}{\partial \dot{q_k}}M_{ii}\right) = 0. \tag{5.9}$$

Here also M_{ii} satisfies (2.6) when the last sum of (5.9) vanishes, which may be ensured by choosing the w_k's to be velocity independent. Under these circumstances all the M_{ij}'s satisfy the equation

$$\frac{\partial M_{ii}}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q_k}M_{ii})+\frac{\partial}{\partial \dot{q_k}}(w_k M_{ii})\right)=0, \tag{5.10}$$

as they should, provided $\partial w_k/\partial \dot{q_j}=0$ for all $k, j=1,\ldots,n$. With this assumption equations (5.7) and 5.10) always admit the solution $M_{ij}=$ constant. The following examples illustrate how simple choices of M_{ij} can be made to obtain the Lagrangians of second-order ODEs satisfying the above velocity-independent criterion.

Example 1. Consider the system

$$\ddot{x}+\frac{\alpha}{x^2}g(y/x)-\frac{\lambda}{x^3}=0,$$
$$\ddot{y}+\frac{\beta}{x^2}f(y/x)-\frac{\mu}{y^3}=0.$$

Here $w_1(x,y)=-\alpha g(y/x)/x^2+\lambda/x^3$ and $w_2(x,y)=-\beta f(y/x)/x^2+\mu/y^3$ respectively. On the other hand α,β,λ and μ are arbitrary parameters while g and f are functions with argument $u=y/x$. Notice that w_1 and w_2 are independent of the velocities. The Jacobi Last Multiplier for this system is therefore a solution of the equation,

$$\frac{\partial M}{\partial t}+\frac{\partial (M\dot{x})}{\partial x}+\frac{\partial (M\dot{y})}{\partial y}+\frac{\partial (Mw_1)}{\partial \dot{x}}+\frac{\partial (Mw_2)}{\partial \dot{y}}=0,$$

and admits constant solutions. We choose them as follows:

$$M_{xy}=M_{yx}=0 \quad \text{and} \quad M_{xx}=M_{yy}=1.$$

These yield the Lagrangian

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+h_1(t,x,y)\dot{x}+h_2(t,x,y)\dot{y}+h_3(t,x,y).$$

Substitution of this into the Euler–Lagrange equations for x and y gives, upon using the original equations of motion,

$$h_{1t}-h_{3x}+w_1+(h_{1y}-h_{2x})\dot{y}=0, \tag{5.11}$$
$$h_{2t}-h_{3y}+w_2+(h_{2x}-h_{1y})\dot{x}=0. \tag{5.12}$$

Equating the coefficients of $\dot{x}$ and $\dot{y}$ respectively we get the following set of equations:

$$h_{1y}-h_{2x}=0 \quad \text{which implies} \quad h_1=G_x, \quad h_2=G_y \quad \text{and} \tag{5.13}$$
$$h_{1t}-h_{3x}+w_1=0, \tag{5.14}$$
$$h_{2t}-h_{3y}+w_2=0. \tag{5.15}$$

These in turn give

$$h_{3x}=G_{xt}+w_1 \text{ or } h_3=G_t+\int w_1\,dx+r(y), \tag{5.16}$$
$$h_{3y}=G_{yt}+w_2 \text{ or } h_3=G_t+\int w_2\,dy+s(x). \tag{5.17}$$

Consistency for h_3 requires that

$$h_{3xy}=h_{3yx}$$

and translates into the requirement that $w_{1y}=w_{2x}$. This imposes the following condition on the functions f and g which define the second-order system:

$$\frac{\alpha}{\beta}g'(u)+uf'(u)+2f(u)=0, \quad \text{where } u=\frac{y}{x}.$$

One can rewrite this as

$$\frac{\alpha}{\beta}ug'(u)+\frac{d}{du}\left(u^2 f(u)\right)=0. \tag{5.18}$$

When we use the explicit forms of w_1 and w_2 and make use of the last condition, the form of the functions $r(y)$ and $s(x)$ occurring in (5.16) and (5.17) may be fixed and the functional form of h_3 is found to be

$$h_3(t,x,y) = G_t - \left[\frac{\alpha}{2x^2} + \frac{\mu}{2y^2} - \frac{1}{x}\left(\alpha g(y/x) + \beta \frac{y}{x} f(y/x)\right)\right].$$

Therefore the Lagrangian is given by

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left[\frac{\alpha}{2x^2} + \frac{\mu}{2y^2} - \frac{1}{x}\left(\alpha g(y/x) + \beta \frac{y}{x} f(y/x)\right)\right] + \frac{dG}{dt}. \tag{5.19}$$

Again the total derivative term, being of little physical significance in the classical case, may be safely discarded. It is interesting to note that the above second-order system, though similar in some respects to equations of the Ermakov system, is not merely a mathematical artifact. It is similar in structure to the system studied in [25] in the context of the dynamics of stellar systems.

A similar exercise may be carried out for the following:

Example: Generalized Van der Waals Potential

$$\ddot{x} = -\left(2\gamma x + \frac{x}{r^3}\right) = w_1(x,y),$$
$$\ddot{y} = -\left(2\gamma \beta^2 y + \frac{y}{r^3}\right) = w_2(x,y) \quad \text{where } r = \sqrt{x^2 + y^2},$$

and γ, β are parameters. In this case the Lagrangian is given by

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left[\gamma\left(x^2 + \beta^2 y^2\right) - \frac{1}{r}\right] + \frac{dG}{dt}. \tag{5.20}$$

Similarly for the

Example: Henon–Heiles system,

$$\ddot{x} = -(Ax + 2\alpha xy),$$
$$\ddot{y} = -\left(By + \alpha x^2 - \beta y^2\right), \tag{5.21}$$

the Lagrangian is given by

$$L(t,x,\dot{x}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left(A\frac{x^2}{2} + B\frac{y^2}{2} + \alpha x^2 y - \beta \frac{y^3}{3}\right) + \frac{dG}{dt}. \tag{5.22}$$

6. Outlook

In this paper we have discussed applications of the Jacobi Last Multiplier for the deduction of Lagrangian functions for the second-order ODEs of the Painlevé–Gambier classification. We have specifically deduced the Lagrangians for the majority of the six Painlevé equations as also other prototype equations of the Painlevé–Gambier classification. We have also dwelt on the geometrical background involving the last multiplier. This is an on-going endeavour and we propose to perform further investigations in our future works. In addition we have used the above technique to analyse a particular class of coupled second-order equations. Besides the well-known Henon–Heiles system we have obtained the Lagrangian for a relatively less studied systems occurring in the context of stellar dynamics. The Lagrangians discussed here are found to admit a Noetherian symmetry, with an associated first integral, which are the Hamiltonians of the equations concerned.

Acknowledgments

We wish to thank Peter Leach, Basil Grammaticos, Clara Nucci and K.M. Tamizhmani for enlightening discussions. In addition AGC wishes to acknowledge the support provided by the S.N. Bose National Centre for Basic Sciences, Kolkata, in the form of an Associateship.

References

[1] S. Lie, Theorie des Integrabilitetsfaktors, Christiania Forh. (1874) 242–254, reprint, Gessammelte Abhandlungen, vol. III, no. XIII, pp. 176–187;
Also, S. Lie, Verallgemeinerung und neue Verwertung der Jacobischen Multiplikatortheorie, Christiania Forh. (1874) 255–274, reprint, Gesammelte Abhandlungen, vol. III, no. XIV, pp. 188–206.

[2] S. Lie, Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten, Math. Ann. 32 (1888) 213–281.

[3] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, Extended Prelle–Singer method and integrability/solvability of a class of nonlinear nth order ordinary differential equations, J. Nonlinear Math. Phys. 12 (1) (2005) 184–201.
[4] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, A unification in the theory of linearization of second-order nonlinear ordinary differential equations, J. Phys. A 39 (3) (2006) L69–L76.
[5] V.K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, A simple and unified approach to identify integrable nonlinear oscillators and systems, J. Math. Phys. 47 (2) (2006) 023508, 37 pp.
[6] M.J. Prelle, M.F. Singer, Elementary first integrals of differential equations, Trans. Amer. Math. Soc. 279 (1) (1983) 215–229.
[7] L.G.S. Duarte, S.E.S. Duarte, L.A.C. da Mota, J.E.F. Skea, Solving second-order ordinary differential equations by extending the Prelle–Singer method, J. Phys. A 34 (14) (2001) 3015–3024.
[8] L.G.S. Duarte, S.E.S. Duarte, L.A.C. da Mota, Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution, J. Phys. A 35 (4) (2002) 1001–1006, J. Phys. A 35 (14) (2002) 3015–3024.
[9] B.S. Madhava Rao, On the reduction of dynamical equations to the Lagrangian form, Proc. Benaras Math. Soc. (N.S.) 2 (1940) 53–59.
[10] E.T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge Univ. Press, Cambridge, 1988.
[11] Z. Oziewicz, José R. Zeni, Ordinary differential equation: Symmetries and last multiplier, in: Clifford Algebras and Their Applications in Mathematical Physics, vol. 1, Ixtapa–Zihuatanejo, 1999, in: Progr. Phys., vol. 18, Birkhäuser Boston, Boston, MA, 2000, pp. 425–433.
[12] M. Crasmareanu, Last multipliers as autonomous solutions of the Liouville equation of transport, Houston J. Math. 34 (2) (2008) 455–466.
[13] M. Taylor, Partial Differential Equations: Basic Theory, Text Appl. Math., vol. 23, Springer, New York, 1996.
[14] J.E. Marsden, Well-posedness of the equations of a nonhomogeneous perfect fluid, Comm. Partial Differential Equations 1 (3) (1976) 215–230.
[15] P. Morando, Liouville condition, Nambu mechanics, and differential forms, J. Phys. A 29 (13) (1996) L329–L331.
[16] M.C. Nucci, P.G.L. Leach, Jacobi's last multiplier and the complete symmetry group of the Euler–Poinsot system, J. Nonlinear Math. Phys. 9 (s2) (2002) 110–121.
[17] K. Okamoto, On the τ-function of the Painlevé equations, Phys. D 2 (1981) 525–535.
[18] T. Wolf, A. Brand, Examples of the investigation of differential equations with modularized programs. Algorithms and software for symbolic analysis of nonlinear systems, Math. Comput. Modelling 25 (1997) 133–139.
[19] M.C. Nucci, P.G.L. Leach, Jacobi's last multiplier and symmetries for the Kepler problem plus a lineal story, J. Phys. A 37 (2004) 7743–7753.
[20] M.C. Nucci, K.M. Tamizhmani, Lagrangian for dissipative nonlinear oscillators: The method of Jacobi last multiplier, arXiv:0809.0022.
[21] M.C. Nucci, K.M. Tamizhmani, Using an old method of Jacobi to derive Lagrangians: A nonlinear dynamical system with variable coefficients, arXiv:0807.2791.
[22] M.C. Nucci, P.G.L. Leach, Jacobi's last multiplier and Lagrangians for multidimensional systems, arXiv:0709.3231v1.
[23] C.G.J. Jacobi, Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, Giornale Arcadico di Scienze, Lettere ed Arti 99 (1844) 129–146.
[24] C.G.J. Jacobi, Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi, J. Reine Angew. Math. (Crelle J.) 27 (1844) 199–268, J. Reine Angew. Math. (Crelle J.) 29 (1845) 213–279, 333–376.
[25] S. Sridhar, R. Nityananda, Undamped oscillations of collisionless systems: Spheres, spheroids and discs, J. Astrophys. Astr. 10 (1989) 279–293.

Publications not included in the Thesis

IOP PUBLISHING INVERSE PROBLEMS

Inverse Problems **25** (2009) 085002 (11pp) doi:10.1088/0266-5611/25/8/085002

Canonical Bäcklund transformation for the DST model under open boundary conditions

Barun Khanra[1] **and A Ghose Choudhury**[2]

[1] Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Calcutta-700 004, India
[2] Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta-700032, India

E-mail: barunkhanra@rediffmail.com and a-ghosechoudhury@rediffmail.com

Received 13 January 2009, in final form 22 May 2009
Published 24 June 2009
Online at stacks.iop.org/IP/25/085002

Abstract
We study a Bäcklund transformation for the dimer self-trapping (DST) model under open boundary conditions. As in the periodic case, the transformation is found to be canonical with a corresponding generating function. The spectrality property of the transformation is investigated. Finally, as an application of Bäcklund transformations we study its connection with discrete-time dynamics.

1. Introduction

Classical discrete integrable systems have attracted the interest of many researchers over the recent years. The dimer self-trapping (DST) model, the Toda lattice and the XXX spin chain are some of the most well-known discrete lattice systems which have been extensively studied from various perspectives [1, 2]. A novel approach for obtaining parameter-dependent Bäcklund transformations for discrete lattice models was first discussed by Sklyanin in [3, 4]. Subsequently, many authors have made various contributions in this regard [5–7]. Sklyanin's initial work was focussed on systems obeying periodic boundary conditions. However, in [8] an attempt was made to investigate the problem of deriving Bäcklund transformations under more general quasi-periodic and open boundary conditions. Recently, there has been a renewal of interest in this aspect [9]. In this brief paper, we examine the DST model under open boundary conditions and derive a parameter-dependent Bäcklund transformation for the model. The canonical nature of the transformation is explicitly deduced, and the property of spectrality is investigated. Finally, the connection of such Bäcklund transformations to discrete-time dynamics of the system is briefly discussed.

2. Formulation

The DST model may be described by a system of difference equations:

$$\dot{q}_n = q_{n+1} - q_n^2 r_n, \qquad \dot{r}_n = -r_{n-1} + q_n r_n^2, \qquad n = 1, \ldots, N. \qquad (2.1)$$

These may be obtained from the consistency of the following linear system:

$$\Psi_{n+1} = \ell_n \Psi_n, \qquad \dot{\Psi}_n = M_n \Psi_n, \tag{2.2}$$

where

$$\ell_n(u) = \begin{pmatrix} u + q_n r_n & q_n \\ r_n & 1 \end{pmatrix} \quad \text{and} \quad M_n(u) = \begin{pmatrix} \frac{u}{2} & q_n \\ r_{n-1} & -\frac{u}{2} \end{pmatrix}, \tag{2.3}$$

and u is, in general, a complex spectral parameter. Equations (2.1) define a Hamiltonian system with a symplectic structure given by

$$\begin{pmatrix} \dot{q}_n \\ \dot{r}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta q_n} \\ \frac{\delta H}{\delta r_n} \end{pmatrix} \quad \text{and} \quad H = \frac{1}{2} \sum_n (q_{n+1} r_n + q_n r_{n-1} - q_n r_n). \tag{2.4}$$

Usually, we consider a chain of N lattice points and assume that the lattice variables obey periodic boundary conditions:

$$q_{n+N} = q_n, \qquad r_{n+N} = r_n.$$

Consistency of the linear system (2.2) implies

$$\dot{\ell}_n(u) = M_{n+1}(u)\ell_n(u) - \ell_n(u)M_n(u), \qquad n = 1, \ldots, N, \tag{2.5}$$

which yields the equation of motion (2.1). The r-matrix formalism has proved to be a very powerful and useful tool in the analysis of integrable Hamiltonian systems. Given the Hamiltonian structure (2.4), it is easy to verify that

$$\{\ell_n(u) \overset{\otimes}{,} \ell_n(v)\} = [r(u - v), \ell_n(u) \otimes \ell_n(v)], \tag{2.6}$$

where the classical r-matrix is given by

$$r(u - v) = -\frac{1}{u - v} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

The monodromy matrix is then defined as the ordered product,

$$T(u) = \ell_n(u) \cdots \ell_1(u), \tag{2.8}$$

and it satisfies the Sklyanin quadratic algebra:

$$\{T(u) \overset{\otimes}{,} T(v)\} = [r(u - v), T(u) \otimes T(v)]. \tag{2.9}$$

The generator of the integrals of motion follows from the trace of (2.9) and, in the periodic case, is given by

$$t(u) = \operatorname{tr} T(u) = u^N + u^{N-1} H_1 + u^{N-2} H_2 + \cdots, \tag{2.10}$$

where

$$H_1 = \sum_{i=1}^{N} q_i r_i, \qquad H_2 = \sum_{i=1}^{N} q_{i+1} r_i + \sum_{i<j} (q_i r_i)(q_j r_j).$$

The Hamiltonian of the system is a nonlinear combination of the above constants of motion, namely

$$H_{\text{periodic}} = \sum_{i=1}^{N} q_{i+1} r_i - \frac{1}{2} \sum_{i=1}^{N} q_i^2 r_i^2. \tag{2.11}$$

2.1. Open boundary conditions

To study the system (2.1) under open boundary conditions we replace (2.5) by the following system of equations:

$$\dot{\ell}_j(u) = M_{j+1}(u)\ell_j(u) - \ell_j(u)M_j(u),$$

$$\text{with} \qquad M_n(u) = \begin{pmatrix} \frac{u}{2} & q_n \\ r_{n-1} & -\frac{u}{2} \end{pmatrix}, \qquad j = 2, \ldots, (N-1)$$

and

$$\dot{\ell}_N(u) = M_{N+1}(u)\ell_N(u) - \ell_N(u)M_N(u), \qquad \text{and} \qquad \dot{\ell}_1(u) = M_2(u)\ell_1(u) - \ell_1(u)M_1(u), \tag{2.12}$$

with

$$M_{N+1}(u) = \begin{pmatrix} \frac{u}{2} & q_{N+1} \\ r_N & -\frac{u}{2} \end{pmatrix} \qquad \text{and} \qquad M_1(u) = \begin{pmatrix} \frac{u}{2} & q_1 \\ r_0 & -\frac{u}{2} \end{pmatrix}.$$

The break with periodic boundary conditions is achieved by demanding

$$q_{N+1} = \theta_+ \qquad \text{and} \qquad r_0 = \theta_-, \tag{2.13}$$

($\theta_\pm$ being constants) so that the equations of motion for the end points of the chain become

$$\dot{q}_1 = q_2 - q_1^2 r_1, \qquad \dot{r}_1 = -\big(\theta_- - q_1 r_1^2\big), \tag{2.14}$$

$$\dot{q}_N = \big(\theta_+ - q_N^2 r_N\big), \qquad \dot{r}_N = -r_{N-1} + q_N r_N^2. \tag{2.15}$$

The Hamiltonian of the system under open boundary conditions is given by

$$H = \sum_{i=1}^{N-1} q_{i+1} r_i - \frac{1}{2}\sum_{i=1}^{N} q_i^2 r_i^2 + q_1\theta_- + r_N\theta_+. \tag{2.16}$$

It may be shown that in such a case [10, 11],

$$\tau(u) = \text{tr}[K_+(u)U(u)] = \text{tr}\,\mathcal{T}(u), \qquad \text{where} \qquad U(u) = T(u)K_-(u)T^{-1}(-u) \tag{2.17}$$

serves as a generator of the conserved quantities, with $U(u)$ satisfying the following reflection equation (RE) algebra:

$$\begin{aligned}\{U(u) \overset{\otimes}{,} U(v)\} &= [r(u-v), U(u) \otimes U(v)] \\ &\quad + U^{(1)}(u) r(u+v) U^{(2)}(v) - U^{(2)}(v) r(u+v) U^{(1)}(u).\end{aligned} \tag{2.18}$$

The matrices $K_\pm(u)$ involve the non-dynamical parameters determining the boundary conditions imposed on the system and are, in addition, required to satisfy the following relations:

$$K_+(u)M_{N+1}(u) = M_{N+1}(-u)K_+(u), \qquad M_1(u)K_-(u) = K_-(u)M_1(-u). \tag{2.19}$$

In the present case, the following forms of $K_\pm(u)$ are found to be admissible [8]:

$$K_-(u) = \begin{pmatrix} \theta_- & u \\ 0 & \theta_- \end{pmatrix} \qquad \text{and} \qquad K_+(u) = \begin{pmatrix} \theta_+ & 0 \\ u & \theta_+ \end{pmatrix}. \tag{2.20}$$

Having briefly recounted the basic relations governing discrete systems under open boundary conditions, we shall now describe a systematic procedure for obtaining Bäcklund transformations for the DST model, under such conditions.

3. Bäcklund transformation

It is worthwhile to briefly review the methodology proposed by Sklyanin in [3], for the periodic case. This is based on establishing a similarity transformation for the monodromy matrix, determined by local transformations of the form

$$b_\lambda^{(i)} : \ell_i(u; q_i, r_i) \longrightarrow \ell_i(u; \tilde{q}_i, \tilde{r}_i), \qquad i = 1, \ldots, N, \tag{3.1}$$

and being defined by

$$W_{i+1}(u, \lambda)\ell_i(u; \tilde{q}_i, \tilde{r}_i) = \ell_i(u; q_i, r_i)W_i(u, \lambda). \tag{3.2}$$

Here $W_i(u, \lambda)$ is a suitable non-singular matrix obeying the quadratic algebra (2.9), and λ is a parameter. The dynamical variables in $W_i(u, \lambda)$ serve an auxiliary role and are assumed to satisfy periodic boundary conditions. As a consequence of (3.2), one can easily derive the following relation:

$$W_{N+1}(u, \lambda)T(u; \tilde{q}, \tilde{r}) = T(u; q, r)W_1(u, \lambda). \tag{3.3}$$

Therefore, in the periodic case when $W_{N+1}(u, \lambda) = W_1(u, \lambda)$, one has

$$\operatorname{tr} T(u; \tilde{q}, \tilde{r}) = \operatorname{tr}\left\{W_{N+1}^{-1}(u, \lambda)T(u; q, r)W_1(u, \lambda)\right\} = \operatorname{tr} T(u; q, r) \tag{3.4}$$

implying that the Hamiltonian (2.11) is invariant under the transformation (3.2).

In the case of open boundary conditions we shall look for a local one-parameter transformation of the form (3.2), which manages to keep $\tau(u)$ as defined in (2.17) invariant. This will then guarantee the invariance of the commuting integrals of motion, even after affecting the local Bäcklund transformation.

Let us write $T(u; \tilde{q}, \tilde{r}) = \widetilde{T}(u)$, after suppressing the arguments, so that (3.3) may be concisely expressed as

$$W_{N+1}(u, \lambda)\widetilde{T}(u) = T(u)W_1(u, \lambda). \tag{3.5}$$

Note that

$$\det \ell_n(u) = \det\begin{pmatrix} u + q_n r_n & q_n \\ r_n & 1 \end{pmatrix} = u \text{ which, in turn, implies} \qquad \det T(u) = u^n.$$

This requires in view of (3.2) that

$$\det W_{i+1}(u, \lambda) = \det W_i(u, \lambda)$$

or in other words $\det W_i(u, \lambda)$ be independent of the lattice variables. Furthermore, as $W_i(u, \lambda)$ is assumed to be non-singular, we have from (3.5)

$$\begin{aligned} &\widetilde{T}^{-1}(-u)W_{N+1}^{-1}(-u, \lambda) = W_1^{-1}(-u, \lambda)T^{-1}(-u) \\ \text{or} \qquad &W_1(-u, \lambda)\widetilde{T}^{-1}(-u) = T^{-1}(-u)W_{N+1}(-u, \lambda). \end{aligned} \tag{3.6}$$

Consider now the product

$$K_+(u)U(u) = K_+(u)T(u)K_-(u)T^{-1}(-u).$$

Using (3.5) and (3.6) this may be written as

$$K_+(u)U(u) = K_+(u)\left\{W_{N+1}(u, \lambda)\widetilde{T}(u)W_1^{-1}(u, \lambda)\right\}K_-(u)\left\{W_1(-u, \lambda)\widetilde{T}^{-1}(-u)W_{N+1}^{-1}(-u, \lambda)\right\}.$$

Hence

$$\begin{aligned} \tau(u) &= \operatorname{tr}[K_+(u)U(u)] \\ &= \operatorname{tr}\left[\left\{W_{N+1}^{-1}(-u, \lambda)K_+(u)W_{N+1}(u, \lambda)\right\}\widetilde{T}(u)\left\{W_1^{-1}(u, \lambda)\right\}K_-(u)\{W_1(-u, \lambda)\}\widetilde{T}^{-1}(-u)\right]. \end{aligned} \tag{3.7}$$

Invariance of the trace follows immediately by demanding

$$W_{N+1}(-u,\lambda)K_+(u) = K_+(u)W_{N+1}(u,\lambda), \tag{3.8}$$

$$W_1(u,\lambda)K_-(u) = K_-(u)W_1(-u,\lambda). \tag{3.9}$$

It is interesting to note that these equations are similar to (2.19).

To summarize therefore, in the case of open boundary conditions, the single-parameter Bäcklund transformation for the chain is determined by (3.2) no doubt, but in addition the boundary matrices $K_\pm(u)$ should satisfy the additional conditions (3.8) and (3.9). The essential problem is to find out a suitable gauge matrix, $W_i(u,\lambda)$, which fulfils all these requirements. We use the following ansatz for the gauge matrix [9]:

$$W_i(u,\lambda) = \begin{pmatrix} u-\lambda+s_iS_i & s_i^2S_i - 2\lambda s_i \\ S_i & -u-\lambda+s_iS_i \end{pmatrix}, \qquad \det W_i(u,\lambda) = \lambda^2 - u^2, \tag{3.10}$$

where it is assumed that $u \neq \pm\lambda$. Substituting this ansatz for the gauge matrix in (3.2) we get

$$q_i = s_{i+1} - \frac{\lambda s_i}{1+r_is_i}, \qquad \tilde{q}_i = -s_{i+1} - \frac{\lambda s_i}{1-\tilde{r}_is_i}, \tag{3.11}$$

$$S_i = \lambda\left\{\frac{r_i}{1+r_is_i} - \frac{\tilde{r}_i}{1-\tilde{r}_is_i}\right\}, \qquad S_{i+1} = r_i + \tilde{r}_i \quad \text{where} \quad i = 1,\ldots N. \tag{3.12}$$

On the other hand from (3.8) and (3.9) we find

$$S_1 = 2\theta_- \qquad \text{and} \qquad S_{N+1} = \frac{2(\theta_+ + \lambda s_{N+1})}{s_{N+1}^2}. \tag{3.13}$$

Equation (3.11) defines the requisite Bäcklund transformation, provided we can express s_i as functions of the $r_i, \tilde{r}_i$'s. This is achieved by noting that (3.12) allows us to eliminate S_i:

$$S_i = \lambda\left\{\frac{r_i}{1+r_is_i} - \frac{\tilde{r}_i}{1-\tilde{r}_is_i}\right\} = r_{i-1} + \tilde{r}_{i-1},$$

and, in turn, yields the following quadratic equation determining s_i:

$$s_i^2 - \left(\frac{1}{\tilde{r}_i} + \frac{2\lambda}{r_{i-1}+\tilde{r}_{i-1}} - \frac{1}{r_i}\right)s_i + \left(\frac{\lambda\left(\tilde{r}_i^{-1} - r_i^{-1}\right)}{r_{i-1}+\tilde{r}_{i-1}} - \frac{1}{r_i\tilde{r}_i}\right) = 0, \qquad i = 2,\ldots,N-1. \tag{3.14}$$

Similarly equating S_1 and S_{N+1} from (3.6) with their corresponding values from (3.13) we get

$$s_1^2 - \left(\frac{\lambda}{\theta_-} - \frac{1}{r_1} + \frac{1}{\tilde{r}_1}\right)s_1 - \left(\frac{1}{r_1\tilde{r}_1} + \frac{\lambda(r_1^{-1} - \tilde{r}_1^{-1})}{2\theta_-}\right) = 0, \tag{3.15}$$

$$s_{N+1}^2 - \frac{2\lambda}{r_N+\tilde{r}_N}s_{N+1} - \frac{2\theta_+}{r_N+\tilde{r}_N} = 0. \tag{3.16}$$

These equations implicitly define the Bäcklund transformations (3.11). In fact, one may consider the entire set of equations (3.11)–(3.16) as defining the Bäcklund transformation relations.

3.1. Canonicity

It is evident that the manner in which we have defined the Bäcklund transformation (by demanding the invariance of $\mathrm{tr}[K_+(u)U(u)]$) automatically ensures that the transformed Hamiltonian $H(\tilde{q}, \tilde{r}) = H(q, r)$ for all values of the parameter λ.

Second, the canonical nature of the transformation is best demonstrated by finding explicitly a suitable generating function, $F_\lambda(\tilde{r}; r)$, such that

$$q_j = \frac{\partial F_\lambda}{\partial r_j}, \qquad \tilde{q}_j = -\frac{\partial F_\lambda}{\partial \tilde{r}_j}, \qquad S_j = -\frac{\partial F_\lambda}{\partial s_j}. \tag{3.17}$$

In the present case, such a generating function is given by

$$F_\lambda(\tilde{r}; r) = \sum_{i=1}^{N} [s_{i+1}(r_i + \tilde{r}_i) - \lambda \ln(1 + r_i s_i)(1 - \tilde{r}_i s_i)] \\ - 2\theta_- s_1 + 2\theta_+ s_{N+1}^{-1} - 2\lambda \ln s_{N+1} + \text{const}. \tag{3.18}$$

Note that the constant, say C, is independent of dynamical variables, but may depend on the parameter λ.

3.2. Spectrality

The notion of spectrality is a new feature in the theory of Bäcklund transformations. In a sense it is a generalization of the concept of canonicity, where, in addition to the canonical variables (q_i, r_i) and $(\tilde{q}_i, \tilde{r}_i)$, we define another pair of canonical variables (μ, λ), with μ being conjugate to the parameter λ of the Bäcklund transformation. We define μ by

$$\mu = \frac{\partial F_\lambda(\tilde{r}; r)}{\partial \lambda}. \tag{3.19}$$

Consider the eigenvalue problem for the matrix $\mathcal{T}(u = \lambda) = K_+(\lambda)U(\lambda)$:

$$\mathcal{T}(\lambda)\Omega = \Lambda\Omega. \tag{3.20}$$

It turns out that the eigenvalue Λ is proportional to the exponential of μ [10]. This means that the pair $(\mathrm{e}^\mu, \lambda)$ lies on the characteristic curve of the matrix $\mathcal{T}(\lambda)$ i.e.,

$$\det(\mathrm{e}^\mu - \mathcal{T}(\lambda)) = 0. \tag{3.21}$$

In general, if B_λ denotes a family of Bäcklund transformations, then it is said to be associated with a Lax operator $\mathcal{T}(\lambda)$, if for some function $f(\mu)$, the pair $(\lambda, f(\mu))$ lies on the spectral curve of the Lax operator $\mathcal{T}(\lambda)$:

$$\det(f(\mu) - \mathcal{T}(\lambda)) = 0. \tag{3.22}$$

In the present case, we shall show that $f(\mu) = \mathrm{e}^\mu$ explicitly. It will be observed from (3.5), (3.6), (3.8) and (3.9) that

$$W_{N+1}(-u, \lambda)\widetilde{\mathcal{T}}(u) = W_{N+1}(-u, \lambda)\{K_+(u)\widetilde{T}(u)K_-(u)\widetilde{T}^{-1}(-u)\} = \mathcal{T}(u)W_{N+1}(-u, \lambda). \tag{3.23}$$

Setting $u = -\lambda$, we find from (3.10) that $W_i(-\lambda, \lambda)$ is a projector:

$$W_i(-\lambda, \lambda) = \begin{pmatrix} -2\lambda + s_i S_i & s_i^2 S_i - 2\lambda s_i \\ S_i & s_i S_i \end{pmatrix} = \begin{pmatrix} s_i S_i - 2\lambda \\ S_i \end{pmatrix} (1\ s_i). \tag{3.24}$$

Furthermore, there exists a null vector,

$$\Omega_i = \begin{pmatrix} -s_i \\ 1 \end{pmatrix}, \tag{3.25}$$

such that

$$W_i(-\lambda,\lambda)\Omega_i = \begin{pmatrix} s_iS_i - 2\lambda \\ S_i \end{pmatrix}(1 s_i)\begin{pmatrix} -s_i \\ 1 \end{pmatrix} = 0. \tag{3.26}$$

Consequently, it follows from (3.23) and (3.26):

$$W_{N+1}(-\lambda,\lambda)\widetilde{T}(\lambda)\Omega_{N+1} = 0. \tag{3.27}$$

Hence $\widetilde{T}(\lambda)\Omega_{N+1}$ is proportional to Ω_{N+1}, i.e.,

$$\widetilde{T}(\lambda)\Omega_{N+1} = \Lambda\Omega_{N+1}. \tag{3.28}$$

The determination of the eigenvalue Λ may be carried out as follows. Since

$$\widetilde{T}(\lambda)\Omega_{N+1} = K_+(\lambda)\widetilde{T}(\lambda)K_-(\lambda)(\widetilde{T}^{-1}(-\lambda)\Omega_{N+1})$$

and

$$\widetilde{T}^{-1}(-\lambda)\Omega_{N+1} = \widetilde{\ell}_1^{-1}(-\lambda)\cdots\widetilde{\ell}_{N-1}^{-1}(-\lambda)\{\widetilde{\ell}_N^{-1}(-\lambda)\Omega_{N+1}\},$$

we shall need to determine a recurrence relation for the Ω_i's, so as to shift them from right to left through the string of $\ell_j^{-1}(-\lambda)$'s. Note that

$$\widetilde{\ell}_j^{-1}(-\lambda)\Omega_{j+1} = -\frac{1}{\lambda}\begin{pmatrix} -s_{j+1} - \tilde{q}_j \\ \tilde{r}_j s_{j+1} - \lambda + \tilde{q}_j\tilde{r}_j \end{pmatrix} = \frac{1}{1-\tilde{r}_j s_j}\Omega_j,$$

where explicit use of the Bäcklund transformations for $\tilde{q}_j$ from (3.11) has been made. This allows us to run the vectors Ω_i from right to left and to obtain, as a result,

$$\widetilde{T}^{-1}(-\lambda)\Omega_{N+1} = \left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_j s_j}\right)\Omega_1.$$

Next

$$K_-(\lambda)\widetilde{T}^{-1}(-\lambda)\Omega_{N+1} = \left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_j s_j}\right)K_-(\lambda)\Omega_1 = -\frac{1}{2}\left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_j s_j}\right)\begin{pmatrix} s_1S_1 - 2\lambda \\ -S_1 \end{pmatrix}, \tag{3.29}$$

where we have used the fact that $S_1 = 2\theta_-$. Note that $W_i(u,\lambda)$ is also singular at $u=\lambda$ and may be expressed as

$$W_i(\lambda,\lambda) = \begin{pmatrix} s_iS_i & s_i^2S_i - 2\lambda s_i \\ S_i & -2\lambda + s_iS_i \end{pmatrix} = \begin{pmatrix} s_i \\ 1 \end{pmatrix}(S_i\ s_iS_i - 2\lambda). \tag{3.30}$$

Hence its action on the vector,

$$\widetilde{\Omega}_i = \begin{pmatrix} s_iS_i - 2\lambda \\ -S_i \end{pmatrix}, \tag{3.31}$$

is null, i.e.,

$$W_i(\lambda,\lambda)\widetilde{\Omega}_i = 0.$$

This allows us to recast (3.29) as

$$K_-(\lambda)\widetilde{T}^{-1}(-\lambda)\Omega_{N+1} = -\frac{1}{2}\left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_j s_j}\right)\begin{pmatrix} s_1S_1 - 2\lambda \\ -S_1 \end{pmatrix} = -\frac{1}{2}\left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_j s_j}\right)\widetilde{\Omega}_1. \tag{3.32}$$

Applying $\widetilde{T}(\lambda)$ to the left-hand side yields

$$\widetilde{T}(\lambda)K_-(\lambda)\widetilde{T}^{-1}(-\lambda)\Omega_{N+1} = -\frac{1}{2}\left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_j s_j}\right)\widetilde{T}(\lambda)\widetilde{\Omega}_1. \tag{3.33}$$

But, $\widetilde{T}(\lambda)\widetilde{\Omega}_1 = \widetilde{\ell}_n(\lambda)\cdots\widetilde{\ell}_1(\lambda)\widetilde{\Omega}_1$ and we have

$$\widetilde{\ell}_1(\lambda)\widetilde{\Omega}_1 = \begin{pmatrix}(\lambda+\tilde{q}_1\tilde{r}_1)(s_1S_1-2\lambda)-\tilde{q}_1S_1\\ \tilde{r}_1(s_1S_1-2\lambda)-S_1\end{pmatrix}.$$

Upon using the Bäcklund transformation relations (3.11)–(3.16) to eliminate the variables $\tilde{q}_1, \tilde{r}_1$ we find that

$$\begin{pmatrix}(\lambda+\tilde{q}_1\tilde{r}_1)(s_1S_1-2\lambda)-\tilde{q}_1S_1\\ \tilde{r}_1(s_1S_1-2\lambda)-S_1\end{pmatrix} = \frac{\lambda}{1+r_1s_1}\widetilde{\Omega}_2.$$

The generalization of this result is obvious, so that the action of $\widetilde{T}(\lambda)$ on the vector $\widetilde{\Omega}_1$ may be easily calculated, yielding

$$\widetilde{T}(\lambda)K_-(\lambda)\widetilde{T}^{-1}(-\lambda)\Omega_{N+1} = -\frac{1}{2}\left(\prod_{j=1}^{N}\frac{1}{1-\tilde{r}_js_j}\right)\left(\lambda^N\prod_{k=1}^{N}\frac{1}{1+r_is_i}\right)\widetilde{\Omega}_{N+1}. \tag{3.34}$$

Consequently, $\mathcal{T}(\lambda)\Omega_{N+1} = K_+(\lambda)\widetilde{T}(\lambda)K_-(\lambda)\widetilde{T}^{-1}(-\lambda)\Omega_{N+1}$

$$= \left\{-\frac{\lambda^N}{2}\prod_{j=1}^{N}\frac{1}{(1-\tilde{r}_js_j)(1+r_js_j)}\right\}K_+(\lambda)\begin{pmatrix}s_{N+1}S_{N+1}-2\lambda\\ -S_{N+1}\end{pmatrix}.$$

Using the expression for S_{N+1} given in (3.13) one finds that

$$K_+(\lambda)\begin{pmatrix}s_{N+1}S_{N+1}-2\lambda\\ -S_{N+1}\end{pmatrix} = -\frac{2\theta_+^2}{s_{N+1}^2}\begin{pmatrix}-s_{N+1}\\ 1\end{pmatrix}.$$

But the last vector is nothing but Ω_{N+1} as given in (3.25). Thus

$$\mathcal{T}(\lambda)\Omega_{N+1} = \frac{\lambda^N\theta_+^2}{s_{N+1}^2}\prod_{j=1}^{N}\frac{1}{(1+r_js_j)(1-\tilde{r}_js_j)}\Omega_{N+1}. \tag{3.35}$$

Comparison with (3.28) shows that the eigenvalue

$$\Lambda = \frac{\lambda^N\theta_+^2}{s_{N+1}^2}\prod_{j=1}^{N}\frac{1}{(1+r_js_j)(1-\tilde{r}_js_j)}. \tag{3.36}$$

Thus we have explicitly identified one of the eigenvalues of $\mathcal{T}(\lambda)$. The other eigenvalue, say Λ', follows from the fact that

$$\det\mathcal{T}(\lambda) = \theta_+^2\lambda^N\theta_-^2(-\lambda)^N = \Lambda\Lambda' \tag{3.37}$$

and is, therefore, given by

$$\Lambda' = \frac{(-1)^N\theta_-^2}{\lambda^N}\left[\prod_{j=1}^{N}(1-\tilde{r}_js_j)(1+r_js_j)\right]s_{N+1}^2. \tag{3.38}$$

From (3.18) and (3.19) we have

$$\mu = \ln\left[\frac{1}{s_{N+1}^2}\prod_{i=1}^{N}\frac{1}{(1+r_is_i)(1-\tilde{r}_is_i)}\right] + C'(\lambda). \tag{3.39}$$

Using (3.36) this may be written as

$$\mu = \ln\Lambda + \left(C'(\lambda) - N\ln\lambda - \ln\theta_+^2\right). \tag{3.40}$$

Clearly $C(\lambda)$ may be chosen such that

$$\mu = \ln\Lambda, \tag{3.41}$$

and we see that the eigenvalue of $\mathcal{T}(\lambda)$ is nothing but an exponential of the conjugate variable μ, corresponding to the parameter of the Bäcklund transformation. In this case $C(\lambda)$ may be explicitly determined, and is given by

$$C(\lambda) = \ln\left\{\left(\frac{\lambda}{e}\right)^{N\lambda}\theta_+^{2\lambda}\right\}.$$

3.3. Application to discrete-time dynamics

The problem of discretizing a continuous integrable system, without destroying its integrability, has a long and chequered history, involving a number of different approaches. In the case of lattice systems, it is the spatial variable which is discretized, while time is treated as a continuous variable. In deriving Bäcklund transformations by the method described above, we have precisely used this approach. However, it is known that a Bäcklund transformation often provides a discrete-time approximation to a continuous-time integrable system. Indeed, this aspect was studied in great detail by Suris in a number of papers [12, 13].The basic feature of this approach is that it keeps intact the Lax pair of the continuous time system, and is, in principle, applicable to any system admitting an r-matrix interpretation. It will be observed that the local one-parameter Bäcklund transformation, $b_\lambda^{(i)}$, given by (3.1), gives rise to a family of maps, $B_\lambda : (q, r) \mapsto (\tilde{q}, \tilde{r})$, depending on the parameter λ, which preserve the Hamiltonian. Suppose now there exists a point, $\lambda = \lambda_0$, at which B_λ reduces to the identity mapping. Furthermore, if in a neighbourhood of λ_0, the infinitesimal mapping, $B_{\lambda_0+\epsilon}$, goes as $\epsilon\{H, .\}$ and is able to reproduce the Hamiltonian dynamics under investigation, then B_λ may be considered as a discrete-time approximation of the model under consideration. In this section, we investigate this aspect for the DST model.

Let us consider an infinitesimal change in r_i and set

$$\tilde{r}_i = r_i + O(\epsilon), \qquad i = 1, \ldots, N, \qquad \text{where} \qquad \epsilon = \lambda^{-1}. \tag{3.42}$$

Using (3.42) in (3.14) and (3.15) yields up to first order in ϵ the following:

$$s_i \approx -\epsilon(1 + 2r_{i-1})r_i^{-2}, \qquad i = 2, \ldots, N, \qquad \text{and} \qquad s_1 \approx -\epsilon(1 + 2\theta_-)r_1^{-2}. \tag{3.43}$$

Since $r_0 = \theta_-$ by definition, we can combine these results and write

$$s_i = -\epsilon(1 + 2r_{i-1})r_i^{-2} + O(\epsilon^2), \qquad i = 1, 2, \ldots, N. \tag{3.44}$$

Similarly using (3.42) in (3.16) we obtain

$$s_{N+1} = -\epsilon\theta_+ + O(\epsilon^2). \tag{3.45}$$

Now from (3.6) we have

$$S_{i+1} = r_i + \tilde{r}_i = 2r_i + O(\epsilon), \qquad \text{that is,}$$
$$S_i = 2r_{i-1} + O(\epsilon) \quad \text{where} \quad i = 2, \ldots, (N + 1). \tag{3.46}$$

Note that S_1 has already been exactly determined and has the value $2\theta_-$ (see (3.13)). In view of the fact that $r_0 = \theta_-$ we may write therefore

$$S_i = 2r_{i-1} + O(\epsilon) \quad \text{where} \quad i = 1, \ldots, (N + 1). \tag{3.47}$$

In addition, we note that up to first order in ϵ the expression for q_i in (3.11) reduces to

$$q_i = (1 + 2r_{i-1})r_i^{-2} + O(\epsilon).$$

When the above approximations for s_i and S_i are substituted in to the gauge matrix $W_i(u, \lambda)$ given in (3.10), it assumes the following form:

$$-\epsilon W_i(u) \approx I - 2\epsilon \begin{pmatrix} \frac{u}{2} & (1 + 2r_{i-1})r_i^{-2} \\ r_{i-1} & -\frac{u}{2} \end{pmatrix}. \tag{3.48}$$

Consequently,

$$-\epsilon W_i(u) = I - 2\epsilon \begin{pmatrix} \frac{u}{2} & q_i \\ r_{i-1} & -\frac{u}{2} \end{pmatrix} = I - 2\epsilon M_i(u) + O(\epsilon^2). \tag{3.49}$$

Hence from the defining relation for the local Bäcklund transformation, namely

$$W_{i+1}(u, \lambda)\ell_i(u; \tilde{q}_i, \tilde{r}_i) = \ell_i(u; q_i, r_i)W_i(u, \lambda),$$

we find

$$\ell(u, \tilde{q}_i, \tilde{r}_i) = \ell(u; q_i, r_i) + 2\epsilon\,(M_{i+1}\ell(u; q_i, r_i) - \ell(u; q_i, r_i)M_i) + O(\epsilon^2). \tag{3.50}$$

However, as the system is Hamiltonian in nature, the flows are given by (2.5), i.e.,

$$\dot{\ell}(u; q_i, r_i) = M_{i+1}\ell(u; q_i, r_i) - \ell(u; q_i, r_i)M_i = \{H, \ell(u; q_i, r_i)\}.$$

Thus we have the following property of the Bäcklund transformation in the neighbourhood of $\epsilon = 0$, that is, as $\lambda \to \infty$,

$$B_\epsilon \ell(u; q_i, r_i) = \ell(u; \tilde{q}_i, \tilde{r}_i) = \ell(u; q_i, r_i) + 2\epsilon\{H, \ell(u; q_i, r_i)\} + O(\epsilon^2). \tag{3.51}$$

This shows that the Bäcklund transformation derived serves as an approximate discrete-time dynamics for the continuous-time dynamics generated by the system's Hamiltonian.

4. Discussion

In this paper, we have shown the construction of a canonical Bäcklund transformation for the DST model, in the presence of open boundary conditions. This builds on our previous work, in which we had dealt with a quasi-periodic boundary condition and also compliments [9], in the sense that the ansatz for the gauge matrix, $W_i(u, \lambda)$, given in (3.10) would probably work for a large class of discrete models. The canonical nature of the transformations has explicitly been shown through the derivation of the corresponding generating function. The commutativity of such transformations is well known, but the connection with discrete-time dynamics is an interesting feature of Bäcklund transformations derived in the above manner. The quantum analogues of classical Bäcklund transformations are known to lead to Baxter's Q-operator and have been the object of an intense study during the last few years. However, it would been interesting to derive the analogous results in the case of systems described by open boundary conditions. This matter is being investigated and will be communicated in due course.

References

[1] Enol'skii V Z, Kuznetsov V B and Salerno M 1993 On the quantum inverse scattering method for the DST dimer *Physica* D **68** 138
[2] Toda M and Wadati M 1975 Canonical transformation for the exponential lattice *J. Phys. Soc. Japan* **39** 1204
Wadati M and Toda M 1975 Bäcklund transformation for the exponential lattice *J. Phys. Soc. Japan* **39** 1196

[3] Sklyanin E K 2000 Bäcklund transformations and Baxter's Q-operator *Integrable Systems: from Classical to Quantum* (1999, Montreal, QC) *CRM Proc. Lecture Notes* vol 26 (Providence, RI: Amer. Math. Soc.) p 227

[4] Kuznetsov V B, Salerno M and Sklyanin E K 2000 Quantum Bäcklund transformation for DST dimer model *J. Phys. A: Math Gen.* **33** 171

[5] Roy Choudhury A and Ghose Chowdhury A 2004 *Quantum Integrable systems* (*Research Notes in Mathematics* vol 435) (London: Chapman and Hall)

[6] Kuznetsov V B and Sklyanin E K 1998 On Bäcklund transformations for many-body systems *J. Phys. A: Math. Gen.* **31** 2241

[7] Kuznetsov V B, Petrera M and Ragnisco O 2004 Separation of variables and Bäcklund transformations for the symmetric Lagrange top *J. Phys. A: Math. Gen* **37** 8495

[8] Ghose Choudhury A and Roy Chowdhury A 2001 On Bäcklund transformation and boundary conditions associated with the quantum inverse problem for a discrete nonlinear integrable system and its connection to Baxter's Q-operator *Adv. Theor. Math. Phys.* **5** 679–96 (arXiv:math-ph/0202027)

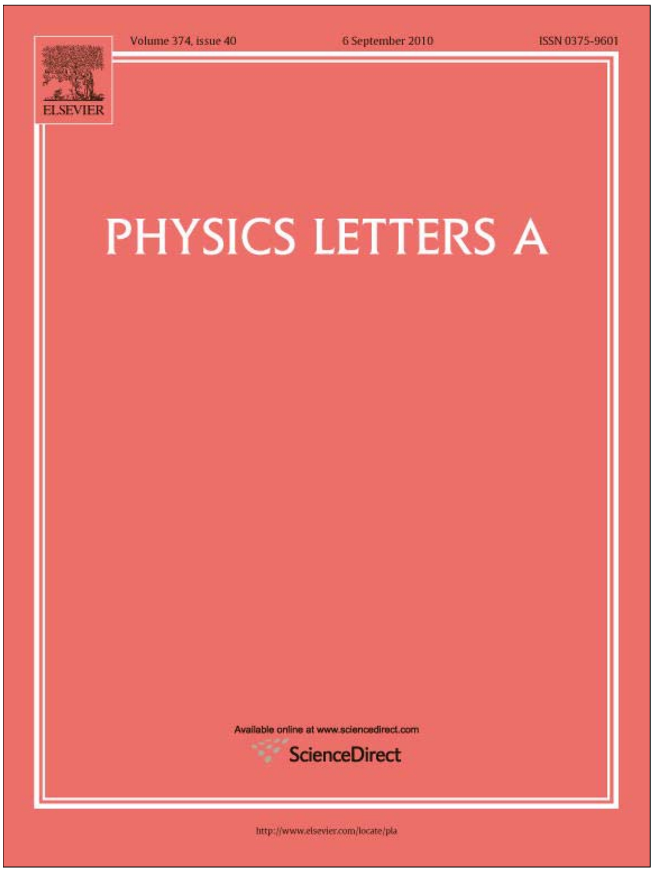
[9] Kuznetsov V B and Sklyanin E K 2007 Bäcklund transformation for the BC-type Toda lattice arXiv: 0707.1950[nlin.Si]

[10] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math Gen.* **21** 2375

[11] Kajinaga Y and Wadati M 1997 Integrable boundary conditions of the modified Volterra model *J. Phys. Soc. Japan.* **66** 3021

[12] Suris Yu B 1996 A discrete time relativistic Toda lattice *J. Phys A: Math. Gen.* **29** 451

[13] Suris Yu B 1999 Integrable discretizations for lattice systems: local equations of motion and their Hamiltonian properties *Rev. Math. Phys.* **11** 727


Volume 374, issue 40
6 September 2010
ISSN 0375-9601
ELSEVIER
PHYSICS LETTERS A
Available online at www.sciencedirect.com
ScienceDirect
http://www.elsevier.com/locate/pla

Physics Letters A 374 (2010) 4120–4127

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

On Bäcklund transformation of D_n type Toda lattice

Barun Khanra [a,*], A. Ghose Choudhury [b]

[a] *Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Calcutta – 700004, India*
[b] *Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta – 700009, India*

ARTICLE INFO

Article history:
Received 15 August 2009
Received in revised form 18 March 2010
Accepted 6 August 2010
Available online 14 August 2010
Communicated by A.P. Fordy

Keywords:
Lax matrix
Bäcklund transformation
Generating function

ABSTRACT

In this Letter we study the Bäcklund transformation for the discrete D_n type Toda lattice with dynamic boundary conditions. As in the periodic case, the transformation is found to be canonical with a corresponding generating function.

1. Introduction

It is well known that many integrable lattice systems retain their integrability even under the imposition of non-periodic boundary conditions [1–4]. The key to understanding their integrability under such circumstances is the reflection equation algebra which plays an important role in the study of classical integrable system [5]. The Toda lattice is among the more well-known classical Liouville integrable systems, which has been extensively studied over the years. Its different versions, corresponding to the various root systems of the affine algebras, may be summarized by the Hamiltonian

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2} + \sum_{j=1}^{n-1} e^{q_j - q_{j+1}} + V(q), \quad \text{with } \{p_j, q_j\} = \delta_{jk}, \tag{1.1}$$

where the additional potential term $V(q)$ has the following forms for the loop algebras $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$ respectively [6]:

$$V_{A_n^{(1)}} = e^{q_n - q_1}, \tag{1.2}$$

$$V_{B_n^{(1)}} = e^{q_n} + e^{-q_1 - q_2}, \tag{1.3}$$

$$V_{C_n^{(1)}} = e^{2q_n} + e^{-2q_1}, \tag{1.4}$$

$$V_{D_n^{(1)}} = e^{q_n + q_{n-1}} + e^{-q_1 - q_2}. \tag{1.5}$$

In a recent communication Kuznetsov et al. [1] have shown how one may derive a Bäcklund transformation (BT) for the BC-type Toda lattice under open boundary conditions and have commented on the applicability of their method to a wider class of problems including the D_n type Toda lattice. In the existing literature on D_n type Toda lattice, there is a generic version of the periodic Toda lattice, due to Kuznetsov [6] in which the potential term in the Hamiltonian has the following appearance:

$$V(q) = \sum_{j=1}^{n-1} e^{q_j - q_{j-1}} + e^{-q_1 - q_2} + e^{q_n + q_{n-1}} + \frac{A}{\sinh^2 \frac{q_1}{2}} + \frac{B}{\sinh^2 q_1} + \frac{C}{\sinh^2 \frac{q_n}{2}} + \frac{D}{\sinh^2 q_n}. \tag{1.6}$$

* Corresponding author. Tel.: +91 9433069860.
E-mail addresses: barunkhanra@rediffmail.com (B. Khanra), a_ghosechoudhury@rediffmail.com (A. Ghose Choudhury).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.08.019

This system is integrable and includes the preceding potentials as limiting cases. We present a derivation of the Bäcklund transformation for the D_n type Toda lattice with extra parameters (Inozemtsev case) [7], by employing the 2×2 Lax matrix given by Kuznetsov in [6]. In this connection it may be remarked that, Kuznetsov et al. had, in an earlier paper [8] shown how the same system could be generated by coupling two e(3) tops interacting with the A_n type Toda lattice.

The Letter is organized as follows. In Section 2 we introduce the D_n type Toda lattice and outline the basic features of the Reflection Equation Algebra (REA). In addition we introduce the generator of the integrals of motion for the REA. In Section 3 we deduce the Bäcklund transformation for the model under consideration with dynamic boundaries. Section 4 contains a discussion of the canonicity of such BT. This is finally followed by a modest outlook. In a short Appendix A we consider the explicit case of four lattice points and exhibit the inherent implicit nature of the BT under the given circumstances.

2. Lax matrix and the Reflection Equation Algebra

In the following we will use the following version of the local Lax matrix of the periodic Toda lattice

$$\ell_j(u) = \begin{pmatrix} 0 & -x_j^{-1} \\ x_j & u + ip_j x_j \end{pmatrix}, \quad j = 3, \ldots, n, \tag{2.1}$$

with p_j and x_j satisfying the canonical Poisson brackets

$$\{p_j, x_j\} = \delta_{jk}.$$

It is easy to verify that $\ell_j(u)$ satisfies the Sklyanin quadratic algebra

$$\left\{\ell_j^1(u), \ell_k^2(v)\right\} = \left[r(u-v), \ell_j^1(u)\ell_k^2(v)\right]\delta_{jk}, \tag{2.2}$$

where $\ell_j^1(u) = \ell_j(u) \otimes I$ and $\ell_j^2(u) = I \otimes \ell_j(u)$ are the standard tensor products of $\ell_j(u)$ with the 2×2 unit matrix I and

$$r(u-v) = \frac{i}{u-v}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

Furthermore we consider the following 2×2 Lax matrices at the first and second lattice sites [6]:

$$\ell_i(u) = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \quad i = 1, 2, \tag{2.4}$$

where

$$A_1 = u^2 x_1 + u\left\{i\left(x_1^2 - 1\right)p_1 + c_1 x_1 + c_2\right\} + c_1 c_2, \tag{2.5}$$

$$B_1 = u\left(x_1^2 - 1\right), \tag{2.6}$$

$$C_1 = u\left[u^2 + \left(x_1^2 - 1\right)p_1^2 - 2ip_1(c_1 x_1 + c_2) - c_1^2\right], \tag{2.7}$$

$$D_1 = u^2 x_1 - u\left\{i\left(x_1^2 - 1\right)p_1 + c_1 x_1 + c_2\right\} + c_1 c_2 \tag{2.8}$$

and

$$A_2 = -u^2 x_2 + u\left\{i\left(x_2^2 - 1\right)p_2 + c_3 x_2 + c_4\right\} - c_3 c_4, \tag{2.9}$$

$$B_2 = u\left[u^2 + \left(x_2^2 - 1\right)p_2^2 - 2ip_2(c_3 x_2 + c_4) - c_3^2\right], \tag{2.10}$$

$$C_2 = u\left(x_2^2 - 1\right), \tag{2.11}$$

$$D_2 = -u^2 x_2 - u\left\{i\left(x_2^2 - 1\right)p_2 + c_3 x_2 + c_4\right\} - c_3 c_4. \tag{2.12}$$

These two matrices satisfy the Reflection Equation Algebra (REA), *viz.*

$$\left\{\ell_\alpha^1(u), \ell_\beta^2(v)\right\} = \left(\left[r(u-v), \ell_\alpha^1(u)\ell_\beta^2(v)\right] + \ell_\alpha^1(u) r(u+v) \ell_\beta^2(v) - \ell_\beta^2(v) r(u+v) \ell_\alpha^1(u)\right)\delta_{\alpha\beta}, \tag{2.13}$$

where $\alpha, \beta = 1, 2$; and c_k $(k = 1, \ldots, 4)$ are constant parameters. The local Lax matrices $\ell_j(u)$ with $j = 1, \ldots, n$ may be used to define the following global Lax matrix, which serves as the generator of the integrals of motion:

$$\mathcal{T}(u) = T(u)\ell_1(u)T^{-1}(-u)\ell_2(u). \tag{2.14}$$

Note that $T(u)$ here is the classical monodromy matrix defined by

$$T(u) := \ell_3(u) \ldots \ell_n(u). \tag{2.15}$$

The Hamiltonian $\mathcal{H}_1$ may be obtained from the spectral curve of $\mathcal{T}(u)$, which is defined by $\det(\mathcal{T}(u) - \lambda I) = 0$, and leads to the following equation, namely

$$\lambda^2 - \lambda\left[(-1)^n u^{2n+2} + (-1)^n \mathcal{H}_1 u^{2n} + \cdots\right] + \prod_{i=1}^{4}\left(u^2 - c_i^2\right) = 0, \tag{2.16}$$

with

$$\mathcal{H}_1 = \sum_{j=3}^{n}(p_j x_j)^2 + p_1^2\left(x_1^2 - 1\right) + p_2^2\left(x_2^2 - 1\right) - 2\sum_{j=3}^{n-1}\frac{x_j}{x_{j+1}} + \frac{2x_2}{x_3} + 2x_1x_n - 2ip_1(c_1x_1 + c_2) - 2ip_2(c_3x_2 + c_4). \tag{2.17}$$

This Hamiltonian may be mapped to that of the periodic D_n type Toda lattice with an additional singular term if we perform the following change of variables $x_1 = \cosh q_1$, $x_2 = \cosh q_2$, $x_j = e^{q_j}$, $j = 3, \ldots, n$, together with a gauge type canonical transformation of (x_1, p_1) and (x_2, p_2) to eliminate the terms linear in p_1 and p_2 in (2.17).

The significance of the Lax matrix $\mathcal{T}(u)$ is that its spectrum is invariant under the dynamics generated by the Hamiltonian (2.17). The equations of motion following from the Hamiltonian (2.17) are as follows:

$$\dot{x}_1 = 2p_1\left(x_1^2 - 1\right) - 2i(c_1x_1 + c_2), \tag{2.18}$$

$$\dot{p}_1 = -2p_1^2x_1 + 2ic_1p_1 - 2x_n, \tag{2.19}$$

$$\dot{x}_2 = 2p_2\left(x_2^2 - 1\right) - 2i(c_3x_1 + c_4), \tag{2.20}$$

$$\dot{p}_2 = 2ip_2c_3 - 2p_2^2x_2 - \frac{2}{x_3}, \tag{2.21}$$

$$\dot{x}_j = 2p_jx_j^2, \quad j = 3, \ldots, n-1, \tag{2.22}$$

$$\dot{p}_j = \frac{2}{x_{j+1}} - 2\frac{x_{j-1}}{x_j^2} - 2p_j^2x_j, \quad j = 3, \ldots, n-1, \tag{2.23}$$

$$\dot{x}_n = 2x_n^2p_n, \tag{2.24}$$

$$\dot{p}_n = -2x_np_n^2 - 2\frac{x_{n-1}}{x_n^2} - 2x_1. \tag{2.25}$$

It is appropriate at this juncture to recollect the basic formalism for discrete integrable systems described by the open boundary conditions. One can show [4] that the generator of the conserved quantities in such cases is given by

$$\mathcal{T}(u) = \operatorname{tr} K_+(u)U(u) \quad \text{where } U(u) = T(u)K_-(u)T^{-1}(-u). \tag{2.26}$$

The matrices $K_\pm$ are referred to as the boundary matrices and generally depend on the values of the dynamical variables at the boundaries, besides on the spectral parameter u.

Now as

$$\mathcal{T}(u) = \operatorname{tr}\left\{K_+(u)T(u)K_-(u)T^{-1}(-u)\right\} = \operatorname{tr}\left\{T(u)K_-(u)T^{-1}(-u)K_+(u)\right\}, \tag{2.27}$$

a straightforward comparison of (2.27) with (2.14) reveals that, in our case

$$K_1(u) \equiv \ell_1(u) \quad \text{and} \quad K_2(u) \equiv \ell_2(u), \tag{2.28}$$

with the boundary matrices $K_\pm(u)$ depending on the dynamical variables (x_1, p_1) and (x_2, p_2). Thus in this sense we may look upon (2.1) as an open system with dynamic boundary conditions. In order that $\frac{d\mathcal{T}(u)}{dt} = 0$, one requires $K_\pm(u)$ to satisfy the REA (2.13) with the left-hand side equal to zero. In addition, to ensure involution of the integrals of motion it is necessary that

$$\left\{U^1(u), U^2(v)\right\} = r(u-v)U^1(u)U^2(v) - U^2(v)U^1(u)r(u-v) - U^1(u)r(u+v)U^2(v) + U^2(v)r(u+v)U^1(u). \tag{2.29}$$

3. Bäcklund transformation

In this section we will construct a Bäcklund Transformation (BT) for the D_n type Toda lattice based on the Hamiltonian approach. The method proposed in [6,3] relies firstly on our ability to find an invertible matrix $M_j(u, \lambda)$ satisfying the following gauge transformation:

$$M_j(u, \lambda)\ell_j(u; x, p) = \ell_j(u; y, q)M_{j+1}(u, \lambda), \quad j = 3, \ldots, n, \tag{3.1}$$

where we have, for the sake of brevity, dispensed with the subscripts on the x's and y's.

Proposition 3.1. *If $M_j(u, \lambda)$ is an invertible 2×2 matrix satisfying the gauge transformation (3.1) then the trace of the generator of the conserved quantities $\mathcal{T}$, as given in (2.27), is an invariant.*

Proof. Eq. (3.1) leads to the following relation for the monodromy matrix $T(u)$:

$$M_3(u, \lambda)T(u; x, p) = T(u; y, q)M_{n+1}(u, \lambda). \tag{3.2}$$

Note that here x stands for the collective set of variables $\{x_3, \ldots, x_n\}$ and similarly for y. Upon using (3.1) and (2.14) we obtain

$$M_3(u, \lambda)\mathcal{T}(u; x, p) = M_3(u, \lambda)T(u; x, p)\ell_1(u)T^{-1}(-u; x, p)\ell_2(u) = T(u; y, q)M_{n+1}(u, \lambda)\ell_1(u)T^{-1}(-u; x, p)\ell_2(u). \tag{3.3}$$

Let us now demand that

$$M_{n+1}(u,\lambda)\ell_1(u;x,p) = \ell_1(u;y,q)M_{n+1}(-u,\lambda), \tag{3.4}$$

then it follows from (3.3) that

$$\begin{aligned} M_3(u,\lambda)\mathcal{T}(u;x,p) &= T(u;y,q)\ell_1(u;y,q)M_{n+1}(-u,\lambda)T^{-1}(-u;x,p)\ell_2(u,x,p) \\ &= T(u,y,q)\ell_1(u;y,q)M_{n+1}(-u,\lambda)\big\{\ell_n^{-1}(-u;x,p)\ldots\ell_3^{-1}(-u;x,p)\big\}\ell_2(u). \end{aligned} \tag{3.5}$$

However from the local gauge transformation (3.1) we have

$$M_{j+1}(-u,\lambda)\ell_j^{-1}(-u;x,p) = \ell_j^{-1}(-u;y,q)M_j(-u,\lambda), \quad j = 3,\ldots,n. \tag{3.6}$$

Using (3.6), the r.h.s. of (3.5) may be simplified to yield

$$T(u,y,q)\ell_1(u;y,q)T^{-1}(-u;y,q)M_3(-u,\lambda)\ell_2(u;x,p).$$

Finally, demanding that

$$M_3(-u,\lambda)\ell_2(u;x,p) = \ell_2(u;y,q)M_3(u,\lambda) \tag{3.7}$$

we arrive at the relation

$$M_3(u,\lambda)\mathcal{T}(u;x,p) = \mathcal{T}(u;y,q)M_3(u,\lambda). \tag{3.8}$$

Since $M_3(u,\lambda)$ is invertible it follows that

$$\operatorname{tr}\mathcal{T}(u;x,p) = \operatorname{tr}\mathcal{T}(u;y,q). \tag{3.9}$$

Thus the conserved quantities are invariant under the BT defined by (3.1). □

To derive the form of the BT we assume the following ansatz for the matrix $M_j(u,\lambda)$, which is in the form of the Lax matrix for the Isotropic Heisenberg magnet (XXX model) [1,5], *viz.*

$$M_j(u,\lambda) = \begin{pmatrix} u-\lambda+r_jR_j & r_j^2R_j - 2\lambda r_j \\ R_j & -u-\lambda+r_jR_j \end{pmatrix}, \quad j = 3,\ldots,n, \tag{3.10}$$

with $\det(M_j(u,\lambda)) = \lambda^2 - u^2 \neq 0$. The same gauge transformation is used in [8] for constructing a Q-operator for the quantum XXX-magnet.

Substitution of (3.10) into (3.1) yields the following relations for $j = 3,\ldots,n$

$$p_j = -\frac{i\lambda}{x_j} - \frac{i}{r_jx_j^2} - ir_{j+1}, \tag{3.11}$$

$$q_j = \frac{i\lambda}{y_j} + \frac{i}{r_jy_j^2} + ir_{j+1}, \tag{3.12}$$

$$R_j = \frac{2\lambda}{r_j} + \frac{1}{r_j^2x_j} + \frac{1}{r_j^2y_j}, \tag{3.13}$$

$$R_{j+1} = -(x_j+y_j), \quad j = 3,\ldots,n. \tag{3.14}$$

From (3.13) and (3.14) we obtain a quadratic equation for r_j in terms of x_j and y_j with $j = 4,\ldots,n$, namely

$$r_j^2(x_{j-1}+y_{j-1}) + 2\lambda r_j + \big(x_j^{-1}+y_j^{-1}\big) = 0. \tag{3.15}$$

On the other hand from (3.4) and (3.10) we obtain the following relations, upon equating coefficients of the different powers of u:

$$R_{n+1} = \frac{2\lambda}{r_{n+1}} - \frac{x_1}{r_{n+1}^2} - \frac{y_1}{r_{n+1}^2}, \tag{3.16}$$

$$A'_1p_1^2 + A'_2p_1 + A'_3 = 0, \tag{3.17}$$

$$B'_1q_1^2 + B'_2q_1 + B'_3 = 0, \tag{3.18}$$

where the A'_i $(i = 1,2,3)$ are given by the following expressions:

$$A'_1 = \big(x_1^2-1\big)(x_1+y_1), \tag{3.19}$$

$$A'_2 = \left[-2i(c_1x_1+c_2)(x_1+y_1) - 2i\big(x_1^2-1\big)\left(\lambda - \frac{x_1}{r_{n+1}} - \frac{y_1}{r_{n+1}}\right)\right], \tag{3.20}$$

$$\begin{aligned} A'_3 &= (y_1-x_1)\left(\lambda - \frac{x_1}{r_{n+1}} - \frac{y_1}{r_{n+1}}\right)^2 - 2(c_1x_1+c_2)\left(\lambda - \frac{x_1}{r_{n+1}} - \frac{y_1}{r_{n+1}}\right) \\ &\quad + \big(y_1^2-1\big)\left(\frac{2\lambda}{r_{n+1}} - \frac{x_1}{r_{n+1}^2} - \frac{y_1}{r_{n+1}^2}\right) - c_1^2(x_1+y_1) - 2c_1c_2, \end{aligned} \tag{3.21}$$

and the B_i' $(i = 1, 2, 3)$ are as follows:

$$B_1' = \left(y_1^2 - 1\right)(x_1 + y_1), \tag{3.22}$$

$$B_2' = -2i(c_1 y_1 + c_2)(x_1 + y_1) + 2i\left(y_1^2 - 1\right)\left(\lambda - \frac{x_1}{r_{n+1}} - \frac{y_1}{r_{n+1}}\right), \tag{3.23}$$

$$B_3' = -(y_1 - x_1)\left(\lambda - \frac{x_1}{r_{n+1}} - \frac{y_1}{r_{n+1}}\right)^2 + 2(c_1 y_1 + c_2)\left(\lambda - \frac{x_1}{r_{n+1}} - \frac{y_1}{r_{n+1}}\right) + \left(x_1^2 - 1\right)\left(\frac{2\lambda}{r_{n+1}} - \frac{x_1}{r_{n+1}^2} - \frac{y_1}{r_{n+1}^2}\right) - c_1^2(x_1 + y_1) - 2c_1 c_2. \tag{3.24}$$

Note that the equations determining p_1 and q_1 being quadratic in nature we get two values of p_1 and q_1.

From (3.16) and (3.14) we get

$$(x_n + y_n)r_{n+1}^2 - 2\lambda r_{n+1} - (x_1 + y_1) = 0, \tag{3.25}$$

while using (3.10) in (3.7) we obtain upon equating coefficients of powers of u the following:

$$R_3 = x_2 + y_2, \tag{3.26}$$

$$\tilde{A}_1 p_2^2 + \tilde{A}_2 p_2 + \tilde{A}_3 = 0, \tag{3.27}$$

$$\tilde{B}_1 q_2^2 + \tilde{B}_2 q_2 + \tilde{B}_3 = 0, \tag{3.28}$$

where the $\tilde{A}_i$ $(i = 1, 2, 3)$ are given by the following:

$$\tilde{A}_1 = \left(x_2^2 - 1\right)(x_2 + y_2), \tag{3.29}$$

$$\tilde{A}_2 = -\left[2i(c_3 x_2 + c_4)(x_2 + y_2) + 2i\left(x_2^2 - 1\right)X_3\right], \tag{3.30}$$

$$\tilde{A}_3 = -\left[-(y_2 - x_2)X_3^2 + 2(c_3 x_2 + c_4)X_3 + \left(y_2^2 - 1\right)Y_3 + c_3^2(x_2 + y_2) - 2c_3 c_4\right] \tag{3.31}$$

and the $\tilde{B}_i$ $(i = 1, 2, 3)$ are:

$$\tilde{B}_1 = \left(y_2^2 - 1\right)(x_2 + y_2), \tag{3.32}$$

$$\tilde{B}_2 = -\left[2i(c_3 y_2 + c_4)(x_2 + y_2) - 2i\left(y_2^2 - 1\right)X_3\right], \tag{3.33}$$

$$\tilde{B}_3 = \left[(y_2 - x_2)X_3^2 - 2(c_3 y_2 + c_4)X_3 + \left(x_2^2 - 1\right)Y_3 + c_3^2(x_2 + y_2) - 2c_3 c_4\right]. \tag{3.34}$$

Here

$$X_3 = -\lambda + r_3(x_2 + y_2) \tag{3.35}$$

and

$$Y_3 = r_3^2(x_2 + y_2) - 2\lambda r_3. \tag{3.36}$$

From (3.26) and (3.14) we obtain

$$(x_2 + y_2)r_3^2 - 2\lambda r_3 - \left(x_3^{-1} + y_3^{-1}\right) = 0. \tag{3.37}$$

So substituting the values of r_j, $j = 4, \ldots, n$ from (3.15), of r_3 from (3.37) and of r_{n+1} from (3.25) into (3.11) and (3.12) one can determine the p_j and q_j implicitly for $j = 3, \ldots, n$ as functions of the x_i's and y_i's and λ. On the other hand since (3.25) determines r_{n+1} as a function of (x_1, y_1) and (x_n, y_n), the quantities A_i' appearing in (3.19)–(3.21) can be expressed in terms of the variables (x_1, y_1, x_n, y_n) and this in turn allows p_1 and q_1 to be determined from (3.17) and (3.18). A similar procedure when applied to (3.27) and (3.28) determines p_2 and q_2 implicitly as functions of x_i's and y_i's and λ from the quadratic equation (3.37) and upon using Eqs. (3.29)–(3.34), thereby completing the Bäcklund transformation for $j = 1, \ldots, n$. The explicit expressions for p_1, q_1, p_2, and q_2 as determined by the above procedure are given below:

$$p_1 = \frac{i(c_1 x_1 + c_2)(x_1 + y_1) + i(x_1^2 - 1)(\lambda - \frac{x_1 + y_1}{r_{n+1}}) \pm \sqrt{D_{p_1}}}{(x_1^2 - 1)(x_1 + y_1)}, \tag{3.38}$$

$$q_1 = \frac{i(c_1 y_1 + c_2)(x_1 + y_1) - i(y_1^2 - 1)(\lambda - \frac{x_1 + y_1}{r_{n+1}}) \pm \sqrt{D_{q_1}}}{(y_1^2 - 1)(x_1 + y_1)}, \tag{3.39}$$

$$p_2 = \frac{i(c_3 x_2 + c_4)(x_2 + y_2) + i(x_2^2 - 1)\{-\lambda + r_3(x_2 + y_2)\} \pm \sqrt{D_{p_2}}}{(x_2^2 - 1)(x_2 + y_2)}, \tag{3.40}$$

$$q_2 = \frac{i(c_3 y_2 + c_4)(x_2 + y_2) - i(y_2^2 - 1)\{-\lambda + r_3(x_2 + y_2)\} \pm \sqrt{D_{q_2}}}{(y_2^2 - 1)(x_2 + y_2)}. \tag{3.41}$$

Here

$$D_{p_1} = -\lambda^2(x_1^2-1)(y_1^2-1) - (c_1x_1+c_2)^2(x_1+y_1)^2 + c_1^2(x_1^2-1)(x_1+y_1)^2 + 2c_1c_2(x_1^2-1)(x_1+y_1), \tag{3.42}$$

$$D_{q_1} = -\lambda^2(x_1^2-1)(y_1^2-1) - (c_1y_1+c_2)^2(x_1+y_1)^2 + c_1^2(y_1^2-1)(x_1+y_1)^2 + 2c_1c_2(y_1^2-1)(x_1+y_1), \tag{3.43}$$

$$D_{p_2} = -\lambda^2(x_2^2-1)(y_2^2-1) - (c_3x_2+c_4)^2(x_2+y_2)^2 + c_3^2(x_2^2-1)(x_2+y_2)^2 - 2c_3c_4(x_2^2-1)(x_2+y_2), \tag{3.44}$$

$$D_{q_2} = -\lambda^2(x_2^2-1)(y_2^2-1) - (c_3y_2+c_4)^2(x_2+y_2)^2 + c_3^2(y_2^2-1)(x_2+y_2)^2 - 2c_3c_4(y_2^2-1)(x_2+y_2) \tag{3.45}$$

and r_3 and r_{n+1} are the solutions of the quadratic equations (3.37) and (3.25) respectively.

Remark. Note that unlike in the case of periodic boundary conditions, where the BT can be explicitly derived [2,8,9], here the BT is defined in an implicit manner. In Appendix A we explain the implicit nature of the transformation more clearly by considering a lattice with four points. This appears to be an inherent feature associated with the Bäcklund transformation derived in the above manner (see also [1,3]).

4. Canonicity of the Bäcklund transformation

In order that the Bäcklund transformation derived in the manner of the preceding section be a canonical transformation one must show that the variables $y(x,p;\lambda)$ and $q(x,p;\lambda)$ have the same canonical Poisson brackets as (x,p), as stated in (1.1). One method of proving the canonicity is to present explicitly the generating function $\Phi_\lambda(y;x)$ of the canonical transformation. This requires from the definition of canonical transformations, that

$$p_j = \frac{\partial \Phi_\lambda}{\partial x_j} \quad \text{and} \quad q_j = -\frac{\partial \Phi_\lambda}{\partial y_j}, \quad j = 1,\dots,n. \tag{4.1}$$

The generating function may be formally written as:

$$\Phi_\lambda(y;x) = \sum_{j=3}^{n} f_\lambda(y_j, r_{j+1}; x_j, r_j) + \Phi_\lambda^{(1)}(x_1, y_1, r_{n+1}) + \Phi_\lambda^{(2)}(x_2, y_2, r_3), \tag{4.2}$$

where $f_\lambda(y_j, r_{j+1}; x_j, r_j)$ represents the generating function of the bulk of the chain, i.e., for $j = 3,\dots,n$ and is given by

$$f_\lambda(y_j, r_{j+1}; x_j, r_j) = -i\lambda\log(x_j+y_j) - 2\lambda\log(r_j) + \frac{i}{r_j}(x_j^{-1}+y_j^{-1}) - ir_{j+1}(x_j+y_j). \tag{4.3}$$

To find $\Phi_\lambda^{(1)}(x_1; y_1, r_3)$ and $\Phi_\lambda^{(2)}(x_2; y_2, r_{n+1})$, one has to integrate partially Eqs. (3.38) to (3.41) w.r.t. p_1, q_1, p_2, and q_2 respectively and check that the results satisfy the definition (4.1). This turns out to be a rather nontrivial computation. Using Maple 12 we have found a complicated result in terms of elliptic functions, but apparently the result is not very illuminating. To arrive at a more concrete result we will next calculate an explicit expression for the generating function for the special case of $c_1 = c_2 = c_3 = c_4 = 0$ below.

4.1. *A special case* $c_1 = c_2 = c_3 = c_4 = 0$

Under the above simplifying assumption, we find from (3.17) and (3.18) upon using (3.19)–(3.24) the following values of p_1 and q_1, namely

$$p_1 = \frac{i\lambda}{x_1+y_1} - \frac{i}{r_{n+1}} \pm i\lambda\frac{\sqrt{y_1^2-1}}{(x_1+y_1)\sqrt{x_1^2-1}}, \tag{4.4}$$

$$q_1 = -\frac{i\lambda}{x_1+y_1} + \frac{i}{r_{n+1}} \pm i\lambda\frac{\sqrt{x_1^2-1}}{(x_1+y_1)\sqrt{y_1^2-1}}. \tag{4.5}$$

Similarly from (3.27) and (3.28) we find p_2 and q_2 to be given by

$$p_2 = -\frac{i\lambda}{x_2+y_2} + ir_3 \pm i\lambda\frac{\sqrt{y_2^2-1}}{(x_2+y_2)\sqrt{x_2^2-1}}, \tag{4.6}$$

$$q_2 = \frac{i\lambda}{x_2+y_2} - ir_3 \pm i\lambda\frac{\sqrt{x_2^2-1}}{(x_2+y_2)\sqrt{y_2^2-1}}. \tag{4.7}$$

Using the definition (4.1) we now find the generating function for the above simple case to be the following:

$$\Phi_\lambda(y;x) = \sum_{j=3}^{n} f_\lambda(y_j, r_{j+1}; x_j, r_j) + \Phi_\lambda^{(1)}(x_1; y_1, r_{n+1}) + \Phi_\lambda^{(2)}(x_2; y_2, r_3), \tag{4.8}$$

where

$$f_\lambda(y_j; x_j) = -i\lambda \log(x_j + y_j) - 2\lambda \log(r_j) + \frac{i}{r_j}\left(x_j^{-1} + y_j^{-1}\right) - ir_{j+1}(x_j + y_j), \tag{4.9}$$

$$\Phi_\lambda^{(1)}(x_1; y_1) = i\,\lambda \log(x_1 + y_1) - \frac{i}{r_{n+1}}(x_1 + y_1)\ \pm \log \frac{(x_1 + y_1)(x_1^2 - 1)(y_1^2 - 1)}{\sqrt{x_1^2 - 1}\sqrt{y_1^2 - 1}\sqrt{x_1 - y_1} - (x_1 y_1 + 1)} \tag{4.10}$$

and

$$\Phi_\lambda^{(2)}(x_2; y_2) = -i\,\lambda \log(x_2 + y_2) + ir_3(x_2 + y_2)\ \pm \log \frac{(x_2 + y_2)(x_2^2 - 1)(y_2^2 - 1)}{\sqrt{x_2^2 - 1}\sqrt{y_2^2 - 1}\sqrt{x_2 - y_2} - (x_2 y_2 + 1)}, \tag{4.11}$$

and $r_j(x, y; \lambda)$ are defined implicitly through (3.15), (3.25) and (3.37).

5. Discussion

In this Letter we have shown the construction of a canonical Bäcklund transformation for the D_n type Toda lattice. The canonical nature of the transformations has been explicitly shown through the derivation of the corresponding generating function. The commutativity of such transformations is well known. The quantum analog of classical Bäcklund transformations, are known to lead to Baxter's Q-operator [9] and have been the object of an intense study during the last few years. However, it would be interesting to derive the analogous results for the case of systems described in this Letter. This matter is being investigated and will be communicated in due course.

Appendix A

As remarked earlier, we examine in this appendix the nature of the BT more explicitly for the case $n = 4$, i.e., when there are only four lattice points. As before the first and second lattice sites are identified with the dynamic boundary points of the chain and expressions for p_1, q_1, p_2 and q_2 as given in (3.38)–(3.41) stand. Setting $j = 3, 4$ in (3.11) and (3.12) we have

$$p_3 = -\frac{i\lambda}{x_3} - \frac{i}{r_3 x_3^2} - ir_4, \tag{A.1}$$

$$q_3 = \frac{i\lambda}{y_3} + \frac{i}{r_3 y_3^2} + ir_4, \tag{A.2}$$

$$p_4 = -\frac{i\lambda}{x_4} - \frac{i}{r_4 x_4^2} - ir_5, \tag{A.3}$$

$$q_4 = \frac{i\lambda}{y_4} + \frac{i}{r_4 y_4^2} + ir_5. \tag{A.4}$$

Next from (3.13) we find that

$$R_3 = \frac{2\lambda}{r_3} + \frac{1}{r_3^2}\left(x_3^{-1} + y_3^{-1}\right), \tag{A.5}$$

$$R_4 = \frac{2\lambda}{r_4} + \frac{1}{r_4^2}\left(x_4^{-1} + y_4^{-1}\right), \tag{A.6}$$

while from (3.14) we have

$$R_4 = -(x_3 + y_3) \quad \text{and} \quad R_5 = -(x_4 + y_4). \tag{A.7}$$

Finally (3.16) with $n = 4$ gives

$$R_5 = \frac{2\lambda}{r_5} - \frac{1}{r_5^2}(x_1 + y_1)$$

while (3.26) stipulates that $R_3 = (x_2 + y_2)$. By equating the respective expressions for R_3, R_4 and R_5 we are led to the following quadratic equations determining r_3, r_4 and r_5 namely:

$$(x_2 + y_2)r_3^2 - 2\lambda r_3 - \left(x_3^{-1} + y_3^{-1}\right) = 0,$$
$$(x_3 + y_3)r_4^2 + 2\lambda r_4 + \left(x_4^{-1} + y_4^{-1}\right) = 0,$$
$$(x_4 + y_4)r_5^2 - 2\lambda r_5 - (x_1 + y_1) = 0.$$

Solving these quadratic equations for r_3, r_4 and r_5 and substituting their values in (A.1)–(A.4) we arrive at the following relations:

$$p_3(x_3+y_3) = \mp i\left[\left(\frac{y_3}{x_3}\right)\sqrt{\lambda^2 + \frac{(x_2+y_2)(x_3+y_3)}{x_3y_3}} + \sqrt{\lambda^2 - \frac{(x_3+y_3)(x_4+y_4)}{x_4y_4}}\right], \tag{A.8}$$

$$p_4(x_4+y_4) = -2i\lambda \pm i\left(\frac{y_4}{x_4}\right)\sqrt{\lambda^2 - \frac{(x_3+y_3)(x_4+y_4)}{x_4y_4}} \mp i\sqrt{\lambda^2 + (x_4+y_4)(x_1+y_1)}, \tag{A.9}$$

$$q_3(x_3+y_3) = \pm i\left[\left(\frac{x_3}{y_3}\right)\sqrt{\lambda^2 + \frac{(x_2+y_2)(x_3+y_3)}{x_3y_3}} + \sqrt{\lambda^2 - \frac{(x_3+y_3)(x_4+y_4)}{x_4y_4}}\right], \tag{A.10}$$

$$q_4(x_4+y_4) = 2i\lambda \mp i\left(\frac{x_4}{y_4}\right)\sqrt{\lambda^2 - \frac{(x_3+y_3)(x_4+y_4)}{x_4y_4}} \pm i\sqrt{\lambda^2 + (x_4+y_4)(x_1+y_1)}. \tag{A.11}$$

An explicit form of the BT necessitates that we be able to solve for say, q_3, q_4, y_3 and y_4 in terms of p_3, p_4, x_3 and x_4 from the above set of equations. It is evident from (A.10) and (A.11) that q_3 and q_4 can be expressed in terms of x_3, x_4, y_3 and y_4 (we do not worry about the presence of x_1, x_2, y_2 and y_2 as they are associated with the boundary sites). But we still need to find y_3 and y_4 in terms of p_3, p_4, x_3 and x_4 together with the boundary variables, which can in principle be achieved by solving (A.8) and (A.9). However in practice this is a nontrivial task. For instance, introducing the variables

$$U = \frac{y_3}{x_3} \quad \text{and} \quad V = \frac{y_4}{x_4}$$

we find that (A.8) and (A.9) assume the following forms:

$$p_3x_3(1+U) = \mp i\left[U\sqrt{\lambda^2 + \left(\frac{x_2+y_2}{x_3}\right)\left(\frac{1+U}{U}\right)} + \sqrt{\lambda^2 - \frac{x_3}{x_4}\frac{(1+U)(1+V)}{V}}\right],$$

$$p_4x_4(1+V) = -2i\lambda \pm i\left[V\sqrt{\lambda^2 - \frac{x_3}{x_4}\frac{(1+U)(1+V)}{V}}\right] \mp i\sqrt{\lambda^2 + x_4(x_1+y_1)(1+V)}.$$

One can attempt to solve for V from the first of these equations in terms of U and then insert the resulting expression into the last equation to obtain a single equation involving U. However, it is a moot point whether the resulting equation for U, or essentially for y_3 can be solved explicitly, especially in the general case of n lattice sites. It is in view of these considerations that we claim that the BT in this case is an implicit one.

References

[1] V.B. Kuznetsov, E.K. Sklyanin, SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007) 080.
[2] A. Roy Chowdhury, A. Ghose Choudhury, Quantum Integrable Systems, Res. Notes Math., vol. 435, Chapman & Hall/CRC, Boca Raton, 2004.
[3] Barun Khanra, A. Ghose Choudhury, Inverse Problems 25 (2009) 085002, 11 pp.
[4] E.K. Sklyanin, J. Phys. A: Math. Gen. 21 (10) (1998) 2375.
[5] V.B. Kuznetsov, E.K. Sklyanin, J. Phys. A: Math. Gen. 31 (1998) 2241.
[6] V.B. Kuznetsov, J. Phys. A: Math. Gen. 30 (1997) 2127.
[7] V.I. Inozemtsev, Comm. Math. Phys. 121 (1989) 629.
[8] V.B. Kuznetsov, M. Salerno, E.K. Sklyanin, J. Phys. A: Math. Gen. 33 (2000) 171.
[9] E.K. Sklyanin, Bäcklund transformation and Baxter's Q operator, in: Integrable Systems from Classical to Quantum, Montreal, QC, 1999, in: CRM Proc. Lecture Notes, vol. 26, 2000, p. 227.