A STUDY ON CHARACTERIZATION AND DETECTION OF QUANTUM ENTANGLEMENT

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by

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To my parents and teachers

All knowledge that the world has ever received comes from the mind; the infinite library of the universe is in our own mind.

Swami Vivekananda

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Abstract

Quantum entanglement which is at the heart of quantum information science marked one of the most significant departures of quantum mechanics from classical formalism. It remains a source of heated debates and posed questions pertaining to the foundational issues of quantum mechanics. A plethora of work followed the inception of entanglement, which addressed fundamental aspects.

However, towards the end of the last century the focus on entanglement gradually shifted from its foundational attributes to practical implementations. This resulted in state of the art technologies unimaginable within classical limits. Entanglement started to be counted as a major resource for quantum information and computational tasks.

The ubiquitous role of entanglement in tasks of communication and computation and its various manifestations , make its detection and characterization all the more important. This thesis is an attempt to encapsulate some features and characterizations of entanglement.

Here we have laid down prescriptions to construct hermitian operators , namely entanglement witnesses which can detect states admitting a positive partial transpose. We have discussed the experimental implementation of such witnesses. This is followed up by a study on the methods of construction of common witnesses which can detect a large set of entangled states. This is extended to the case of common Schmidt witnesses. A comparative study is made between decomposable and non-decomposable witnesses.

Teleportation, an archetypal quantum information processing task is made possible through the use of entangled states as quantum channels. However,not all entangled states are suitable for the purpose . We have introduced the notion of teleportation witnesses which detect those entangled states which can be utilized as teleportation channels. The question of optimality of such witnesses is addressed next, where we have constructed some optimal teleportation witnesses in $d \otimes d$ dimensions.

We have introduced the notion of hermitian operators that can detect states which are not separable from spectrum and also discussed their relevance in entanglement creation through quantum gates and non-local unitary operations.

The thesis concludes with a summary and discussion on possible courses of future work.

Chapter 1

Introduction and Outline

"There are two ways of doing calculations in theoretical physics. One way and this is the way I prefer is to have a clear physical picture of the process that you are calculating. The other way is to have a precise and self consistent mathematical formalism."

Enrico Fermi

Quantum entanglement [EPR35, Sch35] brought a significant distinction to quantum mechanics. It signalled one of the foremost departures of quantum mechanics from classical mechanics.Entanglement refers to quantum correlations between separated physical systems much stronger than correlations allowed in classical mechanics. The notion of entanglement was envisaged by Einstein , Podolsky and Rosen (EPR) in their seminal paper [EPR35] to demonstrate the incompleteness of quantum theory. The nomenclature "Entanglement" was however coined by Schrodinger [Sch35] in a follow-up to the EPR contribution. Three decades later Bell formulated inequalities that any theory fulfilling the fundamental assumptions of the EPR , has to obey[Bel64]. He even showed that some entangled states violate these inequalities.

Gradually, the attitude towards entanglement changed from the exploitation of its foundational principles to its practical implementations. Feynman [Fey82] motivated the quest to use quantum mechanics to simulate computations unimaginable classically. Subsequently, Deutsch [Deu85] laid down the quantum counterpart to the Church-Turing thesis [Chu36, Tur36] to pioneer quantum computation , later built upon by Barenco *et. al.*[BDEJ95]. Quantum algorithms [Sho94, Gro97] followed ,demonstrating exponential speed up over classical computations once again buttressing the significance of entanglement.
In parallel , the information theoretical aspects of entanglement[NC10] were also being exploited upon leading to milestone applications like teleportation [BBC+93], superdense coding[BW92]. The study of secured messages shifted paradigms with the advent of quantum cryptography [BB84, Eke91]. These state of the art technologies , gradually established the notion of entanglement as a physical resource much like energy.
Although entanglement is a vital resource its detection is a hard task. For low dimensional ($2\otimes 2$ and $2\otimes 3$) states there exist simple necessary and sufficient conditions for separability [Per96, HHH96], which is based on the fact that separable states have a positive-partial transpose(PPT). For higher dimensional systems all states with negative partial transpose(NPT) are entangled but there are entangled states which have a positive-partial transpose. This paradoxical behaviour of quantum entanglement in higher dimensions makes it difficult to lay down a single necessary and sufficient condition for its detection.
A silver lining though came in the form of entanglement witnesses(EW) [HHH96, Ter00, GT09]. An outcome of the celebrated Hahn-Banach theorem in functional analysis[Hol75], entanglement witnesses are hermitian operators with at least one negative eigenvalue. Entanglement witness provides a necessary and sufficient condition to detect entanglement . More specifically a given state is entangled if and only if there is an EW that detects it[HHH96]. Entanglement witnesses can also be realized in an experimental setup thus making it a potent tool in the detection of entanglement. Teleportation [BBC+93], a quintessential quantum information processing task,

in its standard protocol, requires entangled states which can give a fidelity higher than the classical optimum. It is an intriguing fact that not all entangled states suit the purpose. In fact literature has instances where even though a state is entangled, it cannot provide for a fidelity higher than the classical limit. This necessitates a criterion to check whether an entangled state is eligible to be used as a quantum teleportation channel. On the other side, the set of separable states contains a special class of states termed as absolutely separable which remain separable under any factorization. The complementary class containing non-absolutely separable states can be used successfully for entanglement creation, which has been a scope of major research [SKK+00, RNO+00].

1.1 Plan of the thesis

The thesis follows the scheme given below :

- Chapter 2 revisits some mathematical and physical requisites needed to follow the later study. It gives a platform for the analysis in the chapters that follow.

- In chapter 3 we prescribe methods to construct entanglement witnesses for PPT entangled states both in the bipartite and multipartite cases. We compare their efficacy with other known witnesses and discuss their experimental relevance.

- In chapter 4 we lay down procedures to construct common witnesses, following which one may detect a large number of entangled states both for the PPT and NPT case. We investigate the same for Schmidt witnesses and follow it up with a discussion on common decomposable and non-decomposable witnesses.

- In chapter 5 we prove the existence of hermitian operators which can identify useful resources for performing teleportation which we name as teleportation witness. We propose such a witness and lay down its measurement prescription.

- We construct optimal teleportation witnesses in chapter 6 and illustrate its experimental representation.

- In chapter 7 we propose hermitian operators that can detect separable states that are not separable from spectrum. These states can give rise to entangled states on the use of global unitary operations on them. The practical relevance of such operators are discussed in conjunction with entanglement creation through quantum gates.

- The thesis ends with a summary of the works with speculations on directions of future work.

Chapter 2

Mathematical and Physical prerequisites

"In mathematics you do not understand things , you just get used to them".

John von Neumann

In this chapter we revisit some mathematical and physical ideas closely associated with the later study. It will give us a framework to analyze and follow the later chapters.

2.1 Mathematical preliminaries

Vector spaces form the foundation of linear algebra. A vector space V is a collection of members known as *vectors* $\{v_i\}$, equipped with two fundamental operations namely,
(i) Vector addition $v_i + v_j$, where $v_i, v_j \in V$.
(ii) Scalar multiplication $c.v_i$, where $c \in F$, F denotes the scalar field.
and which obeys the following axioms:

1. $v_i + v_j \in V, \forall v_i, v_j \in V$
2. $v_i + (v_j + v_k) = (v_i + v_j) + v_k, \forall v_i, v_j, v_k \in V$
3. $\exists \theta \in V,$ `such that` $v + \theta = v = \theta + v, \forall v \in V$
4. $\forall v \in V, \exists(-v) \in V$ `such that` $v + (-v) = \theta = (-v) + v$
5. $\forall v_i, v_j \in V, v_i + v_j = v_j + v_i$
6. $\forall c \in F$ `and` $v \in V, c.v \in V$
7. $(c_1.c_2).v = c_1.(c_2.v)$, where $c_1, c_2 \in F$
8. $\exists \mathbf{1} \in F$ `such that` $\mathbf{1}.v = v$
9. $c.(v_i + v_j) = c.v_i + c.v_j$ and $(c_1 + c_2).v = c_1.v + c_2.v$

Generally , C which denotes the set of complex numbers, is taken to be the field.
We will henceforth use $|..\rangle$ as our quantum mechanical notation for a state vector.

2.1.1 Linear Independence and basis vectors

Suppose we have a set of vectors $|v_i\rangle$ in a vector space . Then by a linear combination of the vectors we mean a vector v such that $v = \sum_i c_i|v_i\rangle, c_i \in C$. If in a vector space V , any vector v can be expressed as a linear combination of the vectors from the set $\{|v_i\rangle\}$, then $\{|v_i\rangle\}$ is called a spanning set of V.
A set of vectors $|v_i\rangle$ is said to be linearly dependent, if there exists at least one $c_i \neq 0$ such that $\sum_i c_i|v_i\rangle = \theta$. A set of vectors is linearly independent if it is not linearly dependent.
A linearly independent set of vectors that span a vector space is known as a basis and the number of vectors in the set is the dimension of the space. Basis, however is not unique but any two basis sets have the same number of vectors.

2.1.2 Linear Operators and Matrices

A operator T from a vector space V to W is said to be linear if

$$T(\sum_i c_i v_i) = \sum_i c_i T v_i \tag{2.1}$$

We say that a linear operator T is defined on a vector space V if T is a linear operator from V to V. Matrices are conveniently used in quantum information science to represent linear operators. Suppose $T : V \rightarrow W$ is a linear operator between vector spaces V and W. Suppose $\{|v_1\rangle, |v_2\rangle, ..., |v_m\rangle\}$ is a basis for $V$and $\{|w_1\rangle, |w_2\rangle, ..., |w_n\rangle\}$ is a basis for W. Then for each j in the range $1, .., m$ one can find complex numbers T_{1j} .... T_{nj} such that $T|v_j\rangle = \sum_i T_{ij}|w_i\rangle$. The matrix whose elements are the complex numbers T_{ij} form a matrix representation of the operator T.

2.1.3 Inner Product and Norm

An inner product between two vectors v and w , denoted by (v, w), is a function $V \times V \rightarrow C$, which satisfies the following axioms:

1. $(u + v, w) = (u, w) + (v, w)$
2. $(\alpha v, w) = \alpha(v, w)$
3. $(v, w) = (w, v)^*$ (`* denotes complex conjugation`)
4. $(v, v) > 0, (v, v) = 0$ `iff` $v = \theta$(`zero vector`)

In quantum mechanical notation , the inner product between two state vectors is written as $\langle v|w\rangle$. Norm of a vector $v \in$ vector space V, denoted by $\|v\|$, is defined as $\|v\| = \sqrt{(v, v)}$. A unit vector v is such that $\|v\| = 1$. It is also called a normalized vector . A Hilbert space is an inner product space complete with respect to the norm induced [Sim63].

2.1.4 Normal , Unitary and Hermitian operators

Suppose A is a linear operator on a Hilbert space H, then there exists a unique linear operator $A^\dagger$, such that $(Av, w) = (v, A^\dagger w)$. $A^\dagger$ is known as the hermitian conjugate of A. An operator A is called:

- Normal iff $AA^\dagger = A^\dagger A$.
- Unitary iff $AA^\dagger = A^\dagger A = I$.
- Hermitian iff $A = A^\dagger$.

An eigenvector $|v\rangle$ is a non-zero vector corresponding to an operator A such that $A|v\rangle = \lambda|v\rangle$, where λ is the corresponding eigenvalue of the operator.

2.1.5 Convex sets

A set V is convex if for any $v_1, v_2, ..., v_n \in V$

$$\sum_{i=1}^{n} \lambda_i v_i \in V \tag{2.2}$$

where $\sum_{i=1}^{n} \lambda_i = 1$.

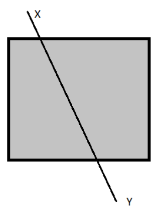

Figure 2.1: A convex set

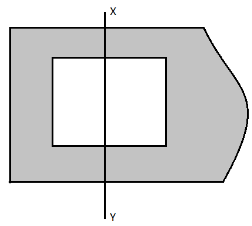

Figure 2.2: A non convex set

Convex sets play a significant role in quantum information where very important structures can be classified as convex sets and elucidate inherent geometrical concepts.Convex sets can be separated from each other as well as from points which do not belong to the set by using separation axioms.This will form the framework for our later analysis to separate entangled states from separable ones using entanglement witnesses. There is a very important separation axiom pertaining to convex sets , namely the geometric form of the Hahn-Banach theorem [Hol75] which states that:

Hahn-Banach Theorem:
Let S be a convex and compact subset of a Hilbert space. Then a $p \notin S$ can be separated from S by a hyperplane.

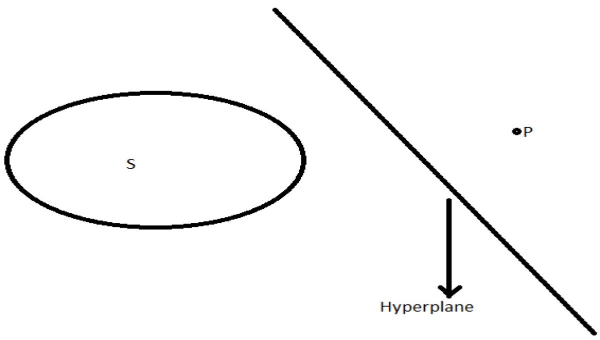

Figure 2.3: Hyperplane Separation

2.2 Qubits and Bloch Sphere

Analogous to classical states 0 and 1, a quantum mechanical system can also be in a state. In the Dirac notation we write them as $|0\rangle$ and $|1\rangle$. Here $|0\rangle$ and $|1\rangle$ represent an orthonormal set of basis vectors. However, a quantum system can be in a superposition of the two basis vectors which has no analogue in a classical situation. Therefore, the simplest quantum mechanical system can be written as :

$$|\psi_{qubit}\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2.3}$$

where α, β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. In the classical scenario we can always determine whether a state is in 0 or 1 which however is not possible in quantum systems. When we measure a qubit we get either "0" with probability $|\alpha|^2$ or "1" with probability $|\beta|^2$.
The squares of the modulus of the complex numbers add to 1. Thus,equation 2.3 can be rewritten as(ignoring an insignificant factor) :

$$|\psi_{qubit}\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\omega}\sin\frac{\theta}{2}|1\rangle \tag{2.4}$$

The above gives an excellent intuitive picture as ω and θ correspond to a point on the unit 3-dimensional sphere. This sphere is known as the Bloch Sphere which forms a convenient language in quantum computation and quantum information. However, the intuition is impeded by the lack of elegant generalization to multiqubit systems.
The notion of a qubit is based on a two dimensional space spanned by $|0\rangle$ and $|1\rangle$, which can be suitably extended to higher dimensions. For example a general single qutrit spanned by $|0\rangle, |1\rangle, |2\rangle$ can be expressed as:

$$|\psi_{qutrit}\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \tag{2.5}$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

2.3 Density Matrix

The density matrix is a convenient language to describe a quantum system. If the state $|\psi_i\rangle$ occurs with a probability p_i, then the density matrix of the system can be expressed as

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{2.6}$$

ρ is a hermitian operator which is non-negative and $Tr(\rho) = 1$(Normalization condition). A state ρ is said to be pure if $Tr(\rho^2) = 1$ and mixed if $Tr(\rho^2) < 1$. Since a density matrix ρ operates on a state vector $|\psi\rangle \in H$, $\rho \in B(H)$, where

$B(H)$ denotes the set of bounded linear operators on H, which again forms a Hilbert space.

2.4 Pauli matrices

The Pauli matrices are extremely important in quantum information and computation. They are given by the following matrices:

$$X \equiv \sigma_1 \equiv \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y \equiv \sigma_2 \equiv \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$Z \equiv \sigma_3 \equiv \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.7}$$

These matrices together with the identity matrix I forms a basis in the space of single qubit density matrices. The higher dimensional extension of these matrices are the Gell-Mann matrices [BK08] which is taken up in our later chapters.

In quantum mechanics, there is an operator that corresponds to each of the three spin observables for a spin 1/2 particle i.e., the component of the angular momentum along x, y and z axes respectively. The three operators are :

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x, \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y, \quad \hat{S}_z = \frac{\hbar}{2}\sigma_z \tag{2.8}$$

2.5 Schmidt decomposition and Schmidt number

Schmidt decomposition is an important tool in quantum information science. Suppose $|\Psi\rangle$ is a pure state of a composite system AB, then there exists orthonormal states $\{|i_A\rangle\}$ for A and $\{|i_B\rangle\}$ for system B, such that [NC10]

$$|\Psi\rangle = \sum_i^k \lambda_i |i_A\rangle |i_B\rangle \tag{2.9}$$

where $\sum_i \lambda_i^2 = 1$, are known as the Schmidt coefficients which are non-negative real numbers. The number k is the Schmidt rank of the composite state. The Schmidt number for a pure state is its Schmidt rank.
Mixed states however do not admit an unique decomposition as in eq. (2.9). An extension of the Schmidt number for mixed states was provided in [TH00], stated as : A bipartite density matrix ϱ has Schmidt number k if (i) for any decomposition of ϱ with probabilities p_i and vectors $|\psi_i\rangle$, atleast one of the vectors $|\psi_i\rangle$ has atleast Schmidt rank k and (ii) there exists a decomposition of ϱ with all vectors $|\psi_i\rangle$ having Schmidt rank at most k.

2.6 Entanglement

We come to the section where we will discuss quantum entanglement. We will discuss here, entanglement in bipartite and multipartite systems, their various detection methods and one of its seminal applications namely quantum teleportation.

2.6.1 Bipartite Entanglement

Suppose a pure state $|\psi\rangle \in H$. If one can find $|\psi\rangle_A \in H_A$ and $|\psi\rangle_B \in H_B$ (H_A and H_B refer to the subsystems), such that $|\psi\rangle$ can be written as,

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \tag{2.10}$$

then $|\psi\rangle$ is known to be separable. However, states which cannot be written as in (2.10) are known as entangled states. A bipartite mixed composite system ρ is said to be in an entangled state if ρ cannot be expressed in a convex combination as

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1 \tag{2.11}$$

where ρ_i^A and ρ_i^B pertain to the respective subsystems.

2.6.2 Entanglement of three qubits

Consider first pure three-qubit states. There are two different categories of separability: the fully separable states that can be written as

$$|\chi^{fs}\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C \tag{2.12}$$

and the biseparable states that can be written as a product state in the bipartite system. A biseparable state can be formed if two of the three qubits form one group. Thus, the three possibilities are :

$$|\chi^{bs}\rangle_{A|BC} = |\alpha\rangle_A \otimes |\xi\rangle_{BC}, |\chi^{bs}\rangle_{B|AC} = |\beta\rangle_B \otimes |\xi\rangle_{AC}, |\chi^{bs}\rangle_{C|AB} = |\gamma\rangle_C \otimes |\xi\rangle_{AB}$$

A pure state is called genuine tripartite entangled if it is neither fully separable nor biseparable. Examples of such states are the Greenberger Horne Zeilinger (GHZ) state [GHZ89] and the W state [ZHG92, DVC00].
The classification for mixed states follows from [ABLS01, DC00], where a mixed state is said to be fully separable if it can be expressed as convex combination of fully separable states.

$$\rho^{fs} = \sum_i p_i |\chi_i^{fs}\rangle\langle\chi_i^{fs}| \tag{2.13}$$

Similarly, for biseparable states

$$\rho^{bs} = \sum_i p_i |\chi_i^{bs}\rangle\langle\chi_i^{bs}| \tag{2.14}$$

States which are neither fully separable nor biseparable are fully entangled. There are two important classes of fully entangled states. A fully entangled state is said to belong to the W class if can be written as a convex combination of W-type pure states,

$$\rho^{W} = \sum_i p_i |\chi_i^{W}\rangle\langle\chi_i^{W}| \tag{2.15}$$

on the other hand a state is said to belong to GHZ class if it can be written as a convex combination of GHZ-type states,

$$\rho^{GHZ} = \sum_i p_i |\chi_i^{GHZ}\rangle\langle\chi_i^{GHZ}| \tag{2.16}$$

2.6.3 Bound Entanglement

Different methods to detect entanglement actually try to find the the quantum-classical demarcation line. Entanglement is not very robust in nature. It is vulnerable against the decoherence effect of the environment.Thus it is important to analyze those entangled states which have a close proximity with the set of separable states and also inspect if some entangled states with a high quantity of entanglement can be created from weakly entangled states for the sake of quantum information processing tasks.In fact this is actually the process known as distillation of entanglement where many entangled states which possess a weak entanglement can be processed to obtain a fewer number of states having a high degree of entanglement.

Entanglement distillation [BBP+96] is the process of transforming N copies of an arbitrary entangled state ρ into approximately $S(\rho)N$ Bell pairs (where $S(\rho)$ is the von Neumann entropy of ρ), using only local operations and classical communication (LOCC)[BBPS96].Distillation [BBP+96] can in this way overcome the degenerative influence of noisy quantum channels by transforming previously shared less entangled pairs into a smaller number of maximally entangled pairs (Bell states). However there are entangled states from which no entanglement can be distilled, accordingly called bound entangled states [HHH98] and hence cannot be directly used for quantum information processing .Interestingly, entangled states that are positive under partial transpose (PPT) (i.e PPTES) were shown to be bound entangled. There are examples of bound entangled states both in bipartite[Hor97] and multipartite case [ABLS01]. Of specific importance in this regard are edge states[LKCH00] which lie at the boundary of PPT and NPT entangled states.

2.6.4 Different detection methods

Quantum entanglement in general is not easy to detect with the situation getting more involved in high dimensions. However, literature in quantum information contains various prescriptions to detect entanglement, some of which we discuss in this segment[HHHH09, GT09].

1. The positive partial transposition(PPT) Criterion : Any density matrix of a composite quantum system in a chosen basis can be written in the form

$$\rho = \sum_{i,j}^{N} \sum_{k,l}^{M} \rho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{2.17}$$

Given this decomposition the partial transposition with respect to one subsystem is given by

$$\rho^{T_A} = \sum_{i,j}^{N} \sum_{k,l}^{M} \rho_{ji,kl} |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{2.18}$$

Any bipartite separable state ρ has a positive partial transposition [Per96] , i.e $\rho^{T_A} \geq 0$.
If ρ is a state in $2 \otimes 2$ or $2 \otimes 3$ system , then ρ is a separable state iff $\rho^{T_A} \geq 0$[HHH96, Stø63, Wor76]. For higher dimensions this is not the case, i.e there are entangled states with positive partial transposition.

2. Computable Cross norm and realignment criterion[CCNR][CW03, Rud00]: For a density matrix ρ , the Schimdt decomposition can be expressed as :

$$\rho = \sum_{k} \lambda_k O_k^A \otimes O_k^B \tag{2.19}$$

where $\lambda_k \geq 0$ and O_k^A, O_k^B are orthonormal bases of the spaces H_A and H_B.
The criteria states that if the state ρ is separable , then the sum of all

λ_k in the above equation is smaller than one.

$$\sum_k \lambda_k \leq 1$$

Hence $\sum_k \lambda_k > 1$ signals that the state must be entangled.

3. The range criteria provides an approach to detect entangled states in some cases when the PPT criteria fails[Hor97].It states that if a state ρ is separable,then there is a set of product vectors $|a_i b_i\rangle$ such that the set $|a_i b_i\rangle$ spans the range of ρ as well as the set $|a_i b_i^*\rangle$ spans the range of ρ^{T_B}. Here $*$ denotes conjugation.

4. The majorization criterion gives a separation criteria based on the relation of the density matrix of the state to the density matrix of the reduced state[NK01]. Consider $P = \{p_1, p_2, ...\}$ to be the decreasingly ordered eigenvalues of ρ and $Q = \{q_1, q_2, ...\}$ the decreasingly ordered eigenvalues of the reduced density matrix ρ_A. The majorization criteria states that , if ρ is separable then

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \tag{2.20}$$

holds for all k.

5. There some strong algorithmic approaches to detect entanglement,one of them being the method of symmetric extensions [DPS02, Wer89a].

6. The PPT criteria is a part of a more general method of detection of entanglement based on positive maps. Let H_B and H_C be the Hilbert spaces and $B(H_i)$ be the space of bounded linear operators on it. Then a linear map $\Lambda : B(H_B) \to B(H_C)$ is said to be positive if it preserves positivity and hermiticity. A positive map Λ is called a completely positive map if $I_A \otimes \Lambda$ is also positive. A state ρ is separable iff for all positive maps Λ , the following relation holds [HHH96]

$$I_A \otimes \Lambda(\rho) \geq 0 \tag{2.21}$$

Another example of a positive map which is not completely positive is the reduction map [CAG99, HH99], defined on one subsystem as

$$\Lambda^r(X) = Tr(X)I - X \tag{2.22}$$

Consequently, a separable state fulfills $I_A \otimes \Lambda^r(\rho) = \rho_A \otimes I - \rho \geq 0$.

The methods underlined above constitute a small portion of a substantial literature on detection of entanglement. For a more detailed discussion one may consult [HHHH09, GT09].

2.7 Entanglement Witnesses

Entanglement witnesses constitute a very general method to distinguish entangled states from the separable ones. Since, entanglement witnesses arise from the geometry of entangled states, it can be used to characterize the set of separable states. They form a potent tool in detection of entanglement, as being hermitian,they provide experimentally viable procedures of entanglement detection. This section studies entanglement witnesses and their different methods of construction.
Entanglement witnesses [HHH96, Ter00, GT09, HHHH09] rely on the Hahn-Banach theorem. Since the set of separable states is convex and compact , one can construct a hyperplane which separates an entangled state from the set of all separable states. In fact, the entanglement witnesses provide for a strong criterion because of its completeness, i.e, a state is entangled if and only if there is a witness that detects it[HHH96].

2.7.1 Definition

Let SEP and ENT denote the set of all separable states and entangled states respectively.A hermitian operator W with atleast one negative eigenvalue will be called an entanglement witness if the following two conditions are satisfied

$$(i) Tr(W\sigma) \geq 0, \quad \forall \sigma \in SEP$$
$$(ii) \exists \rho \in ENT \quad s.t \quad Tr(W\rho) \quad < 0 \tag{2.23}$$

Therefore a negative expectation value of a witness on a state indicates the presence of entanglement. The hyperplane is constituted by all those states χ for which the expectation value of the witness is zero, i.e, represented by the equation $Tr(W\chi) = 0$. However entanglement witnesses are not universal as there does not exist a single witness that detects all entangled states. If a witness can be expressed as

$$W = P + Q^{T_B}, P \geq 0, Q \geq 0 \tag{2.24}$$

then the witness is called decomposable otherwise it is called non-decomposable or indecomposable. Decomposable witnesses cannot detect entangled states with a positive partial transpose.

2.7.2 Different methods of construction of Entanglement Witnesses

There are numerous methods to construct entanglement witnesses [GT09],some of them are enumerated below:

1. Let ρ be a state having a negative partial transposition . Let $|\eta\rangle$ be an eigenvector corresponding to a negative eigenvalue of the partial transposed state ρ^{T_A}. Then $(|\eta\rangle\langle\eta|)^{T_A}$ is a decomposable witness that detects ρ as $Tr((|\eta\rangle\langle\eta|)^{T_A}\rho) = Tr((|\eta\rangle\langle\eta|)\rho^{T_A}) < 0$.

2. When ρ is a state that violates the CCNR criterion , then $W = I - \sum_k O_k^A \otimes O_k^B$ is an entanglement witness for ρ, where O_k^A, O_k^B are the observables from the Schimdt decomposition [YL05, GMTA06].

3. The states in the neighbourhood of an entangled state may also be entangled.Witness operators can be constructed from this point of view.

Thus $W = \alpha I - |\psi\rangle\langle\psi|$ is an witness , where $\alpha = max_\sigma Tr(\sigma|\psi\rangle\langle\psi|)$ and $|\psi\rangle$ is an entangled state . The maximization is done over all separable states σ.

4. Edge states(a special class of PPT entangled states and we will define them later) was studied extensively in [LKCH00] and a non-decomposable witness was proposed exclusively for edge states δ which was of the form

$$W^\delta = P + Q^{T_B} - \varepsilon I, \quad P \geq 0, \quad Q \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0 \tag{2.25}$$

Since edge states are not of full rank neither are their partial transpose so P and Q can always be chosen as projectors from the respective kernels of δ and δ^{T_B}. ε_0 was defined as

$$\varepsilon_0 = inf_{|e,f\rangle}\langle e, f|P + Q^{T_B}|e, f\rangle \tag{2.26}$$

The above mentioned choices entailed that

$$\begin{aligned} &Tr(W^\delta\sigma) \geq 0 \quad \forall \text{ separable } \sigma \text{ and} \\ &Tr(W^\delta\delta) < 0 \end{aligned} \tag{2.27}$$

5. Geometry provides another perspective to the construction of entanglement witnesses [PR02, BNT02, BDHK05]. If χ_e is an entangled state and σ_c is the closest separable state to χ_e in the Hilbert Schmidt norm ($\|X\| = \sqrt{Tr(X^\dagger X)}$, then $\frac{1}{N}(\sigma_c - \chi_e + Tr[\sigma_c(\chi_e - \sigma_c)]I)$, with $N = \|\chi_e - \sigma_c\|$ is an entanglement witness. However, finding the nearest separable state is a computationally hard task except for some special states.

6. Indecomposable witnesses are indispensable for detecting entangled states having a positive partial transpose. Some of those constructions can be found in [CK08, CK07].

7. Semidefinite programs used to implement the algorithm for symmetric extension[DPS02] give an entanglement witness as a by product.

2.7.3 Relation between Entanglement witnesses and Positive maps

Jamiolkowski Isomorphism[Jam72]: This isomorphism gives the very important relation between entanglement witness and positive maps. Given an operator $W \in \boldsymbol{B}(H_B \otimes H_C)$ and a map ε , then the relation is

$$\varepsilon(\rho) = Tr_B(W\rho^{T_B}) \tag{2.28}$$

The following are very important observations[Lew04]:
(i) $W \geq 0$ iff ε is a completely positive map.
(ii) W is an entanglement witness iff ε is a positive map(but not completely positive).
(iii) W is a decomposable entanglement witness iff ε is decomposable.
(iv) W is a non-decomposable entanglement witness iff ε is non-decomposable and positive.

2.8 Bell's Inequality

In any discussion of entanglement witnesses, Bell's inequalities occupy an indispensable space as the inequalities are considered to be the foremost entanglement witnesses. Although originally it was Bell's reaction[Bel64] to the EPR paradox three decades earlier[EPR35], yet in the years that followed it came up with various manifestations. From the days of its inception it has constituted a major discussion point of the foundations of quantum mechanics. However, the practical implementations of the inequalities match their foundational importance. One such domain is the theory of entanglement witness.
Observers located at remote positions make measurements on entangled pairs of particles which had an interaction at some point of time. On the assumption of hidden variables, they lead to stringent bounds on the possible values of the correlation of subsequent measurements that can be obtained from the particle pairs. Bell discovered that the predictions of quantum mechanics in

certain cases, constitute a violation of these bounds.
Let us consider a simple example of a bipartite system . We assume that Alice can measure two quantities at her part, labelled A_1 and A_2, while Bob can also measure two quantities at his part,called B_1 and B_2. The results of these experiments are a_1, a_2 and b_1, b_2 and we assume that these results can take the values +1,-1. Expectation values can be obtained simply by averaging the measurement results, $\langle A_i B_j \rangle = \frac{1}{M}\sum_{k=1}^{M} a_i(k) b_j(k)$. Now the Bell-CHSH inequality [CHSH69] can be formulated as :

$$\langle A_1 B_1 \rangle + \langle A_2 B_1 \rangle + \langle A_1 B_2 \rangle - \langle A_2 B_2 \rangle \leq 2 \tag{2.29}$$

The above inequality is called a Bell's inequality.
In quantum mechanics the violation implies that the state is entangled. However the converse is not true , there are entangled states which do not violate Bell's inequality.

2.9 States which are separable from spectrum

Even though there are numerous methods to witness the signature of entanglement in quantum systems, verifying separability is a hard task[Gur03]. Thus, the set of separable states too poses significant questions. One very intriguing existence is claimed by the absolutely separable states [KŻ01] within the set of separable states. The absolutely separable states unlike other separable states remain separable under any factorization of the corresponding Hilbert space. These states are also termed as states which are separable from spectrum [Kni03]. In fact a separable state σ is separable from spectrum if $U\sigma U^\dagger$ is separable for any unitary operator U.
On the other hand states which are not absolutely separable are significant in the sense that such states can be used as initial states for entanglement creation[SKK+00, RNO+00, KC01, LL09, KRS11] using global unitary operations. As no global unitary operator can convert an absolutely separable state into an entangled state, non-absolutely separable states are important in this scenario. However, it is important to determine which states are

non-absolutely separable ,a study which we report in chapter 7.

2.10 Entanglement Measures

Quantification of entanglement occupies a very important place in quantum information science. Since an entanglement measure Q quantifies entanglement , certain properties are desirable. However all quantification procedures available do not satisfy all the desired properties. The properties are mentioned below [VPRK97, GT09, HHHH09]:

1. $Q(\sigma)$ must vanish on any separable state σ.
2. An entanglement measure should be invariant under local unitary transformations.
3. An entanglement measure should not increase on average under local operations and classical communications(LOCC).
4. A measure should decrease on mixing two or more states.
5. Requirement of additivity and full additivity i.e .,

$$Q(\rho^{\otimes n}) = nQ(\rho) \tag{2.30}$$

$$Q(\rho_1 \otimes \rho_2) = Q(\rho_1) + Q(\rho_2) \tag{2.31}$$

We now lay down few measures existing in literature[HHHH09, GT09]:

1. Concurrence [HW97, Woo98, RBC+01]: Concurrence is one of the most popular entanglement measures. For pure states it is defined as :

$$C(|\psi\rangle) = \sqrt{2[1 - Tr(\rho_A^2)]} \tag{2.32}$$

where ρ_A is the reduced state of $|\psi\rangle$ given by $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$. For a general two qubit density matrix ρ the concurrence is given as:

$$C(\rho) = max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0) \tag{2.33}$$

where λ_i are the decreasingly ordered eigenvalues of the matrix $A = \sqrt{\rho^{1/2} Y \otimes Y \rho^* Y \otimes Y \rho^{1/2}}$, Y being the Pauli matrix.

2. Negativity [VW02, ŻHSL98]: Negativity measures to what degree the PPT criterion is violated, formally written as :

$$N(\rho) = \frac{\parallel \rho^{T_A} \parallel -1}{2} \tag{2.34}$$

where $\parallel ... \parallel$ is the trace norm.

3. Relative entropy of entanglement [VP98]: The relative entropy of entanglement is given by :

$$Q(\rho) = inf_{\sigma} S(\rho \parallel \sigma) \tag{2.35}$$

where the infimum is taken over all separable states σ and $S(\rho \parallel \sigma) = Tr[\rho log(\rho) - \rho log(\sigma)]$.

4. Entanglement witnesses can be utilized to quantify entanglement. A general expression for the quantification of entanglement via witness operators is [Bra05]:

$$Q(\rho) = max[0, -min_{W \in M} Tr(W\rho)] \tag{2.36}$$

where M is the intersection of the set of entanglement witnesses with another set such that M is compact.

These were just a few of the measures . For a detailed discussion one may refer to [PV07, HHHH09].

2.11 Quantum Teleportation

The notion of separation axiom now no longer remains confined to the mere construction of entanglement witnesses. Witness operators are now being constructed to capture other manifestations of quantum states namely mixedness[MPM13],thermodynamical properties[BV04],cryptography[BHH12], discord[DVB10, BC10, GB13, GA12] etc. Consequently witness operators have gone beyond the realm of mere detection.Witnesses are now gradually being linked with various quantum information processing tasks. One such task that we probe upon in this thesis is Quantum Teleportation [BBC+93]. Quantum teleportation is a seminal information processing task where now the present challenges are to explore the experimental frontiers [BPM+97, UJA+04].We revisit the basic protocol of teleportation[NC10] in this section to facilitate our later study.

Suppose, Alice wants to send an unknown state $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. Let them share a maximally entangled state (say the Bell state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Therefore, the three party system can be written in the form,

$$\frac{1}{\sqrt{2}}[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)] \tag{2.37}$$

where the first two qubits belong to Alice and the last to Bob.

Alice sends her two qubits through the CNOT gate,to obtain the state in the form,

$$\frac{1}{\sqrt{2}}[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)] \tag{2.38}$$

Applying the Hadamard gate to the first qubit she gets

$$\frac{1}{2}[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)] \tag{2.39}$$

Regrouping the terms,

$$\frac{1}{2}[|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)] \tag{2.40}$$

Alice's measurement and Bob's state depending upon her measurement can have the following four possibilities with equal probability: $\{00, \alpha|0\rangle+\beta|1\rangle\}$, $\{01, \alpha|1\rangle+\beta|0\rangle\}$, $\{10, \alpha|0\rangle-\beta|1\rangle\}$, $\{11, \alpha|1\rangle-\beta|0\rangle\}$, wherein, the first member corresponds to Alice's measurement and the second corresponds to Bob's state.

Now, Alice has to communicate two classical bits to Bob to convey her measurement outcome.If she communicates 00 then Bob has to do nothing as his state is already in the state χ. If she communicates 01,then Bob has to apply X gate to his qubit to change it to χ. In case of the communication being 10, Bob has to apply Z gate to his qubit. In the last case Bob has to apply X gate followed by a Z gate to retrieve χ. The gates used in the protocol are transformation brought about by the following operators: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and the Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

The Bell state that is shared between Alice and Bob represents the quantum channel for teleportation. It can be replaced by other entangled states[LK00]. The protocol has been generalized to include multipartite channels[MP08, AP06] and also bipartite channels in higher dimensions. However, different channels have different efficacy represented by the fidelity of teleportation. Fidelity in turn is dictated by a parameter known as fully entangled fraction[BDSW96, HHH99a, ZLFW10]. The fully entangled fraction (FEF) [HHH99a] is defined for a bipartite state ρ in $d \otimes d$ dimensions as

$$F(\rho) = max_U \langle \psi^+ | U^\dagger \otimes I \rho U \otimes I | \psi^+ \rangle \tag{2.41}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ and U is a unitary operator. A quantum channel is useful for teleportation if it can provide a fidelity higher than what can be done classically. The fidelity (f) depends on the FEF of the state, given by $f = \frac{Fd+1}{d+1}$[HHH99a]. A state in $d \otimes d$ dimensions works as a teleportation channel if its FEF $> \frac{1}{d}$ [HHH99a, VJN00, ZLFW10]. It is an intriguing aspect that not all entangled states are useful for teleportation [ZLFW10]. So, one may want to detect entangled states which can be useful for the protocol, an investigation that we address in chapter 5.

2.12 Summary

This chapter captures the basic essentials needed to follow the later study. The physical insights that are exploited in later chapters have been discussed here with the corresponding mathematical aspects to maintain the logical pedagogy. However there remain some finer notions which will be elucidated in the later discussions as and when necessary. Entanglement, being such an important player in quantum information processing tasks, its detection remains a major issue which we address in the next chapter. More so, for those entangled states admitting a positive partial transpose, as they remain unnoticed by most standard detection procedures including Bell's inequalities [Bel64, CHSH69].

Chapter 3

Entanglement Witness Operator for edge states

3.1 Prelude

Quantum entanglement continues to have an enigmatic presence since the days of its inception. Towards the end of the last century and with the turn of the century the theoretical proposals for possible applications of entanglement[EPR35, Sch35] increased manifolds. This brought a sea change in quantum computation and information with the advent of quantum computation [BDEJ95] , teleportation [BBC+93].Experimental challenges [BPM+97] broadened the frontiers of both informational and computational tasks.

The significance of entanglement in state of the art technologies motivated scientists to take up the challenge of its detection[GT09, HHHH09], more so incited by the paradoxical behaviour. For low dimensional ($2 \otimes 2$ and $2 \otimes 3$) states there exist simple necessary and sufficient conditions for separability [Per96, HHH96] which is based on the fact that separable states have a positive partial transpose (PPT). However, in higher dimensional systems all states with negative partial transpose(NPT) are entangled but there are entangled states which have a positive partial transposition [Hor97, BDM+99].

Thus the separability problem can be framed as finding whether states with positive partial transposition are entangled. Of specific importance in this context are the so called edge states [LKCH00] which lie at the boundary of PPT and NPT states. An interesting character that an edge state shows is extreme violation of the range criterion [Hor97] which states that there exists no product vector $|e, f\rangle$ belonging to the range of the edge state ϱ such that $|e, f^*\rangle$(conjugation is done with respect to the second system) belongs to the range of ϱ^{T_B}. Since the edge states are PPT entangled states so partial transposition method fails to detect them and also it is very difficult to identify the edge states by range criterion. So it becomes necessary to find an alternate method to detect the edge states.

Since, standard detection procedures complicates the detection of PPT entangled states, witnesses can be pragmatically used for the purpose. However, witnesses which are non-decomposable in nature are of prime importance in this scenario as decomposable witnesses fail to identify PPT entangled states as the partial transposition is itself decomposable[HHH96]. Terhal first introduced non-decomposable positive linear maps based on entangled quantum states using the notion of unextendible product basis[Ter01]. Thereafter Lewenstein *et. al.* worked extensively on non-decomposable witnesses and provided elegant prescriptions for their optimization[LKCH00]. In this chapter we propose a new non-decomposable witness for edge states and make a comparative analysis of it with other known witness. We also provide an insight as to how an experimental realization can be done of our proposed witness with illustrations of the action of the operator from different dimensions.

The chapter follows the following structure: In section 3.2 we cite certain related definitions and terms. In section 3.3 we revisit the non-decomposable witness operator and find the condition for which it is finer. In section 3.4 we give the construction of the witness, its extension to multipartite edge states and discuss its experimental realization. In section 3.5 we compare our proposed witness with that in [LKCH00]. In section 3.6 we provide explicit examples. Lastly we end with conclusions.

3.2 A few definitions and results

Definition-1: The kernel of a given density matrix $\rho \in B(H_A \otimes H_B)$ is defined as the set of all eigenvectors corresponding to the zero eigenvalue of ρ in the Hilbert space $H_A \otimes H_B$. Mathematically, $ker(\rho) = \{|x\rangle \in H_A \otimes H_B : \rho|x\rangle = 0\}$.
Definition-2: A PPT entangled state δ is called an edge state if for any $\varepsilon > 0$ and any product vector $|e, f\rangle$, $\delta' = \delta - \varepsilon|e, f\rangle\langle e, f|$ is not a PPT state. where P and Q are positive semi-definite operators.
Definition-3: Given two entanglement witnesses W_1 and W_2, a witness W_1 is said to be finer than another witness W_2 if $D_{W_2} \subseteq D_{W_1}$, where the set D_W is defined as $D_W = \{\rho \geq 0, \text{ such that } Tr(W\rho) < 0\}$.
Result-1: Given two non-decomposable witnesses W_1 and W_2, W_1 is finer than W_2, if W_2 can be written as [LKCH00]

$$W_2 = (1 - \lambda)W_1 + \lambda D \tag{3.1}$$

where D is a decomposable witness operator and $0 \leq \lambda < 1$.
Result-2: A witness operator D is decomposable iff [LKCH00]

$$Tr(D\rho) \geq 0, \quad \text{for PPT entangled state } \rho \tag{3.2}$$

3.3 Revisiting non-decomposable witnesses

Lewenstein et. al. [LKCH00] studied the edge states extensively and introduced a non-decomposable witness exclusively for edge states δ which was of the form

$$W^\delta = P + Q^{T_B} - \varepsilon I, \quad P \geq 0, \quad Q \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0 \tag{3.3}$$

Since edge states are not of full rank neither are their partial transpose,so P and Q can always be chosen as projectors from the respective kernels of δ

and δ^{T_B}. ε_0 was defined as

$$\varepsilon_0 = inf_{|e,f\rangle}\langle e,f|P + Q^{T_B}|e,f\rangle \tag{3.4}$$

The above mentioned choices entailed that

$$\begin{aligned} &Tr(W^{\delta}\sigma) \geq 0 \ \ \forall \ \text{ separable } \ \sigma \ \text{ and} \\ &Tr(W^{\delta}\delta) < 0 \end{aligned} \tag{3.5}$$

When we are willing to detect PPT entangled states which are not edge states through the witness operator W^{δ} then in this situation the task becomes very difficult in choosing the positive semi-definite operators P and Q. This is because of the fact that the given PPT entangled state ρ (not edge state) or the state described by its partial transposition can be of full rank. Therefore the detection of PPT entangled state (excluding edge states) using W^{δ} turned out to be a difficult task. So our focus should be on searching the witness operator which can be easily constructed and also detects PPT entangled state together with edge states. We start our search by considering a PPT entangled state ρ. Next we impose two assumptions on ρ:

A1: The PPT entangled state ρ is not an edge state.

A2: ρ is not of full rank but ρ^{T_B} is.

With these assumptions, P can be chosen as mentioned earlier i.e. P can be chosen as a projector on the kernel of ρ.Since there are no vectors in the kernel of ρ^{T_B}(ρ^{T_B} is of full rank), we take Q as a null operator. These choices of P and Q reduces W^{δ} to W^{ρ}, which is given by

$$W^{\rho} = P - \varepsilon' I, \quad P > 0, \quad 0 < \varepsilon' \leq \varepsilon_1 \tag{3.6}$$

where

$$\varepsilon_1 = inf_{|e,f\rangle}\langle e,f|P|e,f\rangle \tag{3.7}$$

Thus, the PPT entangled state ρ which satisfies the above mentioned assumptions can be detected by the non-decomposable witness operator W^{ρ}.

Next our task is to show that W^ρ is finer than W^δ.
The result given in (3.1) clearly authenticates that the witness operator (3.6) is finer than its counterpart (3.3) because (3.3) can be written as

$$W^\delta = (1-\lambda)W^\rho + \lambda D, \quad 0 \le \lambda < 1 \tag{3.8}$$

taking $D = Q^{T_B}$.
Thus, W^ρ gives us a more general entanglement witness which can detect some PPT entangled states along with edge states, or in other words, W^ρ is finer than W^δ.
Illustration: As an illustration we consider the PPT entangled state [HHH99b]

$$\rho_\alpha = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{7}\rho_+ + \frac{5-\alpha}{7}\rho_- \tag{3.9}$$

where

$$\begin{aligned}\rho_+ &= \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)\\ \rho_- &= \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)\\ |\psi^+\rangle &= \frac{1}{\sqrt{3}}\sum_{i=0}^{2}|ii\rangle\end{aligned} \tag{3.10}$$

The state is PPT entangled for $3 < \alpha \le 4$ and edge state for $\alpha = 4$. The rank of ρ_α is 7 whereas the rank of $\rho_\alpha^{T_B}$ is 9. Now using the prescription described above for the construction of the witness operator (3.6), we can easily construct the witness operator for the PPT entangled state ρ_α as

$$W^{\rho_\alpha} = \begin{pmatrix} 1-\varepsilon' & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon' & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2-\varepsilon' & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon' & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1-\varepsilon' \end{pmatrix} \tag{3.11}$$

We observe that $Tr(W^{\rho_\alpha}\rho_\alpha) = -\varepsilon' < 0$.

3.4 Construction of the witness and its experimental realization

In this section we propose a new non-decomposable witness operator and thereafter show that it is an operator which detects the edge states. Also we study its extension in the multipartite system and further discuss its experimental realization.

Theorem 3.1. *An operator W is a non-decomposable witness operator for an edge state δ if it can be expressed in the form*

$$W = Q^{T_B} - k(I - P) \tag{3.12}$$

where P is a positive semi-definite operator and Q is a positive definite operator and T_B denotes the partial transposition over the second subsystem.

Proof: To prove that W is a non-decomposable witness operator for an edge state δ, it is sufficient to verify the two witness inequalities given in (2.23) for W.

(i) We have to show that $Tr(W\sigma) \geq 0 \quad \forall$ separable state σ.

$$\begin{aligned} Tr(W\sigma) &= Tr((Q^{T_B} - k(I-P))\sigma) \\ &= Tr(Q\sigma^{T_B}) - k(1 - Tr(P\sigma)) \quad (\text{since } \; Tr(Q^{T_B}\sigma) = Tr(Q\sigma^{T_B})) \\ &= ((1 - Tr(P\sigma)))[\frac{Tr(Q\sigma^{T_B})}{(1 - Tr(P\sigma))} - k] \end{aligned} \tag{3.13}$$

We can always select a value of k from the interval $0 < k \leq k_0$ so that $Tr(W\sigma) \geq 0$, where k_0 is given by

$$k_0 = \min \frac{Tr(Q\sigma^{T_B})}{1 - Tr(P\sigma)} \tag{3.14}$$

Here the minimum is taken over all separable states σ.
(ii) Now it remains to be shown that $Tr(W\delta) < 0$ for an edge state δ. Since δ and δ^{T_B} have some vectors in their kernel so we get some freedom to choose the operators P and Q as the projectors on $ker(\delta)$ and $ker(\delta^{T_B})$ respectively. Therefore, we have $Tr(P\delta) = 0$ and $Tr(Q\delta^{T_B}) = 0$.

$$\begin{aligned} Tr(W\delta) &= Tr(Q^{T_B}\delta) - kTr((I-P)\delta) \\ &= Tr(Q\delta^{T_B}) - k(1 - Tr(P\delta)) \\ &= -k \end{aligned} \tag{3.15}$$

Now using the inequality $0 < k \leq k_0$ and exploiting equations (3.14) and (3.15), we find that $Tr(W\delta) < 0$. Hence we are able to prove that the non-decomposable witness operator proposed in the theorem detects an edge state.
Corollary: The non-decomposable witness can also be constructed as

$$W' = P - k(I - Q^{T_B}), \quad 0 < k \leq k_0, \quad P > 0, \quad Q \geq 0 \tag{3.16}$$

where

$$k_0 = \min_\sigma \frac{Tr(P\sigma)}{1 - Tr(Q\sigma^{T_B})} \tag{3.17}$$

With similar arguments it can be shown that W' also detects edge states. Particularly if $Q^{T_B} = 0$, i.e. if the state described by the partially transposed density operator has no vectors in its kernel then witness operator (3.16) reduces to (3.6). Hence in this case the witness operator (3.16) detects not only edge states but also other PPT entangled states.

Extension of the witness for edge states in 3 qubits: Since edge states are also found in tripartite systems so we extend the prescription of our proposed entanglement witness operator in 3-qubit systems.

For a given tripartite edge state $\delta_{tri} \in B(H_1 \otimes H_2 \otimes H_3)$, we define the non-decomposable witness operator as:

$$W_{tri} = Q^{T_X} - k_0(I - P), \quad X = 1, 2, 3 \tag{3.18}$$

P=Projector on ker(δ_{tri}) and Q= Projector on ker($\delta_{tri}^{T_X}$), where T_X denotes the transpose taken with respect to any one of the subsystems. As before we define

$$k_0 = \min \frac{Tr(Q^{T_X}\sigma)}{Tr((I - P)\sigma)} \tag{3.19}$$

where the minimum is taken over all separable states σ.

If now we take $0 < k \leq k_0$ and use $W_{tri} = Q^{T_X} - k(I - P)$, then we obtain

$$Tr(W_{tri}\delta_{tri}) = -k < 0 \tag{3.20}$$

For the above choice of k_0 given in (3.19),we can always find some k for which $Tr(W_{tri}\sigma) \geq 0$.

Experimental Realization: Our task is now to show that our proposed witness operator can be used in an experimental setup to detect the edge state in a qutrit system.The quantity to be measured is the expectation value

$$\langle W \rangle = Tr(W\rho) \tag{3.21}$$

Here we rewrite the witness operator defined in (3.12) for a certain edge state in a qutrit system in terms of Gell-Mann matrices [BK08] and thereby finding the expectation value of these physical operators in order to experimentally detect entanglement.

The generalized Gell-Mann matrices are higher dimensional extensions of the Pauli matrices (for qubits) and are hermitian and traceless. They form an orthogonal set and basis. In particular, they can be categorized for qutrits as three different types of traceless matrices :

(i) three symmetric Gell-Mann matrices

$$\Lambda_s^{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_s^{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_s^{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.22}$$

(ii) three antisymmetric Gell-Mann matrices

$$\Lambda_a^{01} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_a^{02} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\Lambda_a^{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{3.23}$$

(iii) two diagonal Gell-Mann matrices

$$\Lambda^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^1 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix} \tag{3.24}$$

Let us consider a qutrit described by the density operator (3.9). Our prescribed witness (3.12) for the state with $\alpha = 4$ is given in matrix form as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -k-2 \\ 0 & 1-k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4-k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-k & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k-2 \\ 0 & 0 & 0 & 0 & 0 & 1-k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4-k & 0 \\ -k-2 & 0 & 0 & 0 & -k-2 & 0 & 0 & 0 & k \end{pmatrix} \tag{3.25}$$

Writing the witness A in terms of the Gell-Mann matrices and taking the expectation value we obtain,

$$\begin{aligned} \langle A \rangle &= -\langle \Lambda_s^{01} \otimes \Lambda_s^{01} \rangle + \langle \Lambda_a^{01} \otimes \Lambda_a^{01} \rangle - \frac{k+2}{2}(\langle \Lambda_s^{02} \otimes \Lambda_s^{02} \rangle - \langle \Lambda_a^{02} \otimes \Lambda_a^{02} \rangle) \\ &-\frac{k+2}{2}(\langle \Lambda_s^{12} \otimes \Lambda_s^{12} \rangle - \langle \Lambda_a^{12} \otimes \Lambda_a^{12} \rangle) + \frac{2k-5}{4}\langle \Lambda^0 \otimes \Lambda^0 \rangle \\ &-\frac{9}{4\sqrt{3}}(\langle \Lambda^0 \otimes \Lambda^1 \rangle - \langle \Lambda^1 \otimes \Lambda^0 \rangle) + \frac{22k-45}{36}\langle \Lambda^1 \otimes \Lambda^1 \rangle \\ &-\frac{k}{9}(\langle \Lambda^1 \otimes I \rangle + \langle I \otimes \Lambda^1 \rangle) + \frac{15-5k}{9}\langle I \otimes I \rangle \end{aligned} \tag{3.26}$$

Thus for an experimental outcome $\langle A \rangle < 0$, the state is entangled.
For qutrits the Gell-Mann matrices can be expressed in terms of eight physical operators , the observables $S_x, S_y, S_z, S_x^2, S_y^2, S_z^2, \{S_x, S_y\}, \{S_y, S_z\}, \{S_z, S_x\}$ of a spin-1 system , where $\overrightarrow{S} = \{S_x, S_y, S_z\}$ is the spin operator and $\{S_i, S_j\} = S_iS_j + S_jS_i$ (with $i, j = x, y, z$) denotes the corresponding anticommutator. The representation of the Gell-Mann matrices in terms of the spin-1 operators

is as follows [BK08]:

$$
\begin{aligned}
\Lambda_s^{01} &= \frac{1}{\sqrt{2}\hbar^2}(\hbar S_x + \{S_z, S_x\}), \qquad \Lambda_s^{02} = \frac{1}{\hbar^2}(S_x^2 - S_y^2), \\
\Lambda_s^{12} &= \frac{1}{\sqrt{2}\hbar^2}(\hbar S_x - \{S_z, S_x\}), \qquad \Lambda_a^{01} = \frac{1}{\sqrt{2}\hbar^2}(\hbar S_y + \{S_y, S_z\}), \\
\Lambda_a^{02} &= \frac{1}{\hbar^2}\{S_x, S_y\}, \qquad \Lambda_a^{12} = \frac{1}{\sqrt{2}\hbar^2}(\hbar S_y - \{S_y, S_z\}), \\
\Lambda^0 &= 2I + \frac{1}{2\hbar^2}(\hbar S_z - 3S_x^2 - 3S_y^2), \\
\Lambda^1 &= \frac{1}{\sqrt{3}}(-2I + \frac{3}{2\hbar^2}(\hbar S_z + S_x^2 + S_y^2))
\end{aligned} \tag{3.27}
$$

All eight physical operators can be represented by the following matrices :

$$
\begin{aligned}
S_x &= \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
S_z &= \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, S_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\
S_y^2 &= \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \{S_x, S_y\} = \hbar^2\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\{S_y, S_z\} &= \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \{S_z, S_x\} = \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{3.28}
$$

Therefore experimental detection of entanglement can also be done by writing the Gell-Mann matrices in terms of spin-1 operators and then taking the expectation value.

3.5 Detection and Comparative analysis

Consider the two witness operators W^{δ} and W given in (3.3) and (3.12) respectively and let us investigate the situation when W^{δ} detects a larger set of PPT entangled state than W or vice-versa. Also we observe that

$$D_W \cap D_{W^{\delta}} \neq \phi \tag{3.29}$$

Equation (3.29) depicts the fact that there exist PPT entangled states which are detected by both W and W^{δ}.

Case-I: If the entanglement witness W be finer than W^{δ} then using (3.1), we can always write

$$\begin{aligned} & W^{\delta} = (1-\lambda)W + \lambda D \\ & \Rightarrow P + Q^{T_B} - \varepsilon I = (1-\lambda)(Q^{T_B} - k(I-P)) + \lambda D \\ & \Rightarrow D = \frac{1-k+\lambda k}{\lambda} P + Q^{T_B} + \frac{k-\varepsilon-\lambda k}{\lambda} I \end{aligned} \tag{3.30}$$

From (3.30) and using the Result-2, we get

$$1-k+\lambda k \geq 0, \qquad k-\varepsilon-\lambda k \geq 0 \tag{3.31}$$

which gives

$$k \leq \frac{1}{1-\lambda}, \qquad k \geq \frac{\varepsilon}{1-\lambda} \tag{3.32}$$

Thus W is finer than W^{δ} when $k \in [\frac{\varepsilon}{1-\lambda}, \frac{1}{1-\lambda}]$.

Case-II: If the entanglement witness W^{δ} be finer than W then we can proceed in similar way as above and find that W^{δ} is finer than W when $k \in [1-\lambda, \varepsilon - \lambda\varepsilon]$.

3.6 Examples

In this section we explicitly construct our proposed witness operator for different edge states living in $C^3 \otimes C^3$ and $C^2 \otimes C^2 \otimes C^2$ and express them in the matrix form.

Example 1: We start with the edge state in $C^3 \otimes C^3$ as proposed in [Hor97]. The state and its partial transpose is :

$$\rho_a = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix} \tag{3.33}$$

$$\rho_a^{T_B} = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix} \tag{3.34}$$

where $0 < a < 1$.
The projector on the kernel of ρ_a is:

$$\begin{aligned}P = |00\rangle\langle 00| + c|00\rangle\langle 20| - |00\rangle\langle 22| + c|20\rangle\langle 00| + \\ c^2|20\rangle\langle 20| - c|20\rangle\langle 22| - |22\rangle\langle 00| - c|22\rangle\langle 20| + \\ |22\rangle\langle 22| + |11\rangle\langle 11| + c|11\rangle\langle 20| - |11\rangle\langle 22| \\ +c|20\rangle\langle 11| + c^2|20\rangle\langle 20| - c|20\rangle\langle 22| - |22\rangle\langle 11| \\ -c|22\rangle\langle 20| + |22\rangle\langle 22|\end{aligned} \tag{3.35}$$

The partial transpose of the projector on the kernel of $\rho_a^{T_B}$ is:

$$\begin{aligned}Q^{T_B} = d^2|02\rangle\langle 02| - d^2|00\rangle\langle 22| - d|02\rangle\langle 22| - d^2|22\rangle\langle 00| \\ +d^2|20\rangle\langle 20| + d|22\rangle\langle 20| - d|22\rangle\langle 02| + d|20\rangle\langle 22| \\ +|22\rangle\langle 22| + |12\rangle\langle 12| - |11\rangle\langle 22| - |22\rangle\langle 11| \\ +|21\rangle\langle 21| + |01\rangle\langle 01| - |00\rangle\langle 11| - |11\rangle\langle 00| + |10\rangle\langle 10|\end{aligned} \tag{3.36}$$

where $c = \frac{\sqrt{1-a^2}}{1+a}$ and $d = \frac{\sqrt{1-a^2}}{a-1}$. Thus the witness is obtained as :

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & ck & 0 & -(d^2+k) \\ 0 & 1-k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d^2-k & 0 & 0 & 0 & 0 & 0 & -d \\ 0 & 0 & 0 & 1-k & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & ck & 0 & -1-k \\ 0 & 0 & 0 & 0 & 0 & 1-k & 0 & 0 & 0 \\ ck & 0 & 0 & 0 & ck & 0 & 2c^2k+d^2-k & 0 & d-2ck \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-k & 0 \\ -(d^2+k) & 0 & -d & 0 & -1-k & 0 & d-2ck & 0 & 1+k \end{pmatrix} \tag{3.37}$$

Using W as constructed in (3.37) we obtain,

$$Tr(W\rho_a) = -k < 0 \tag{3.38}$$

Example 2: Next we construct the witness for the edge state in 3 qubits proposed in [ABLS01]. The edge state was proposed as:

$$\delta_{tri} = \frac{1}{n}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.39}$$

where $n = 2 + a + b + c + 1/a + 1/b + 1/c$ and the basis is taken in the order $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$.The partial transpose with respect to system C is given by:

$$\delta_{tri}^{T_C} = \frac{1}{n}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.40}$$

The vector in the kernel of δ_{tri} is $|000\rangle - |111\rangle$ and the vector in the kernel of $\delta_{tri}^{T_C}$ is $|001\rangle - a|110\rangle$. With these vectors the witness (3.18) is obtained as

:

$$W_{tri} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k-a \\ 0 & 1-k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2-k & 0 \\ -k-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.41}$$

which gives,

$$Tr(W_{tri}\delta_{tri}) = -k < 0 \tag{3.42}$$

3.7 Summary

PPT entangled states share a close proximity with separable states and thereby go unnoticed by most detection procedures. Even in the class of entanglement witnesses,not all witnesses can detect them. We need non-decomposable operators to detect them.Significantly, as entanglement witnesses are not universal in nature,what one operator detects goes undetected by another witness. Therefore,different considerations give birth to different witnesses.

In this chapter, we have prescribed a non-decomposable operator for the detection of edge states.The proposed witness operator is interesting in the sense that it conditionally detects a larger set of PPT entangled state than the non-decomposable witness operator given by Lewenstein et.al.[LKCH00]. However,the fact gets reversed under a different set of conditions. The experimental relevance of the witness further substantiates its construction.

Together as newer methods of detection arise to increasingly recognize entangled states,the geometry of the set of entangled states is also being probed upon.The set of entangled states is not convex i.e a convex combination of two entangled states might not be convex. However,some entangled states combine to form entangled states. A witness that detects the two entangled states at the endpoints might be able to detect all the states in between. This also increases by manifold the number of entangled states detected by a single witness.However,it takes some examination before we construct such common witnesses, an investigation that we report in the next chapter.

Chapter 4

Common Entanglement Witnesses

4.1 Prelude

Detection of quantum entanglement is a rigorous procedure as vindicated by the observation that the separability problem is NP-hard [Gur03].The most involved part in any detection procedure of quantum entanglement is to have a uniform conclusive result for the set of all separable states. This is further aggravated by the complexity of the geometry of the entangled states as it is not convex. Although entanglement witnesses provide a pragmatic detection procedure for entangled states, yet its construction is a difficult task. However, several methods have been suggested in literature [GT09, HHHH09, Ter00, DPS04, SV09].

The notion of entanglement witnesses was further extended to Schmidt number witnesses [SBL01, TH00, SV11]. This is a significant approach as mixed states do not admit a unique Schmidt decomposition.Further, since Schmidt number is an indicator of the amount of entanglement present in a state, Schimdt number witnesses contribute to entanglement quantification.

Since,entanglement witnesses are not universal, one pertinent line of study is to gauge the number of states that can be detected by a single witness. Optimization of witnesses[LKCH00] is a significant approach in this direction. However, another approach is the search for common witnesses for different entangled states. It was proved by Wu and Guo [WG07] that for a given pair of entangled states ρ_1 and ρ_2, a common EW exists if and only if $\lambda\rho_1 + (1-\lambda)\rho_2$ is an entangled state $\forall\lambda \in [0, 1]$. They thus arrived at a sufficient condition for entanglement for pairs of entangled states. Construction of a common entanglement witness for two entangled states not only detects them but also any state which is a convex combination of the two. Thereby one can detect a large class of entangled states if one is able to find a common EW satisfying the above criterion.

In the present chapter our motivation is to propose some methods to construct common EW for certain classes of states making use of the above condition of existence. We first propose some characteristics of common Schmidt number witnesses based on the analysis of common entanglement witnesses, providing suitable examples for our propositions. We then suggest schemes for finding common EW for various categories of states based on their spectral characteristics. The distinction between a common decomposable witness operator and a non-decomposable one is of relevance in the probe for finding common EW. A decomposable operator is unable to detect a PPTES (positive partially transposed entangled state), whereas a non-decomposable witness can successfully detect a PPTES. This distinction propagates to a common witness. Precisely, if one of the entangled states in a convex combination is a PPTES, then the common witness is non-decomposable. Our analysis makes use of some decomposable and nondecomposable witnesses. We illustrate our results through various appropriate examples from qutrit systems. The chapter is organized as follows. In section 4.2, we propose and study some features of common Schmidt number witnesses. Next, in section 4.3 we suggest methods to detect a combined pair of entangled states and construct the common EW for them. We then provide explicit examples demonstrating our methods for finding common entanglement witnesses in section 4.4. In section 4.5, we distinguish between a common decomposable and a non-decomposable witness operator citing examples. We conclude with a brief summary of our results in section 4.6.

4.2 Common Schmidt number witness

Consider the space $B(H^N \otimes H^M)$, with $N < M$. Define S_k to be the set of states whose Schmidt number is $\leq k$. Thus, S_1 is the set of separable states and the different states share the relation $S_1 \subset S_2 \subset S_3.... \subset S_k.. \subset S_N$ and are convex[SBL01, TH00].

A k Schmidt witness(kSW),W^S is defined as [SBL01, TH00, SV11]

$$Tr(W^S\sigma) \geq 0, \quad \forall \sigma \in S_{k-1} \tag{4.1}$$

$$Tr(W^S\rho) < 0 \quad for \quad at \quad least \quad one \quad \rho \in S_k \tag{4.2}$$

A well-known example of a kSW is $I - \frac{m}{k-1}P$ [SBL01] where m and k respectively denote the dimension and Schmidt number and P is a projector on $\frac{1}{\sqrt{m}}\Sigma_{i=0}^{m-1}|ii\rangle$.

Proposition-I: Suppose ρ_1 and ρ_2 are Schmidt number k states. If there exists a common kSW for ρ_1 and ρ_2, then the Schmidt number of their convex combination will also be k. In other words the Schmidt number of $\lambda\rho_1 + (1-\lambda)\rho_2$ is also k ($\lambda \in [0,1]$).

Proof: Since ρ_1,ρ_2 are in S_k and S_k is convex , $\lambda\rho_1 + (1-\lambda)\rho_2$ cannot have a Schmidt number $> k$.

Now, let W^S be the common kSW for ρ_1 and ρ_2 . As a result

$$Tr(W^S(\lambda\rho_1 + (1-\lambda)\rho_2)) = \lambda Tr(W^S\rho_1) + (1-\lambda)Tr(W^S\rho_2) < 0 \tag{4.3}$$

since $Tr(W^S\rho_1) < 0$, $Tr(W^S\rho_2) < 0$. Thus the Schmidt number of $\lambda\rho_1 + (1-\lambda)\rho_2$ is also k.

Proposition-II: Suppose δ_1 and δ_2 are two states with Schmidt number (SN) k_1 and k_2 respectively where $k_1 > k_2$. Then a common witness W_k, if it exists, will be of class k, where $k \leq min(k_1, k_2)$.

Proof: It follows from the definition of Schmidt number witness that there exists a k_1SW, W_{k_1} for which $Tr(W_{k_1}\delta_1) < 0$, but $Tr(W_{k_1}\delta_2) \geq 0$. Therefore a common witness if it exists should be of class k where $k \leq min(k_1, k_2)$.

Example-I: Convex combination of two pure SN 3 states
Consider the states $|\Phi_1\rangle = a|00\rangle + b|11\rangle + \sqrt{1-a^2-b^2}|22\rangle$ and $|\Phi_2\rangle = x|00\rangle + y|11\rangle + \sqrt{1-x^2-y^2}|22\rangle$. A $3SW$ of the form $W^{S3} = I - \frac{3}{2}P$ detects both states for many ranges of a, b, c, x, y, z (one such range is $0.25 \leq a \leq 0.65, 0.25 \leq b \leq 0.65, 0.25 \leq x \leq 0.65, 0.25 \leq y \leq 0.65$). Therefore, for those ranges, W^{S3} is a common witness for the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ and thus their convex combination will have SN 3. (P is a projector on $\frac{1}{\sqrt{3}}\Sigma_{i=0}^{2}|ii\rangle$)

Example-II: Convex combination of a pure SN 3 state and a pure SN 2 state
Consider now the state $|\Phi_1\rangle = a|00\rangle + b|11\rangle + \sqrt{1-a^2-b^2}|22\rangle$ and $|\chi\rangle = t|00\rangle + \sqrt{1-t^2}|11\rangle$. Here a $2SW$ of the form $W^{S2} = I - 3P$ detects both of them whereas the previous $3SW$ fails to qualify as a common witness.

Example-III: Convex combination of a mixed state and a pure SN 2 state
Consider the two-qutrit isotropic state $\Omega = \alpha P + \frac{1-\alpha}{9}I$ with $(-\frac{1}{8} \leq \alpha \leq 1)$. The $2SW$, W^{S2} detects it $\forall 1 \geq \alpha > \frac{1}{4}$, which is exactly the range for which the isotropic state is entangled. As a result, the $2SW$ detects $\lambda\Omega + (1-\lambda)|\chi\rangle\langle\chi| (\lambda \in [0, 1])$.

4.3 Methods to construct common entanglement witness

Case-I: Let us consider that the two states described by the density operators ρ_1 and ρ_2 be negative partial transpose (NPT) states. Let us further assume that the two sets S_1 and S_2 consist of the set of all eigenvectors of $\rho_1^{T_A}$ and $\rho_2^{T_A}$ corresponding to their negative eigenvalues. In set builder notation, S_1 and S_2 can be expressed as $S_1 = \{|x\rangle : \rho_1^{T_A}|x\rangle =$

$\lambda_-|x\rangle$, λ_- is a negative eigenvalue of $\rho_1^{T_A}\}$ and $S_2 = \{|y\rangle : \rho_2^{T_A}|y\rangle = \alpha_-|y\rangle$, α_- is a negative eigenvalue of $\rho_2^{T_A}\}$. Now we propose the following theorem:

Theorem 4.1. *If $S_1 \cap S_2 \neq \phi$, then there exists a common witness detecting not only ρ_1 and ρ_2 both but also all the states lying on the straight line joining ρ_1 and ρ_2*

Proof: Let $S_1 \cap S_2 \neq \phi$. Then there exists a non-zero vector $|\eta\rangle \in S_1 \cap S_2$. Let $W = (|\eta\rangle\langle\eta|)^{T_A}$. This gives

$$Tr(W\rho_1) = Tr((|\eta\rangle\langle\eta|)^{T_A}\rho_1) = Tr((|\eta\rangle\langle\eta|)\rho_1^{T_A}) < 0 \tag{4.4}$$

With similar justifications,

$$Tr(W\rho_2) < 0 \tag{4.5}$$

If now we consider $\rho = \lambda\rho_1 + (1-\lambda)\rho_2, \lambda \in [0,1]$, then $Tr(W\rho) < 0$. Hence the theorem.

Case-II: Let δ_1 and δ_2 be two edge states. We know that a witness operator of the form $W_{edge} = P + Q^{T_A} - \varepsilon I$ can detect an edge state δ if P is a projector on $\ker(\delta)$ and Q a projector on $\ker(\delta^{T_A})$ and $0 < \varepsilon \leq \varepsilon_0 = inf_{|e,f\rangle}\langle e,f|P + Q^{T_A}|e,f\rangle$ where $|e,f\rangle$ is a product vector [LKCH00]. Thus we propose:

Theorem 4.2. *W_{edge} can detect both δ_1 and δ_2 if $\mathtt{dim}(\mathtt{ker}(\delta_1) \cap \mathtt{ker}(\delta_2)) > 0$ or $\mathtt{dim}(\mathtt{ker}(\delta_1^{T_A}) \cap \mathtt{ker}(\delta_2^{T_A})) > 0$.*

Proof: Let $\mathtt{dim}(\mathtt{ker}(\delta_1) \cap \mathtt{ker}(\delta_2)) > 0$, i.e., there exists at least one non-zero eigenvector $|a\rangle \in \mathtt{ker}(\delta_1) \cap \mathtt{ker}(\delta_2)$. We assume $P = |a\rangle\langle a|$.
Further, let $\mathtt{dim}(\mathtt{ker}(\delta_1^{T_A}) \cap \mathtt{ker}(\delta_2^{T_A})) > 0$. We take $|b\rangle \in \mathtt{ker}(\delta_1^{T_A}) \cap \mathtt{ker}(\delta_2^{T_A})$. Assume $Q = (|b\rangle\langle b|)^{T_A}$. On taking $W_{edge} = P + Q^{T_A} - \varepsilon I$ with the above mentioned definition of ε , we obtain

$$Tr(W_{edge}\delta_1) < 0 \quad \text{and} \quad Tr(W_{edge}\delta_2) < 0 \tag{4.6}$$

Consequently, W_{edge} detects $\delta = \lambda\delta_1 + (1-\lambda)\delta_2$ for $0 \leq \lambda \leq 1$ since

$$Tr(W_{edge}\delta) = Tr(W_{edge}(\lambda\delta_1 + (1-\lambda)\delta_2)) < 0 \tag{4.7}$$

Thus W_{edge} is a common witness for δ_1 and δ_2 and detects any convex combination of δ_1 and δ_2.

Case-III: Let δ^1_{tri} and δ^2_{tri} be two tripartite edge states. Using (3.18) we have the following theorem:

Theorem 4.3. *The witness W_{tri} can detect both the tripartite edge states δ^1_{tri} and δ^2_{tri} if $\mathbf{dim}(\mathbf{ker}(\delta^1_{tri}) \cap \mathbf{ker}(\delta^2_{tri})) > 0$ or $\mathbf{dim}(\mathbf{ker}((\delta^1_{tri})^{T_X}) \cap \mathbf{ker}((\delta^2_{tri})^{T_X})) > 0$. Here T_X represents the transposition with respect to any one of the subsystems.*

Proof: Proof is similar to Theorem 4.2.

4.4 Examples from Qutrit systems

Here, we exemplify the methods to find common entanglement witnesses as laid down in section 4.3 for the different classifications.

Example 1: Let us consider the following states in $C^3 \otimes C^3$: $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$, where $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\psi_2\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. On observation we find an eigenvector $|e_-\rangle = |01\rangle - |10\rangle$ common to $\rho_1^{T_A}$ and $\rho_2^{T_A}$ corresponding to their respective negative eigenvalues. On defining $U = |e_-\rangle\langle e_-|$ and $W = U^{T_A}$, we obtain $Tr(W\rho_1) < 0$ and $Tr(W\rho_2) < 0$. Therefore, W is a common witness to the entanglement in ρ_1 and ρ_2. Hence we can conclude that $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ is entangled for all $\lambda \in [0, 1]$ and can be detected by W.

Example 2: The following family of edge states in $C^2 \otimes C^4$ was introduced in [AGKL10].

$$\tau(b,s)=\frac{1}{2(2+b+b^{-1})}\begin{pmatrix} b & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & b^{-1} & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & b^{-1} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & b \end{pmatrix} \tag{4.8}$$

$$(\tau(b,s))^{T_A}=\frac{1}{2(2+b+b^{-1})}\begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 & s \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & b^{-1} & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & b^{-1} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ s & 0 & 0 & 0 & 0 & 0 & 0 & b \end{pmatrix} \tag{4.9}$$

where $0 < b < 1$ and $|s| < b$. We consider $\delta_1 = \tau(0.4, 0)$ and $\delta_2 = \tau(0.5, 0)$. It is observed that the eigenvector $|01\rangle + |12\rangle \in ker(\delta_1) \cap ker(\delta_2)$. Further, the eigenvector $|03\rangle + |12\rangle$ and $|01\rangle + |10\rangle$ lies in $ker(\delta_1^{T_A}) \cap ker(\delta_2^{T_A})$. Taking the projectors as defined in Theorem 4.2, we obtain the witness

$$W_{edge}=\begin{pmatrix} -\varepsilon & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\varepsilon+2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\varepsilon+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon+1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\varepsilon+2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\varepsilon \end{pmatrix} \tag{4.10}$$

This gives $Tr(W_{edge}\delta_1) < 0$ and $Tr(W_{edge}\delta_2) < 0$. Thus, W_{edge} is a common witness and also detects the class of states $\delta = \lambda\delta_1 + (1-\lambda)\delta_2, 0 \leq \lambda \leq 1$.

Example 3: We consider the following class of tripartite edge states as proposed in [ABLS01]:

$$\delta_{tri}(a,b,c) = \frac{1}{n}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.11}$$

where $n = 2+a+b+c+1/a+1/b+1/c$ and the basis is taken in the order $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$. The partial transpose with respect to system C is given by

$$\delta_{tri}^{T_C}(a,b,c) = \frac{1}{n}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.12}$$

Next we take the edge states $\delta_{tri}^1 = \delta_{tri}(1,1,1)$ and $\delta_{tri}^2 = \delta_{tri}(1,2,2)$. It is observed that $|111\rangle - |000\rangle \in \texttt{ker}(\delta_{tri}^1) \cap \texttt{ker}(\delta_{tri}^2)$ and $|110\rangle - |001\rangle \in$

$\texttt{ker}((\delta^1_{tri})^{T_C}) \cap \texttt{ker}((\delta^2_{tri})^{T_C})$.The witness obtained is

$$W_{tri} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k-1 \\ 0 & 1-k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-k & 0 \\ -k-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.13}$$

It is found that W_{tri} detects both δ^1_{tri} and δ^2_{tri} , thus detecting the states $\delta^{12}_{tri} = \lambda\delta^1_{tri} + (1-\lambda)\delta^2_{tri}, \forall\lambda \in [0,1]$.

4.5 Common decomposable and non-decomposable witness operators

Central to the idea of the detection of a PPTES is a non-decomposable witness which can successfully identify a PPTES in contrast to a decomposable witness which fails in this purpose. If we are given two states described by the density operators Δ_1 and Δ_2 then we can construct a witness operator common not only to the states Δ_1 and Δ_2 but also to the states lying on the straight line joining Δ_1 and Δ_2. Naturally, the next question is,as to whether the common witness operator is decomposable or non-decomposable. The answer lies in the nature of the states Δ_1 and Δ_2. The decomposable or non-decomposable nature of the common witness operator depends on the PPT or NPT nature of the states Δ_1 and Δ_2. Let us suppose that Δ_1 and Δ_2 are two entangled states. Now if we consider the convex combination of Δ_1 and Δ_2, i.e., $\Delta = \lambda\Delta_1 + (1-\lambda)\Delta_2, \;\; 0 \leq \lambda \leq 1$, then the common decomposable witness operator and common non-decomposable witness operator can be seen as:

Common decomposable witness operator: If both Δ_1 and Δ_2 are NPT then a decomposable operator is enough to qualify as a common witness.

Common non-decomposable witness operator: If either Δ_1 or Δ_2 or both are PPT then the common witness operator is non-decomposable.

Note that if Δ_1 is PPT and Δ_2 is NPT, or vice-versa, then the state Δ may be NPT and it may be detected by a decomposable witness operator, but such a witness operator will not be common to Δ_1 and Δ_2, because either Δ_1 or Δ_2 is PPT, and a PPT entangled state cannot be detected by a decomposable witness operator. Let us understand the above defined common decomposable and common non-decomposable witness operators by considering the following two cases: (i) convex combination of a class of PPT entangled state and a class of NPT pure entangled state and (ii) convex combination of a class of PPT entangled state and a class of NPT mixed entangled state.

Case-I: Convex combination of a class of PPT entangled state and a class of NPT pure entangled state

Let us consider a class of PPT entangled state [HHH99b]

$$\rho_1^e = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{7}\varrho_+ + \frac{5-\alpha}{7}\varrho_-, \quad 3 < \alpha \leq 4 \tag{4.14}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$, $\varrho_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$ and $\varrho_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)$. Further let us consider a pure entangled state which is described by the density operator

$$\rho_2^e = \beta|00\rangle\langle 00| + \beta\sqrt{1-\beta^2}|00\rangle\langle 11| + \beta\sqrt{1-\beta^2}|11\rangle\langle 00| + (1-\beta^2)|11\rangle\langle 11| \tag{4.15}$$

The convex combination of the above two states can be described by the density operator

$$\rho^e = \lambda\rho_1^e + (1-\lambda)\rho_2^e, 0 \leq \lambda \leq 1 \tag{4.16}$$

Enumerating the eigenvalues of the partial transpose of the state (4.16) it is observed that the state has the following characterization:

Sl. No.	λ	α	β	Nature of ρ^e
1	$0 \leq \lambda < 0.75$	$3 < \alpha \leq 3.9$	$0.07 < \beta \leq 0.99$	$(\rho^e)^{T_B} < 0$
2	$0.75 \leq \lambda \leq 1$	$3 < \alpha \leq 3.9$	$0 \leq \beta \leq 0.01$	$(\rho^e)^{T_B} \geq 0$

Since the state ρ^e is free entangled for the range of three parameters $0 \leq \lambda < 0.75$, $3 < \alpha \leq 3.9$, $0.07 < \beta \leq 0.99$, so a decomposable witness operator is sufficient to detect it and it is given by

$$W^d = (|\chi\rangle\langle\chi|)^{T_B} \tag{4.17}$$

where $|\chi\rangle$ is an eigenvector corresponding to a negative eigenvalue of the state $(\rho^e)^{T_B}$. The witness operator W^d detects ρ^e as well as the state ρ_2^e, but it fails to detect ρ_1^e, as ρ_1^e is PPT and W^d is decomposable. So, in this case we are not able to construct a common decomposable witness operator. However,we can construct a non-decomposable witness operator in the form

$$W^{nd} = |\phi\rangle\langle\phi| - \varepsilon I \tag{4.18}$$

where $|\phi\rangle \in ker(\rho^e)$. With this selection, we obtain

$$Tr(W^{nd}\rho^e) = -\varepsilon < 0, \; Tr(W^{nd}\rho_1^e) = -\varepsilon < 0, \; Tr(W^{nd}\rho_2^e) = -\varepsilon < 0 \tag{4.19}$$

The above non-decomposable witness operator W^{nd} not only detects ρ_2^e but also detects ρ_1^e, and thus, it is a common non-decomposable witness operator. Let us now consider the case when $(\rho^e)^{T_B} \geq 0$ for $0.75 \leq \lambda \leq 1$, $3 < \alpha \leq 3.9$, $0 \leq \beta \leq 0.01$. As $\beta \to 0$, the state ρ_2^e approaches the separable projector $|11\rangle\langle 11|$. Consequently, the convex combination of ρ_1^e and ρ_2^e is PPT. Thus in this scenario, we can conclude that either all the states lying on the straight line joining ρ_1^e and the projector $|11\rangle\langle 11|$ are separable, or we are incapable of detecting the most weak bound entangled state.

Case-II: Convex combination of a class of PPT entangled state and a class of NPT mixed entangled state

Let us consider a class of PPT entangled mixed state and a class of NPT entangled mixed state which are described by the density operators[HHH99b]

$$\Upsilon_1 = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{7}\varrho_+ + \frac{5-\alpha}{7}\varrho_- \quad (3 < \alpha \leq 4) \tag{4.20}$$

and

$$\Upsilon_2 = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\gamma}{7}\varrho_+ + \frac{5-\gamma}{7}\varrho_- \quad (4 < \gamma \leq 5) \tag{4.21}$$

respectively. The convex combination of the states Υ_1 and Υ_2 is given by

$$\Upsilon = \lambda\Upsilon_1 + (1-\lambda)\Upsilon_2 \quad (0 \leq \lambda \leq 1) \tag{4.22}$$

The nature of the resultant state described by the density operator Υ depends on the values of the mixing parameter λ and the other two parameters α and γ, as is given in the table below:

Sl. No.	α	γ	λ	Nature of Υ
1	$3 < \alpha \leq 4$	$4 < \gamma \leq 5$	$0 \leq \lambda < \frac{\gamma-4}{\gamma-\alpha}$	$\Upsilon^{T_B} < 0$
2	$3 < \alpha < 4$	$4 < \gamma \leq 5$	$\frac{\gamma-4}{\gamma-\alpha} \leq \lambda \leq 1$	$\Upsilon^{T_B} \geq 0$
3	$\alpha = 4$	$4 < \gamma \leq 5$	$\lambda = 1$	$\Upsilon^{T_B} \geq 0$

The state Υ is NPT for the range of parameters $3 < \alpha \leq 4$, $4 < \gamma \leq 5$, $0 \leq \lambda < \frac{\gamma-4}{\gamma-\alpha}$, and in this case the common witness operator is a non-decomposable witness which detects Υ, Υ_1 and Υ_2, whereas a decomposable witness fails to detect all the three simultaneously. However, in the remaining two cases where the ranges of three parameters are given by $3 < \alpha < 4, 4 < \gamma \leq 5, \frac{\gamma-4}{\gamma-\alpha} \leq \lambda \leq 1$ and $\alpha = 4, 4 < \gamma \leq 5, \lambda = 1$, we find that the vectors $|v_1\rangle = |11\rangle - |00\rangle \in ker(\Upsilon_1) \cap ker(\Upsilon_2)$ and $|v_2\rangle = |22\rangle - |00\rangle \in ker(\Upsilon_1) \cap ker(\Upsilon_2)$. In this scenario, a non-decomposable witness operator can be constructed, which detects both Υ_1, Υ_2 and hence Υ. Such a non-decomposable witness operator is of the form

$$\Gamma = P - \varepsilon I \tag{4.23}$$

where $P = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$, and $0 < \varepsilon \leq \varepsilon_0 = inf_{|e,f\rangle}\langle e, f|P|e, f\rangle$.

4.6 Summary

To summarize, in this chapter we have investigated the conditions for the existence of common Schmidt number and entanglement witnesses, and proposed methods for the construction of common witness operators. Common entanglement witnesses for pairs of entangled states enable us to detect a large class of entangled states, *viz.*, when a common witness exists for two states, it enables us to detect all states lying on the line segment joining the two. Certain characteristics of the states help us to construct the common witnesses which we have discussed here. We have considered a few interesting examples of states presented earlier in the literature in the context of entanglement witnesses, and these illustrations from qutrit systems buttress our claim of suggesting schemes for finding common witnesses.

Before concluding, let us discuss a couple of relevant issues. First, regarding the non-decomposable witness of the form $P + Q^{T_A} - \varepsilon I$, a method was proposed in [Ter01] to obtain an analytic lower bound of ε and relevant bounds were obtained for some special states which possessed high degrees of symmetry. However the procedure becomes cumbersome when the sets concerned contain few symmetries and also in high dimensions[Ter01].It was stated in [HMABM04], that the calculation of ε would require a multivariable minimization routine. The value of ε was calculated numerically in [GHB+03], albeit for some special states where the discussion was on the measurability of some witness operators. However, the purpose of the present chapter is to present an analytic description of common entanglement witnesses when detection of many entangled states by the same operator is concerned. In the example , we have used the definition of ε as was provided in [LKCH00]. The definition of edge states ensures that $\varepsilon > 0$, hence an expectation value $-\varepsilon$ indicates the presence of entanglement. Thus the positivity of ε is enough to accentuate the aforesaid detection. Secondly, one may also note that Theorem 4.1 can be extended for three or more states, viz., a common witness will exist for three states if there is a common eigenvector corresponding to the negative eigenvalues of their partial transposes.

Finally, our study also shows that the nature of the common witness is significantly dictated by the positivity of the partial transpose of the two states. Specifically, a decomposable witness can never qualify to be a common witness if one of the states is PPTES. Thus, we demarcate between a common decomposable and a nondecomposable witness. In our analysis of common Schmidt number witnesses we find that if the two states are both of SN k and a common SN witness exists for them, then the convex combination will be of SN k. We conclude by noting that an interesting question for further study could be to find whether the converse of the above statement is true.

The study on entanglement witnesses was initially motivated by the quest to demarcate entangled states from the separable ones. Although entanglement has been recognized as a major resource in quantum information processing tasks yet not all entangled states are eligible for the same. This further motivates the search for useful entangled states, an investigation that we report in the next chapter pertaining to a quintessential quantum information processing task namely Teleportation.

Chapter 5

Witness operator for Quantum Teleportation

5.1 Prelude

Quantum information processing is now widely recognized as a powerful tool for implementing tasks that cannot be performed using classical means [NC10]. A large number of algorithms for various information processing tasks such as super dense coding[BW92], teleportation [BBC+93] and key generation [Eke91] have been proposed and experimentally demonstrated. At the practical level information processing is implemented by manipulating states of quantum particles, and it is well known that not all quantum states can be used for such purposes. Hence, given an unknown state, one of the most relevant issues here is to determine whether it is useful for quantum information processing.

The key ingredient for performing many information processing tasks is provided by quantum entanglement. Detection of entanglement is facilitated by entanglement witnesses. However, entanglement witnesses has moved beyond the realm of mere detection of entanglement. Since, the notion of entanglement witnesses arose of separation axioms , they can be also put to use to

distinguish special entangled states for information processing tasks.
Teleportation [BBC+93] is a typical information processing task where at present there is intense activity in extending the experimental frontiers [BPM+97, OMM+09, JRY+10]. However, it is well known that not all entangled states are useful for teleportation. For example, while the entangled Werner state [Wer89b] in $2 \otimes 2$ dimensions is a useful resource [LK00], another class of maximally entangled mixed states [MJWK01], as well as other non-maximally entangled mixed states achieve a fidelity higher than the classical limit only when their magnitude of entanglement exceeds a certain value [AMR+10]. The problem of determining states useful for teleportation becomes conceptually more involved in higher dimensions where bound entangled states [Hor97, HHH98] also exist. The motivation for this study is to enquire how to determine whether an unknown entangled state could be used as a resource for performing information processing tasks. In the present chapter we consider this question for the specific task of quantum teleportation. We propose and demonstrate the existence of measurable witness operators connected to teleportation, by making use of a property of entangled states, *viz*, the fully entangled fraction (FEF) [BDSW96, HHH99a] which can be related to the efficacy of teleportation. In spite of the conceptual relevance of the FEF as a characteristic trait of entangled states [VJN00], its actual determination could be complicated for higher dimensional systems[ZLFW10]. Our proof of the existence of witnesses connected to a relevant threshold value for the FEF enables us to construct a suitable witness operator for teleportation, as is illustrated with certain examples.
The chapter discusses the existence of hermitian operators for teleportation in section 5.2 followed by the proposition of such a witness in $d \otimes d$ dimensions in 5.3. The chapter ends with suitable examples depicting the utility of the proposed operator in various dimensions and speculations on future directions of work.

5.2 Proof of existence of witness

The fully entangled fraction (FEF) [HHH99a] is defined for a bipartite state ρ in $d \otimes d$ dimensions as

$$F(\rho) = max_U \langle\psi^+|U^\dagger \otimes I\rho U \otimes I|\psi^+\rangle \tag{5.1}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$ and U is a unitary operator. A quantum channel is useful for teleportation if it can provide a fidelity higher than what can be done classically. The fidelity depends on the FEF of the state, e.g., a state in $d \otimes d$ dimensions works as a teleportation channel if its FEF $> \frac{1}{d}$ [HHH99a, VJN00, ZLFW10].

Here we propose the existence of a hermitian operator which serves to distinguish between states having FEF higher than a given threshold value from other states. FEF $> \frac{1}{d}$ is a benchmark which measures the viability of quantum states in teleportation. Let us consider the set S of states having FEF $\leq \frac{1}{d}$. A special geometric form of the Hahn-Banach theorem in functional analysis [Hol75] states that if a set is convex and compact, then a point lying outside the set can be separated from it by a hyperplane. The existence of entanglement witnesses are indeed also an outcome of this theorem [HHH96, Ter00]. We now present the proof that the set S of states with FEF $\leq \frac{1}{d}$ is indeed convex and compact, so that the separation axiom in the form of the Hahn-Banach theorem could be applied in order to demonstrate the existence of hermitian witness operators for teleportation.

Theorem 5.1. *The set $S = \{\rho : F(\rho) \leq \frac{1}{d}\}$ is convex and compact.*

Proof: The proof is done in two steps. *(i) We first show that S is convex.* Let $\rho_1, \rho_2 \in S$. Therefore,

$$F(\rho_1) \leq \frac{1}{d}, \quad F(\rho_2) \leq \frac{1}{d}. \tag{5.2}$$

Consider $\rho_c = \lambda\rho_1 + (1-\lambda)\rho_2$, where $\lambda \in [0,1]$ and $F(\rho_c) = \langle\psi^+|U_c^\dagger \otimes I\rho_c U_c \otimes I|\psi^+\rangle$. Now, $F(\rho_c) = \lambda\langle\psi^+|U_c^\dagger \otimes I\rho_1 U_c \otimes I|\psi^+\rangle + (1-\lambda)\langle\psi^+|U_c^\dagger \otimes I\rho_2 U_c \otimes I|\psi^+\rangle$. Let $F(\rho_i) = \langle\psi^+|U_i^\dagger \otimes I\rho_i U_i \otimes I|\psi^+\rangle, \quad (i = 1, 2)$. This is possible since the group of unitary matrices is compact, hence the maximum will be attained for a unitary matrix U. It follows that $F(\rho_c) \leq \lambda F(\rho_1) + (1-\lambda)F(\rho_2)$. Using Eq.(5.2) we have

$$F(\rho_c) \leq \frac{1}{d} \tag{5.3}$$

Thus, ρ_c lies in S, and hence, S is convex.

(ii) We now show that S is compact. Note that in a finite dimensional Hilbert space, in order to show that a set is compact it is enough to show that the set is closed and bounded. The set S is bounded as every density matrix has a bounded spectrum, i.e., eigenvalues lying between 0 and 1. In order to prove that the set S is closed, consider first the following lemma. *Lemma*: Let A and B be two matrices of size $m \times n$ and $n \times r$ respectively. Then $\|AB\| \leq \|A\|\|B\|$, where the norm of a matrix A is defined as $\|A\| = \sqrt{TrA^\dagger A} = \sqrt{\sum_i \sum_j |A_{ij}|^2}$.

Proof of the lemma: Let $A = \begin{pmatrix} A_1 \\ A_2 \\ . \\ . \\ A_m \end{pmatrix}$ and $B = [B^{(1)}B^{(2)}....B^{(r)}]$, where A_i's are row vectors of size n and $B^{(j)}$'s are column vectors of size n respectively. Using the Cauchy-Schwarz inequality, it follows that $|(AB)_{ij}| = |A_i B^{(j)}| \leq \|A_i\|\|B^{(j)}\|$. Therefore, one has

$$\|AB\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{r} |(AB)_{ij}|^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{r} \|A_i\|^2 \|B^{(j)}\|^2 \tag{5.4}$$

The r.h.s of the above inequality can be expressed as $\sum_{i=1}^{m} \|A_i\|^2 \sum_{j=1}^{r} \|B^{(j)}\|^2 = \|A\|^2\|B\|^2$, from which it follows that $\|AB\| \leq \|A\|\|B\|$.

For any two density matrices ρ_a and ρ_b, assume the maximum value of FEF is obtained at U_a and U_b respectively, i.e., $F(\rho_a) = \langle\psi^+|U_a^\dagger \otimes I \rho_a U_a \otimes I|\psi^+\rangle$ and $F(\rho_b) = \langle\psi^+|U_b^\dagger \otimes I \rho_b U_b \otimes I|\psi^+\rangle$. Therefore, we have $F(\rho_a) - F(\rho_b) = \langle\psi^+|U_a^\dagger \otimes I \rho_a U_a \otimes I|\psi^+\rangle - \langle\psi^+|U_b^\dagger \otimes I \rho_b U_b \otimes I|\psi^+\rangle$ from which it follows that $F(\rho_a) - F(\rho_b) \leq \langle\psi^+|U_a^\dagger \otimes I \rho_a U_a \otimes I|\psi^+\rangle - \langle\psi^+|U_a^\dagger \otimes I \rho_b U_a \otimes I|\psi^+\rangle$ since $\langle\psi^+|U_a^\dagger \otimes I \rho_b U_a \otimes I|\psi^+\rangle \leq \langle\psi^+|U_b^\dagger \otimes I \rho_b U_b \otimes I|\psi^+\rangle$. Hence, $F(\rho_a) - F(\rho_b) \leq \langle\psi^+|U_a^\dagger \otimes I(\rho_a - \rho_b)U_a \otimes I|\psi^+\rangle$, implying

$$F(\rho_a) - F(\rho_b) \leq |\langle\psi^+|U_a^\dagger \otimes I(\rho_a - \rho_b)U_a \otimes I|\psi^+\rangle|. \tag{5.5}$$

Now, using the above lemma, one gets $F(\rho_a) - F(\rho_b) \leq \|\langle\psi^+|\|\|U_a^\dagger \otimes I\|\|(\rho_a - \rho_b)\|\|U_a \otimes I\|\||\psi^+\rangle\|$, or $F(\rho_a) - F(\rho_b) \leq C^2 K_1^2 \|\rho_a - \rho_b\|$, where C, K_1 are positive real numbers. The last step follows from the fact that $\|\langle\psi^+|\| = C$. Since the set of all unitary operators is compact, it is bounded, and thus for any U, $\|U \otimes I\| \leq K_1$. Similarly $F(\rho_b) - F(\rho_a) \leq C^2 K_1^2 \|\rho_b - \rho_a\| = C^2 K_1^2 \|\rho_a - \rho_b\|$. So finally, one may write

$$|F(\rho_a) - F(\rho_b)| \leq C^2 K_1^2 \|\rho_a - \rho_b\|. \tag{5.6}$$

This implies that F is a continuous function. Moreover, for any density matrix ρ, with $F(\rho) \in [\frac{1}{d^2}, 1]$, one has $F(\rho) = 1$ iff ρ is a maximally entangled pure state, and $F(\rho) = \frac{1}{d^2}$ iff ρ is the maximally mixed state [ZLFW10]. For the set S in our consideration $F(\rho) \in [\frac{1}{d^2}, \frac{1}{d}]$. Hence, $S = \{\rho : F(\rho) \leq \frac{1}{d}\} = F^{-1}([\frac{1}{d^2}, \frac{1}{d}])$, is closed [Rud64]. This completes the proof of our proposition that the set $S = \{\rho : F(\rho) \leq \frac{1}{d}\}$ is convex and compact.

It now follows from the Hahn-Banach theorem [Hol75], that any $\chi \notin S$ can be separated from S by a hyperplane. In other words, any state useful for teleportation can be separated from the states not useful for teleportation by a hyperplane and thus allows for the definition of a witness. The witness operator, if so defined, identifies the states which are useful in the teleportation protocol, i.e., provides a fidelity higher than the classical optimum.

5.3 A witness operator for teleportation

A hermitian operator W may be called a teleportation witness if the following conditions are satisfied: (i) $Tr(W\sigma) \geq 0$, for all states σ which are not useful

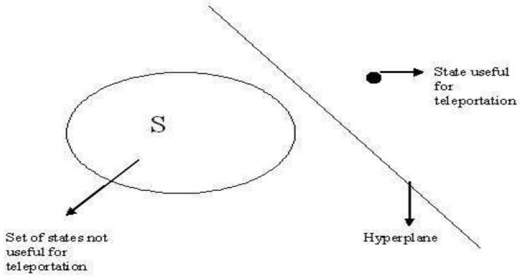

Figure 5.1: The set $S = \{\rho : F(\rho) \leq \frac{1}{d}\}$ is convex and compact, and using the Hahn-Banach theorem it follows that any state useful for teleportation can be separated from the states not useful for teleportation by a hyperplane, thus providing for the existence of a witness for teleportation.

for teleportation. (ii) $Tr(W\chi) < 0$, for at least one state χ which is useful for teleportation. We propose a hermitian operator for a $d \otimes d$ system of the form (using $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$)

$$W = \frac{1}{d} I - |\psi^+\rangle\langle\psi^+| \tag{5.7}$$

In order to prove that W is indeed a witness operator, we first show that the operator W gives a non-negative expectation over all states which are not useful for teleportation. Let σ be an arbitrary state chosen from the set S not useful for teleportation, i.e., $\sigma \in S$. Hence,

$$Tr(W\sigma) = \frac{1}{d} - \langle\psi^+|\sigma|\psi^+\rangle \tag{5.8}$$

from which it follows that $Tr(W\sigma) \geq \frac{1}{d} - max_U \langle\psi^+|U^\dagger \otimes I\sigma U \otimes I|\psi^+\rangle$. Now, using the definition of the FEF, $F(\sigma)$ from Eq.(5.1), and the fact that $\sigma \in S$, one gets

$$Tr(W\sigma) \geq 0 \tag{5.9}$$

Our task now is to show that the operator W detects at least one entangled state χ which is useful for teleportation, i.e., $Tr(W\chi) < 0$, which we do by providing the following illustrations.

Let us first consider the isotropic state

$$\chi_\beta = \beta|\psi^+\rangle\langle\psi^+| + \frac{1-\beta}{d^2}I \qquad (-\frac{1}{d^2-1} \leq \beta \leq 1) \tag{5.10}$$

The isotropic state is entangled $\forall\beta > \frac{1}{d+1}$ [BDHK05]. Now, $Tr(W\chi_\beta) = \frac{(d-1)(1-\beta(d+1))}{d^2}$, from which it follows that $Tr(W\chi_\beta) < 0$, when $\beta > \frac{1}{d+1}$. Therefore, all entangled isotropic states are useful for teleportation. The same conclusion was obtained in [ZLFW10] on explicit calculation of the FEF for isotropic states. We next consider the generalized Werner state [Wer89b, PR00, DC09] in $d \otimes d$ given by

$$\chi_{wer} = (1-v)\frac{I}{d^2} + v|\psi_d\rangle\langle\psi_d| \tag{5.11}$$

where $0 \leq v \leq 1$ and $|\psi_d\rangle = \sum_{i=0}^{d-1}\alpha_i|ii\rangle$, with $\sum_i|\alpha_i|^2 = 1$, for which one obtains $Tr(W\chi_{wer}) = \frac{1}{d} - \frac{1-v}{d^2} - \frac{v}{d}\sum_{i=0}^{d-1}\alpha_i\sum_{i=0}^{d-1}\alpha_i^*$. The witness W detects those Werner states which are useful for teleportation, i.e., $Tr(W\chi_{wer}) < 0$, which is the case when

$$\frac{1}{d} - \frac{1-v}{d^2} - \frac{v}{d}\sum_{i=0}^{d-1}\alpha_i\sum_{i=0}^{d-1}\alpha_i^* < 0 \tag{5.12}$$

In $2\otimes 2$ dimensions with $\alpha_i = 1/\sqrt{2}$, one gets $Tr(W\chi_{wer}) = \frac{1-3v}{4} < 0$, when $v > \frac{1}{3}$. Thus, all entangled Werner states are useful for teleportation, a result which is well-known [LK00].

Now, consider another class of maximally entangled mixed states in $2\otimes 2$ dimensions, which possess the maximum amount of entanglement for a given purity [MJWK01]:

$$\chi_{MEMS} = \begin{pmatrix} h(C) & 0 & 0 & C/2 \\ 0 & 1-2h(C) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C/2 & 0 & 0 & h(C) \end{pmatrix} \tag{5.13}$$

where, $h(C) = C/2$ for $C \geq 2/3$, and $h(C) = 1/3$ for $C < 2/3$, with C the concurrence of χ_{MEMS}. Here we obtain $Tr(W\chi_{MEMS}) = \frac{1}{2} - h(C) - \frac{C}{2}$. For $C > \frac{1}{3}$, the state χ_{MEMS} is suitable for teleportation, as one obtains $Tr(W\chi_{MEMS}) < 0$ in this case, confirming the results derived earlier in the literature [AMR+10]. However, as expected with any witness, our proposed witness operator may fail to identify certain other states that are known to be useful for teleportation. For example, the state (for $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $0 \leq a \leq 1$)

$$\rho_\phi = a|\phi\rangle\langle\phi| + (1-a)|11\rangle\langle 11| \tag{5.14}$$

was recently studied in the context of quantum discord [ARA10]. This class of states is useful for teleportation but the witness W is unable to detect it as $Tr(W\rho_\phi) = \frac{a}{2} \geq 0$.

Let us now briefly discuss the measurability of the witness operator. For experimental realization of the witness it is necessary to decompose the witness into operators that can be measured locally, i.e, a decomposition into projectors of the form $W = \sum_{i=1}^{k} c_i|e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$ [GT09, GHB+03]. For implementation using polarized photons as in [BDMDN+03], one may take $|H\rangle = |0\rangle, |V\rangle = |1\rangle, |D\rangle = \frac{|H\rangle+|V\rangle}{\sqrt{2}}, |F\rangle = \frac{|H\rangle-|V\rangle}{\sqrt{2}}, |L\rangle = \frac{|H\rangle+i|V\rangle}{\sqrt{2}}, |R\rangle = \frac{|H\rangle-i|V\rangle}{\sqrt{2}}$ as the horizontal, vertical, diagonal, and the left and right circular polarization states, respectively. Using a result given in [Hyl05], our witness operator can be recast for qubits into the required form, given by

$$\begin{aligned} W = \frac{1}{2}(&|HV\rangle\langle HV| + |VH\rangle\langle VH| - |DD\rangle\langle DD| \\ &-|FF\rangle\langle FF| + |LL\rangle\langle LL| + |RR\rangle\langle RR|) \end{aligned} \tag{5.15}$$

Using this technique for an unknown two-qubit state χ, the estimation of

$\langle W \rangle$ requires three measurements [Hyl05], as is also evident from the decomposition of our witness operator for qubits in terms of Pauli spin matrices, i.e., $W = \frac{1}{4}[I \otimes I - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z]$, which is far less than the measurement of 15 parameters required for full state tomography[JKMW01]. In higher dimensions, the witness operator may be decomposed in terms of Gell-Mann matrices [BK08], and this difference further increases with the increase in dimensions. Therefore, the utility of the witness operator is indicated as compared to full state tomography when discrimination of useful entangled states for performing teleportation is required.

Before concluding, it may be noted that it is possible to relate the FEF (5.1) with the maximum fidelity for other information processing tasks, such as super dense coding and entanglement swapping [GEJ02]. In the generalized dense coding for $d \otimes d$ systems, one can use a maximally entangled state $|\phi\rangle$ to encode $d^2/2$ bits in d^2 orthogonal states $(I \otimes U_i)|\phi\rangle$ [LLTL02]. If the maximally entangled state is replaced with a general density operator, the dense coding fidelity is defined as an average over the d^2 results. A relation between the maximum fidelity F_{DC}^{max} of dense coding and the FEF was established for $d \otimes d$ systems to be $F_{DC}^{max} = F$. Similarly, for two-qubit systems the maximum fidelity of entanglement swapping[ŻZHE93] F_{ES}^{max} is also related to the FEF by $F_{ES}^{max} = F$ [GEJ02]. However, teleportation is a different information processing task as compared to dense coding where $F > 1/d$ does not guarantee a higher than classical fidelity [BDL+04]. Hence, it is not possible to apply the above witness (5.7) to super dense coding and entanglement swapping.

5.4 Summary

To summarize, in this chapter we have proposed a framework for discriminating quantum states useful for performing teleportation through the measurement of a hermitian witness operator. The ability of an entangled state to act as a resource for teleportation is connected with the fully entangled fraction of the state. The estimation of the fully entangled fraction is difficult in general, except in the case of some known states. We have shown that the set of states having their fully entangled fraction bounded by a certain threshold value required for teleportation is both convex and compact. Exploiting this feature we have demonstrated the existence of a witness operator for teleportation. The measurement of the expectation value of the witness for unknown states reveals which states are useful as resource for performing teleportation. We have provided some illustrations of the applicability of the witness for isotropic and Werner states in $d \otimes d$ dimensions, and another class of maximally entangled mixed states for qubits. The measurability of such a witness operator requires determination of a much lesser number of parameters in comparison to state tomography of an unknown state, thus signifying the practical utility of our proposal. It would be interesting to explore the possibility of existence of witnesses for various other information processing tasks, as well. In this context further studies on finding optimal and common witnesses are called for.

Chapter 6

Construction of optimal teleportation witnesses

6.1 Prelude

Entanglement witnesses provide experimentally viable procedures for detecting the presence of entanglement in composite quantum systems. However,entanglement witnesses are not universal, and hence the question as to how to maximally detect entangled states, i.e., increase the number of states detected by the witness, is of significance. One direction in this line of thought has been provided by the study on common entanglement witnesses [WG07, GAM13]. Another important path in this quest is the study on optimal entanglement witnesses [LKCH00]. Optimization of entanglement witnesses entails the refinement of witnesses , so that a witness can detect the largest class of states within a given set. In fact a witness W_1 is said to be finer than another witness W_2 if W_1 detects some more states in addition to the states that W_2 detects. The possibility of optimization of entanglement witnesses has lead to the construction of optimal witnesses [BDHK05, CP11].

Entanglement witnesses enable experimentally viable procedures to detect the presence of entanglement, a notion that has been carried forward to identify manifestations of various properties of quantum states, such as macroscopic entanglement through thermodynamical witnesses[BV04], as well as witnesses for quantum correlations [DVB10], teleportation [GAMC11, ZFLJ12], cryptography [BHH12] and mixedness [MPM13]. Although entanglement is a key ingredient for teleportation, yet not all entangled states are useful for the purpose of teleportation. The problem gets accentuated in higher dimensions where bound entangled states [HHH98] are also present. The ability of an entangled state to perform teleportation is linked to a threshold value of the fully entangled fraction [BDSW96] which is difficult to estimate except for some known states [ZLFW10]. Based upon the linkage of the threshold value of the fully entangled fraction with teleportation fidelity, and utilising again the separation axioms, the existence of hermitian operators acting as teleportation witness was demonstrated and studied in the previous chapter[GAMC11]. A teleportation witness W_T is a hermitian operator with at least one negative eigenvalue and (i) $Tr(W_T \varpi) \geq 0$, for all states ϖ not useful for teleportation and (ii) $Tr(W_T \vartheta) < 0$ for atleast, one entangled state ϑ which is useful for teleportation. In a following work [ZFLJ12] a teleportation witness with interesting universal properties was proposed, which though depends upon the choice of a unitary operator that may be difficult to find in practice, especially in higher dimensions.The difficulty in identifying useful resources for teleportation necessitates the construction of suitable teleportation witnesses that would be possible to implement experimentally in order to ascertain whether a given unknown state would be useful as a teleportation channel. Moreover, analogous to the theory of entanglement witnesses, maximal detection of states capable for teleportation is a question of significance. This chapter addresses both the issues. In section 6.2 construction of optimal teleportation witnesses is prescribed for qubits, qutrits and qudits. In section 6.3 suitable examples are provided to vindicate the efficacy of such optimal witnesses.

6.2 Optimal teleportation witness

Between two witnesses W_1 and W_2, W_1 is said to be finer than W_2, if $DW_2 \subseteq DW_1$, where $DW_i = \{\chi : Tr(W_i\chi) < 0\}, i = 1, 2$, i.e., the set of entangled states detected by W_i. A witness is said to be optimal if there exists no other witness finer than it [LKCH00]. Further, if the set of product vectors $|e, f\rangle$, $P_W = \{|e, f\rangle : Tr(W|e, f\rangle\langle e, f|) = 0\}$, spans the relevant product Hilbert space, then the witness W is optimal [LKCH00]. It was shown in[ATL11] that if a witness operating on $H_m \otimes H_m$ can be expressed in the form $W = Q^{T_A}$, where Q is the projector on a pure entangled state, then the witness W is optimal.

On the other hand, as stated earlier, the ability of a quantum state in performing teleportation is determined by a threshold value of the fully entangled fraction, given by $F(\rho) = max_U Tr[(U^\dagger \otimes I)\rho(U \otimes I)|\Phi\rangle\langle\Phi|]$ [BDSW96], where $|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|kk\rangle$ and U is an unitary operator. Precisely, in $d \otimes d$ systems if $F(\rho)$ exceeds $\frac{1}{d}$, then the state is considered useful for the protocol [BDSW96].

6.2.1 Optimal teleportation witness for qubits

Consider the entanglement witness, $W^2 = \rho_{\phi^+}^{T_A}$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, acting on two qubit systems. Since, $\rho_{\phi^+} = \frac{1}{4}(I\otimes I + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)$, one thus obtains, $W^2 = \frac{1}{4}(I \otimes I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$, which implies,

$$\begin{aligned} &Tr((W^2 - \frac{1}{4}\sigma_y \otimes \sigma_y)\rho) \\ &= \frac{1}{4}Tr((I \otimes I + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)\rho) \end{aligned} \tag{6.1}$$

for any arbitrary density matrix ρ. Hence,

$$F(\rho) \geq Tr(\rho|\phi^+\rangle\langle\phi^+|) \tag{6.2}$$

The r.h.s of the above equation is given by $\frac{1}{4}Tr((I \otimes I + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z - \sigma_y \otimes \sigma_y)\rho)$, which using Eq.(6.1), becomes $Tr((W^2 - \frac{1}{2}\sigma_y \otimes \sigma_y)\rho)$. This in turn implies using Eq.(6.2) that

$$Tr((\frac{1}{2}\sigma_y \otimes \sigma_y + \frac{1}{2}I - W^2)\rho) \geq \frac{1}{2} - F(\rho) \tag{6.3}$$

If ρ is not useful for teleportation, i.e., $F(\rho) \leq \frac{1}{2}$, then $Tr((\frac{1}{2}\sigma_y \otimes \sigma_y + \frac{1}{2}I - W^2)\rho) \geq 0$, implying that

$$W_{2\otimes 2} = \frac{1}{2}\sigma_y \otimes \sigma_y + \frac{1}{2}I - W^2 \tag{6.4}$$

is a teleportation witness acting on two qubits.

Next, with some straightforward algebraic manipulation it is observed that the witness can be expressed as

$$W_{2\otimes 2} = (|\psi^-\rangle\langle\psi^-|)^{T_A} \tag{6.5}$$

where, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Further the product vectors $(|0\rangle + i|1\rangle) \otimes (|0\rangle - i|1\rangle), (|0\rangle + |1\rangle)^{\otimes 2}, |00\rangle, |11\rangle$ span $C^2 \otimes C^2$ and belong to $P_{W_{2\otimes 2}}$. This establishes the optimality of the teleportation witness [LKCH00, ATL11].

6.2.2 Optimal teleportation witness for qutrits

The generalized Gell-Mann matrices are higher dimensional extensions of the Pauli matrices (for qubits) and are hermitian and traceless. They form an orthogonal set and basis. In particular, they can be categorized for qutrits as the following types of traceless matrices [BK08]:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & \text{-i} & 0 \\ \text{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & \text{-i} \\ 0 & 0 & 0 \\ \text{i} & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \text{-i} \\ 0 & \text{i} & 0 \end{pmatrix}, \lambda^8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

Now, consider the following entanglement witness in qutrits,

$$W^3 = (|\delta\rangle\langle\delta|)^{T_A} \tag{6.6}$$

where, $\delta = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$, yielding,

$$W^3 \quad = \frac{1}{9}(I \otimes I + \frac{3}{2}\Delta) \tag{6.7}$$

with $\Delta = \sum_{i=1}^{8} \lambda^i \otimes \lambda^i$. Therefore, for any arbitrary density matrix $\sigma \in B(H_3 \otimes H_3)$,taking $\Delta_1 = \lambda^2 \otimes \lambda^2 + \lambda^5 \otimes \lambda^5 + \lambda^7 \otimes \lambda^7$ and $\Delta_2 = \lambda^1 \otimes \lambda^1 + \lambda^3 \otimes \lambda^3 + \lambda^4 \otimes \lambda^4 + \lambda^6 \otimes \lambda^6 + \lambda^8 \otimes \lambda^8$, one gets

$$Tr[(W^3 - \frac{1}{6}\Delta_1)\sigma] = \frac{1}{9}Tr[(I \otimes I + \frac{3}{2}\Delta_2)\sigma] \tag{6.8}$$

Hence,

$$F(\sigma) \geq Tr(\sigma|\delta\rangle\langle\delta|) \tag{6.9}$$

The r.h.s. may be expressed as $\frac{1}{9}Tr((I \otimes I + \frac{3}{2}(\Delta_2 - \Delta_1))\sigma)$ which using Eq.(6.8) becomes $Tr((W^3 - \frac{1}{3}\Delta_1)\sigma)$. It follows from Eq.(6.9) that

$$Tr[(\frac{1}{3}\Delta_1 + \frac{1}{3}I - W^3)\sigma] \geq \frac{1}{3} - F(\sigma) \tag{6.10}$$

Hence, if σ is not useful for teleportation , i.e., $F(\sigma) \leq \frac{1}{3}$ [BDSW96], then $Tr[(\frac{1}{3}\Delta_1 + \frac{1}{3}I - W^3)\sigma] \geq 0$. Thus,

$$W_{3\otimes 3} = \frac{1}{3}\Delta_1 + \frac{1}{3}I - W^3 \tag{6.11}$$

is indeed a teleportation witness for qutrits.

Now, let us denote by $P_{W_{3\otimes 3}}$, the set of all product vectors on which the expectation value of the witness $W_{3\otimes 3}$ vanishes, i.e., $P_{W_{3\otimes 3}} = \{|e,f\rangle : \langle e,f|W_{3\otimes 3}|e,f\rangle = 0\}$. If we consider the product vectors $K_1 = |00\rangle, K_2 = |11\rangle, K_3 = |22\rangle, K_4 = (|0\rangle + |1\rangle + |2\rangle)^{\otimes 2}, K_5 = (|0\rangle + i|1\rangle) \otimes (|0\rangle - i|1\rangle), K_6 = (|0\rangle + i|2\rangle) \otimes (|0\rangle - i|2\rangle), K_7 = (|1\rangle + i|2\rangle) \otimes (|1\rangle - i|2\rangle), K_8 = (|0\rangle - |1\rangle - |2\rangle)^{\otimes 2}, K_9 = (|0\rangle + |1\rangle - |2\rangle)^{\otimes 2}$, it is noticed that (i) $\langle K_i|W_{3\otimes 3}|K_i\rangle = 0$, (ii) K_i's are linearly independent, $\forall i \in \{1, 2, ..9\}$. Thus it follows that $P_{W_{3\otimes 3}}$ spans $C^3 \otimes C^3$. This ascertains the optimality of the witness $W_{3\otimes 3}$ [LKCH00].

6.2.3 Teleportation witness for qudits

For general qudit systems the construction of teleportation witnesses from entanglement witnesses may be undertaken in a manner similar to that shown above for qubits or qutrits. Utilising the generalized Gell-Mann matrices for $d \otimes d$ systems, and retracing the steps of an argument similar to that used for qubits and qutrits, one can obtain a teleportation witness for qudits as

$$W_{d\otimes d} = \frac{1}{d}\sum_{j=0}^{d-2}\sum_{k=j+1}^{d-1}(\Lambda_a^{jk} \otimes \Lambda_a^{jk}) + \frac{1}{d}I - (|\Phi\rangle\langle\Phi|)^{T_A} \tag{6.12}$$

where, Λ_a^{jk} = -i$|j\rangle\langle k|$ + i$|k\rangle\langle j|, 0 \leq j < k \leq d-1$ and $|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}|ll\rangle$. Here it may be remarked that there is no general proof of optimality for teleportation witness for qudits, but optimality for a given dimension needs to be checked in the manner above by considering the set of all product vectors on which the expectation value of the witness vanishes.

6.3 Illustrations and Decomposition

We now consider certain classes of states pertaining to qubits and qutrits, which exemplify the action of our constructed witness. Let us first take the class of two qubit states with maximally mixed marginals, given by

$$\eta_{mix} = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} c_i\sigma_i \otimes \sigma_i) \tag{6.13}$$

The expectation value of the witness given by Eq.(6.4) on the above state gives

$$Tr(W_{2\otimes 2}\eta_{mix}) = \frac{1}{4}(1 + c_2 - c_1 - c_3) \tag{6.14}$$

implying that for $1 + c_2 - c_1 - c_3 < 0$, the witness $W_{2\otimes 2}$ detects the states as useful for teleportation. Since $W_{2\otimes 2}$ is optimal, this is the largest set of states useful for teleportation in the given class that can be detected by any witness. Next, we consider the isotropic state in qutrits, given by

$$\eta_{iso} = \alpha|\phi_+^3\rangle\langle\phi_+^3| + \frac{1-\alpha}{9}I \tag{6.15}$$

where, $|\phi_+^3\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ and $-\frac{1}{8} \leq \alpha \leq 1$. Now applying the witness given by Eq.(6.11), it is observed that

$$Tr(W_{3\otimes 3}\eta_{iso}) = \frac{2-8\alpha}{9} \tag{6.16}$$

implying that for $\alpha > \frac{1}{4}$, the states are useful for teleportation. Thus, the witness $W_{3\otimes 3}$ detects all entangled isotropic states as useful for teleportation, in conformity with a result already known in the literature [ZLFW10]. This is a reaffirmation of the optimality of the witness $W_{3\otimes 3}$, as it detects the maximal class of isotropic states as useful for teleportation.

The practical use for teleportation witnesses is that they are experimentally realizable on account of being hermitian. For qubit systems, the decomposition of a proposed teleportation witness in terms of Pauli spin operators has been shown earlier [GAMC11]. The teleportation witness constructed here is expressed in terms of generalized Gell-Mann matrices which are hermitian. However, for $d = 3$, i.e., qutrit systems the teleportation witness can also be expressed in terms of spin-1 operators [BK08] which are the observables $S_x, S_y, S_z, S_x^2, S_y^2, S_z^2, \{S_x, S_y\}, \{S_y, S_z\}, \{S_z, S_x\}$ of a spin-1 system, where $\overrightarrow{S} = \{S_x, S_y, S_z\}$ is the spin operator and $\{S_i, S_j\} = S_iS_j + S_jS_i$ (with $i, j = x, y, z$) denotes the corresponding anticommutator. They are given by, $S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & \text{-i} & 0 \\ \text{i} & 0 & \text{-i} \\ 0 & \text{i} & 0 \end{pmatrix}, S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Expressing the witness given by Eq.(6.11) in terms of spin-1 operators, yields

$$W_{3\otimes 3} = -\frac{2}{9}(I \otimes I) + \Pi \tag{6.17}$$

where

$$\begin{aligned}
\Pi \quad &= \frac{1}{6\hbar^2}(S_y \otimes S_y - S_z \otimes S_z - S_x \otimes S_x) \\
&+ \frac{1}{6\hbar^4}(-\{S_z, S_x\} \otimes \{S_z, S_x\} + \{S_x, S_y\} \otimes \{S_x, S_y\} \\
&+ \{S_y, S_z\} \otimes \{S_y, S_z\}) + \frac{2}{3\hbar^2}(I \otimes S_x^2 + I \otimes S_y^2 \\
&+ S_x^2 \otimes I + S_y^2 \otimes I) - \frac{2}{3\hbar^4}(S_x^2 \otimes S_x^2 + S_y^2 \otimes S_y^2) \\
&- \frac{1}{3\hbar^4}(S_x^2 \otimes S_y^2 + S_y^2 \otimes S_x^2)
\end{aligned} \tag{6.18}$$

Thus, for an experimental outcome,

$$\langle W_{3\otimes 3} \rangle = -\frac{2}{9}\langle I \otimes I \rangle + \langle \Pi \rangle < 0 \tag{6.19}$$

one can detect the given unknown state as useful for teleportation.

6.4 Summary

We have presented here a method to construct teleportation witnesses from entanglement witnesses for general qudit systems. Optimality of the witnesses that we have constructed for qubit and qutrit states ensures a broader perspective in the sense that a maximal class of entangled states can now be recognized to be useful for teleportation. Decomposition of the proposed witness in terms of spin operators authenticates its feasibility in experimental detection of entanglement. The present analysis may be extended in a few directions. One may seek to test the optimality of the witness for two-qudits of any given dimension $d > 3$. Finally, the choice of the entanglement witnesses are not limited to the ones we have taken up here, and other entanglement witnesses may be considered and checked for their viability in the construction of teleportation witnesses using similar methods.

In our discussion so far, we have considered the signature of entanglement in quantum systems. Subsequently, amongst the entangled states we have probed upon useful resources of teleportation through teleportation witness operators. To this end we have used separation axioms analogous to the theory of entanglement witnesses.

However, on the other hand the separable states too have interesting manifestations. As is well known, even though local unitary operations fail to create entanglement between separable systems, global unitary operations can be suitable for the purpose. However, not all separable states give rise to entangled states on non-local unitary operation. States which are separable from spectrum result in separable states on the action of any global operation. Therefore, states which are not separable from spectrum occupies a pertinent place in the creation of entanglement. This is what we probe in the next chapter, namely identification of separable states from which entanglement can be created through non-local unitary operations.

Chapter 7

Witness of mixed separable states useful for entanglement creation

7.1 Prelude

An intriguing feature of the set of separable states is concerning the problem of separability from spectrum [Kni03]. This problem calls for a characterization of those separable states σ for which $U\sigma U^\dagger$ is also separable for all unitary operators U. A possible approach towards this end is to find constraints on the eigenvalues of σ such that it remains separable under any factorization of the corresponding Hilbert space. The states that are separable from spectrum are also termed as absolutely separable states [KŻ01]. There exists a ball of known radius centered at the maximally mixed state $\frac{1}{mn}(I \otimes I)$ (for $mn \times mn$ density matrices), where all the states within the ball are absolutely separable [ŻHSL98, GB02]. However, there exist absolutely separable states outside this ball too[IH00].

The problem of separability from spectrum was first handled in the case of $2 \otimes 2$ systems [VADM01], where it was shown that σ is absolutely separable if and only if (iff) its eigenvalues (in descending order) satisfy $\lambda_1 \leq \lambda_3 + 2\sqrt{\lambda_2\lambda_4}$. A closely related problem is the characterization of the states which have positive partial transpose (PPT) from spectrum, i.e., the states σ_{ppt} with the property that $U\sigma_{ppt}U^\dagger$ is PPT for any unitary operator U. It was shown [Hil07] that $\sigma_{ppt} \in D(H_2 \otimes H_n)$ ($D(X)$ represents the bounded linear operators acting on X) is PPT from spectrum iff its eigenvalues obey $\lambda_1 \leq \lambda_{2n-1} + 2\sqrt{\lambda_{2n-2}\lambda_{2n}}$. It has been recently shown that separability from spectrum is equivalent to PPT from spectrum for states living in $D(H_2 \otimes H_n)$ [Joh13].

The generation of entanglement from separable states is one of the leading experimental frontiers at present [SKK+00, RNO+00, KC01, LL09, KRS11]. As absolutely separable states remain separable under global unitary operations, such states cannot be used as input states for entanglement creation. Though pure product states are not absolutely separable, the same is not true for mixed separable states which become absolutely separable after crossing a given amount of mixedness [TBKN11]. Given the ubiquity of environmental interactions in turning pure states into mixed ones, it is of practical importance to determine whether a state is eligible to be used as input for entanglement generation. The utility of mixed separable states which are not absolutely separable was highlighted in [IH00] for the generation of maximally entangled mixed states. Mixed separable states from which entanglement can be created have also been studied in other works [BPCP03, GHH+14].

Quantum gates have been employed to generate entanglement, especially in the context of quantum computation where unitary gates operate on qubits to perform information processing. Much work has been devoted to study the entangling capacity of unitary gates [DWS03, MKŻ13]. Quantum algorithms use pure product states which can be turned into maximally entangled states using global unitary operations. However, if the state is maximally mixed no benefit can be drawn from it through global unitary operations. States in some vicinity of the maximally mixed state also remain separable as noted in [ŻHSL98, GB02], though such states may have possible implications in nuclear magnetic resonance quantum computation [BCJ+99, SPAM+12]. However, states not close to the maximally mixed state may be useful for entanglement creation. So, it is important to study what happens when one moves from one extreme of a maximally mixed state to the other, i.e., a pure product state, within the set of all separable states.

Given the immense significance of mixed separable states, we present here systematic proposal to identify separable states which are not absolutely separable. Our approach is somewhat different from the objective of imposing restrictions on the spectrum of absolutely separable states [VADM01, Hil07, Joh13]. Our motivation here is to identify those separable states which are not absolutely separable, i.e., the separable states χ for which $U\chi U^\dagger$ is entangled for some unitary operator U. To this end, we characterize the set of all absolutely separable states in any finite dimensional bipartite system as convex and compact. This enables one to construct hermitian operators which identify separable states that are not absolutely separable in any arbitrary dimension Hilbert space. Proposing a general method of construction of witnesses, we illustrate their action on various two-qudit systems. Examples of unitary operations presented here include the celebrated *CNOT* (*Controlled NOT*) gate. We further show that the witnesses can be decomposed in terms of spin operators and locally measureable photon polarizations for qubit states, in order to facilitate their experimental realization.

7.2 Existence, construction and completeness of witness

We begin with some notations and definitions needed. We consider density matrices in any arbitrary dimensional bipartite system, i.e., $\rho \in D(H_m \otimes H_n)$. $\mathbf{S} = \{\rho : \rho \text{ is separable}\}$ is the set of all separable states, and $\mathbf{AS} = \{\sigma \in \mathbf{S} : U\sigma U^\dagger \text{ is separable } \forall \text{ unitary operators } U\}$ is the set of all absolutely separable states. **AS** forms a non-empty subset of **S**, as $\frac{1}{mn}(I \otimes I) \in \mathbf{AS}$. A point x is called a limit point of a set A if each open ball centered on x contains at least one point of A different from x. The set is closed if it contains each of its limit points [Sim63].

Theorem 7.1. **AS** *is a convex and compact subset of* **S**.

Proof: **AS is convex**: Let $\sigma_1, \sigma_2 \in \mathbf{AS}$ and $\sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2$, where $\lambda \in [0,1]$. Consider an arbitrary unitary operator U. Therefore,

$$U\sigma U^\dagger = \lambda U\sigma_1 U^\dagger + (1-\lambda)U\sigma_2 U^\dagger = \lambda\sigma_1^{'} + (1-\lambda)\sigma_2^{'} \tag{7.1}$$

where $\sigma_i^{'} = U\sigma_i U^\dagger, i = 1,2$. $\sigma_1^{'}, \sigma_2^{'} \in \mathbf{S}$ as $\sigma_1, \sigma_2 \in \mathbf{AS}$. Since **S** is convex, $U\sigma U^\dagger \in \mathbf{S}$, which implies that $\sigma \in \mathbf{AS}$. Hence, **AS** is convex.

AS is compact: Consider an arbitrary limit point θ of **AS** (**AS** will always have a limit point. For example, in the neighbourhood of the identity there are other absolutely separable states). The same θ must also be a limit point of **S** as $\mathbf{AS} \subset \mathbf{S}$. Thus $\theta \in \mathbf{S}$, because **S** is closed. Now, let us inductively construct a sequence $\{\theta_n\}$ of distinct states from **AS** such that $\theta_n \to \theta$ as follows:

$$\begin{aligned}
\theta_1 &\in B_1(\theta) \cap \mathbf{AS},\ \ \theta_1 \neq \theta, \\
\theta_2 &\in B_{\frac{1}{2}}(\theta) \cap \mathbf{AS},\ \ \theta_2 \neq \theta, \theta_1 \\
... &\in .... \\
... &\in .... \\
\theta_n &\in B_{\frac{1}{n}}(\theta) \cap \mathbf{AS},\ \ \theta_n \neq \theta, \theta_1, \theta_2 ... \theta_{n-1}
\end{aligned} \tag{7.2}$$

Here $B_r(\theta)$ denotes an open ball of radius r centered at θ. (This construction is possible because each neighbourhood of θ contains infinitely many points of **AS**, θ being a limit point of **AS**). For the above mentioned choice of θ_n's, evidently $\theta_n \to \theta$. Hence, for any unitary operator U, one has $U\theta_n U^\dagger \to U\theta U^\dagger$. Thus, $U\theta_n U^\dagger \in \mathbf{S}$ for each $n \geq 1$, as $\theta_n \in \mathbf{AS}$. Since **S** is a closed set, it must contain the limit of the sequence $\{U\theta_n U^\dagger\}$, which is $U\theta U^\dagger$. Hence, $U\theta U^\dagger \in \mathbf{S}$, for any arbitrary choice of the unitary operator U. Therefore, $\theta \in \mathbf{AS}$ as we already have $\theta \in \mathbf{S}$. Since θ is an arbitrary limit point of **AS**, it follows that **AS** contains all its limit points, thereby implying that **AS** is closed [Sim63]. As any closed subset of a compact set is compact [Sim63], one concludes that **AS** is compact because **S** is compact. Hence, the theorem. ■

In view of the theorem above, we now formally define a hermitian operator

T which identifies separable but not absolutely separable states through the following two inequalities:

$$Tr(T\sigma) \geq 0, \;\; \forall \sigma \in \mathbf{AS} \tag{7.3}$$

$$\exists \chi \in \mathbf{S} - \mathbf{AS}, \;\; \text{s.t.} \;\; Tr(T\chi) < 0 \tag{7.4}$$

Therefore, T identifies those separable states χ that become entangled under some global unitary operation.

Consider $\chi \in \mathbf{S} - \mathbf{AS}$. There exists a unitary operator U_e such that $U_e \chi U_e^\dagger$ is entangled. Consider an entanglement witness W that detects $U_e \chi U_e^\dagger$, i.e., $Tr(W U_e \chi U_e^\dagger) < 0$. Using the cyclic property of the trace, one obtains $Tr(U_e^\dagger W U_e \chi) < 0$. It follows that

$$T = U_e^\dagger W U_e \tag{7.5}$$

is our desired operator. Next, considering its action on an arbitrary absolutely separable state σ, we see that $Tr(T\sigma) = Tr(U_e^\dagger W U_e \sigma) = Tr(W U_e \sigma U_e^\dagger)$. As σ is absolutely separable, $U_e \sigma U_e^\dagger$ is a separable quantum state, and since W is an entanglement witness, $Tr(W U_e \sigma U_e^\dagger) \geq 0$. This implies that T has a non-negative expectation value on all absolutely separable states σ. The completeness of the separation axiom follows from the completeness of entanglement witness, *viz.*, for any entangled state $U_e \chi U_e^\dagger$, there always exists a witness W [HHH96]. Thus, if χ is a separable but not absolutely separable state, then one can always construct an operator T in the above mentioned procedure which distinguishes χ from absolutely separable states.

7.3 Illustrations

As the first example consider the separable state in $D(H_2 \otimes H_2)$ given by [TBKN11]

$$\chi_{2\otimes 2} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{7.6}$$

which becomes entangled on application of the unitary operator

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{7.7}$$

The entanglement witness

$$W_1 = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.8}$$

with $c = \frac{1}{\sqrt{2}+1}$ detects the entangled state $U_1\chi_{2\otimes 2}U_1^\dagger$. Hence, the operator

$$T_1 = U_1^\dagger W_1 U_1 \tag{7.9}$$

gives $Tr(T_1\chi_{2\otimes 2}) < 0$, detecting $\chi_{2\otimes 2}$ to be a state which is not absolutely separable.

Next, consider the following separable density matrix $\chi_{2\otimes 4} \in D(H_2 \otimes H_4)$:

$$\chi_{2\otimes 4} = \begin{pmatrix} 1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 \end{pmatrix} \tag{7.10}$$

The state $\chi^e_{2\otimes 4} = U_2 \chi_{2\otimes 4} U_2^\dagger$, is entangled due to the unitary operator

$$U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.11}$$

Therefore, the operator $T_2 = U_2^\dagger W_2 U_2$ detects the state $\chi_{2\otimes 4}$ as a separable but not absolutely separable state, where W_2 is the entanglement witness for the entangled state $\chi^e_{2\otimes 4}$, given by $W_2 = Q^{T_B}$,with Q being a projector on $|10\rangle - |01\rangle$.

It is hard to classify states separable from spectrum in dimensions other than $2 \otimes n$, due to the absence of suitable methodology in the existing literature. However, through our approach of witnesses we can identify states which are not absolutely separable in any arbitrary dimension. Consider the isotropic state $\in D(H_3 \otimes H_3)$, given by

$$\chi_{3\otimes 3} = \alpha|\phi_3^+\rangle\langle\phi_3^+| + \frac{1-\alpha}{9} I, \tag{7.12}$$

where $|\phi_3^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. This state is separable for $-\frac{1}{8} \leq$

$\alpha \leq \frac{1}{4}$ [BDHK05]. It is observed that the unitary operator $U_3 = I - (\frac{\sqrt{2}-1}{\sqrt{2}})(|00\rangle\langle 00| + |22\rangle\langle 22|) + \frac{1}{\sqrt{2}}(|00\rangle\langle 22| - |22\rangle\langle 00|)$ converts $\chi_{3\otimes 3}$ to an entangled state $\chi^e_{3\otimes 3}$ in the range $\alpha \in (\frac{1}{1+3\sqrt{2}}, \frac{1}{4}]$. So, the operator $T_3 = U_3^\dagger W_3 U_3$ detects $\chi_{3\otimes 3}$ as a state that is not absolutely separable. Here W_3 is the entanglement witness for $\chi^e_{3\otimes 3}$, given by $W_3 = (|\eta\rangle\langle\eta|)^{T_B}$ with $|\eta\rangle$ being the eigenvector of $(\chi^e_{3\otimes 3})^{T_B}$ corresponding to the eigenvalue $-\frac{1}{9}\alpha + \frac{1}{9} - \frac{\sqrt{2}}{3}\alpha$.

Let us now present a construction of the witness operator for general qudit states. Take the unitary operator

$$U_{d\otimes d} = I - (\frac{\sqrt{2}-1}{\sqrt{2}})A + \frac{1}{\sqrt{2}}B \tag{7.13}$$

where $A = |00\rangle\langle 00| + |d-1, d-1\rangle\langle d-1, d-1|$ and $B = |00\rangle\langle d-1, d-1| - |d-1, d-1\rangle\langle 00|$, and the mixed separable state

$$\chi_{d\otimes d} = \frac{1}{4}|00\rangle\langle 00| + \frac{3}{4}|d-1, d-1\rangle\langle d-1, d-1| \tag{7.14}$$

The state $U_{d\otimes d}\chi_{d\otimes d}U^\dagger_{d\otimes d}$ is entangled as detected by the witness $W_{d\otimes d} = \frac{1}{d}I - |P\rangle\langle P|$, where P is the projector on the maximally entangled state $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$. Therefore, in $d\otimes d$ dimensions the operator $T_{d\otimes d} = U^\dagger_{d\otimes d}W_{d\otimes d}U_{d\otimes d}$ detects $\chi_{d\otimes d}$ as a state which is not absolutely separable.

7.4 Entanglement creation using quantum gates

Let us now consider some examples of unitary quantum gates which can produce entanglement by acting on bipartite separable states. Since the construction presented above is valid for any arbitrary dimension, let us consider a case in $d_1 \otimes d_2$ dimensions where $d_1 \neq d_2$. Consider the two qudit hybrid quantum gate U_H acting on $d_1 \otimes d_2$ dimensions, whose action is defined by

$$U_H|m\rangle \otimes |n\rangle = |m\rangle \otimes |m-n\rangle, \tag{7.15}$$

with $m \in \mathbb{Z}_{d_1}, n \in \mathbb{Z}_{d_2}$ [DWS03]. Let us take the initial mixed separable state

$$\chi_{d_1 \otimes d_2} = \frac{1}{4}\chi_x + \frac{3}{4}\chi_y, \tag{7.16}$$

where χ_x is a projector on $\frac{1}{\sqrt{2}}(|0, d_2 - 1\rangle + |1, d_2 - 1\rangle)$, and χ_y a projector on $|d_1 - 1, d_2 - 1\rangle$. The state $U_H \chi_{d_1 \otimes d_2} U_H^\dagger$ is entangled as identified by the witness $W_{d_1 \otimes d_2} = X^{T_B}$ (X being the projector on $|02\rangle - |11\rangle$). Hence, $T_{d_1 \otimes d_2} = U_H^\dagger W_{d_1 \otimes d_2} U_H$ detects $\chi_{d_1 \otimes d_2}$ as a state which is not absolutely separable. The above example again illlustrates the fact that one can construct a hermitian operator for two qudits (for equal or different dimensions) that can recognize useful separable states from which entanglement can be created between the two qudits using global unitary operations.

We finally consider the example of the much discussed $CNOT$ gate. The $CNOT$ gate can generate entanglement between two qubits, if the state under consideration is not absolutely separable. If we now consider the action of U_{CNOT} on a class of mixed separable states of two qubits of the form

$$\chi_{mix} = a|00\rangle\langle 00| + b|00\rangle\langle 10| + b|10\rangle\langle 00| + (1-a)|10\rangle\langle 10| \tag{7.17}$$

where $a, b \in \mathbb{R}$, we find that the states of the form $\chi_{mix}^e = U_{CNOT} \chi_{mix} U_{CNOT}^\dagger$ can be entangled. Such entanglement can be detected by the witness $W_{CNOT} = [(|10\rangle - |01\rangle)(\langle 10| - \langle 01|)]^{T_B}$. A hermitian operator T_{CNOT} constructed according to our prescription, which detects χ_{mix} as a state not absolutely separable, is given by $T_{CNOT} = U_{CNOT}^\dagger W_{CNOT} U_{CNOT}$. Now, $Tr(T_{CNOT}\chi_{mix}) = -2b$, implying that for $b > 0$ the operator detects the class of states as useful for entanglement creation under the action of the $CNOT$ gate. For example, for $a = 3/4$ and $b = 1/4$, we get a state that is not absolutely separable detected by the witness T_{CNOT}. On the other hand, a state of the form [Joh13]

$$\sigma = \frac{1}{11}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{7.18}$$

leads to $Tr(T_{CNOT}\sigma) > 0$, remaining separable under the action of the $CNOT$ gate, as the state σ (7.18) is absolutely separable. Note though, that neither the entanglement witness W_{CNOT}, and nor consequently T_{CNOT} as constructed here, are universal. As a result, the operator T_{CNOT} fails to detect some states which are not absolutely separable that exist even for $b < 0$ in the class of states (7.17). However,through our generic approach, one can construct another suitable witness operator to identify states not absolutely separable in the latter range.

7.5 Decomposition of the witness operator

For the purpose of experimental determination of the expectation value of a witness operator on a given state, it is helpful to decompose it in terms of spin matrices [GHB+02]. As an example, the witness T_{CNOT} which detects the class of states χ_{mix} (7.17) as not absolutely separable, admits the decomposition $T_{CNOT} = \frac{1}{2}(I \otimes I - I \otimes Z - X \otimes Z - X \otimes I)$ where X, Z are the usual Pauli spin matrices. Further, in order that the witness operator can be measured locally, it may be decomposed in the form $T = \sum_{i=1}^{k} c_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$ [GHB+02]. Experimental realization of entanglement witnesses has been achieved using polarized photon states [BDMDN+03]. In case of the operator T_{CNOT}, the decomposition in terms of photon polarization states is given by

$$T_{CNOT} = |HV\rangle\langle HV| + |VV\rangle\langle VV| - |DH\rangle\langle DH| + |FH\rangle\langle FH| \tag{7.19}$$

where, $|H\rangle = |0\rangle, |V\rangle = |1\rangle, |D\rangle = \frac{|H\rangle+|V\rangle}{\sqrt{2}}, |F\rangle = \frac{|H\rangle-|V\rangle}{\sqrt{2}}$ are the horizontal, vertical and diagonal polarization states respectively [BDMDN+03]. The above decomposition suggests a realizable method to experimentally verify whether it is possible to create an entangled state from a mixed separable state through the action of an entangling gate.

7.6 Summary

In this chapter we have proposed a framework to distinguish separable states that remain separable from those that become entangled due to global unitary operations in any arbitrary dimensional Hilbert space [Kni03, KŻ01, ŻHSL98, GB02, IH00, VADM01, Hil07, Joh13]. To this end we have characterized the set of all absolutely separable bipartite states as convex and compact, enabling one to construct suitable hermitian operators for identification of states that are not globally separable. A generic procedure for construction of such operators in any dimensions is suggested, which underlines the completeness of the separation, *viz.*, if χ is not absolutely separable then there will always be an operator which detects it. The action of the operator is demonstrated on states in various dimensions. Observational feasibility of witnesses for qubit states is highlighted through decomposition in terms of locally measureable photon polarizations.

The generation of entanglement from separable initial states is of prime importance in information processing applications [SKK+00, RNO+00, KC01, LL09, KRS11]. Our method helps to identify eligible input states for entanglement creation using global unitary operations in general, and may be of specific relevance in quantum gate operations [MKŻ13] widely used in quantum computation. Though pure product states can be readily entangled through such operations, the inevitability of environmental influences makes the consideration of mixed states highly relevant, and thereby lends practical significance to our proposal for detection of separable mixed states useful for production of entanglement.

Chapter 8

Conclusions

The aim of this chapter is to capture the salient features presented in this dissertation and also to speculate on some future directions of work. Quantum information and its sister area quantum computation claims to make revolutionary changes in computational and information processing tasks. The strength of this claim relies on the phenomenon called entanglement. Although the phenomenon started as a paradox to scientists questioning the foundational attributes of quantum mechanics , yet towards the end of the last century and in the dawn of this century the practical implementations of entanglement took centre stage. It promises to design state of the art technologies like teleportation, superdense coding, quantum algorithms, cryptography and the list goes on.

However,before one utilizes entangled quantum states one needs to detect its presence in them. which operationally is a hard task. The lack of an universal approach further complicates the situation. The situation demands to have a case by case study of quantum states with different approaches to identify signatures of entanglement. Several methods have been laid down over the time for this purpose. One significant approach in this direction is the study on entanglement witnesses. Although entanglement witness is generically a mathematical manifestation yet its practical implementations

to detect entanglement has been quite popular . The strength of entanglement witnesses further lies on the fact that if a state is entangled then there will be a witness that detects it.
Entanglement witnesses, besides performing their primary job of detecting entanglement have now diversified their scope to include various other facets of entanglement. The notion of witnessing entanglement has been carried forward to witness other aspects of quantum states like discord, mixedness etc. Thus, characterization of quantum states plays an important role here. Exploitation of the set of entangled states for different tasks motivates to characterize the set.
This dissertation on one hand studies hermitian operators to detect entanglement and on the other hand also extends the notion of witness operators to identify useful resources for quantum information processing tasks. The thesis also sheds light on the set of separable states through the characterization of one of its special subsets, namely the absolutely separable states.

8.1 Summary of the thesis

In chapter 2 the basic mathematical and physical prerequisites have been given to maintain the logical pedagogy. It also surveys important work related to this dissertation especially the focus being on entanglement witnesses.
Chapter 3 discusses the construction of non-decomposable operators to detect a special class of PPT entangled states, namely edge states. Comparisons are made with previously known non-decomposable operators. The experimental realization of the proposed witness operator is also discussed. The utility of such a witness is vindicated through illustrations.
Common entanglement witnesses, i.e, a single witness which can detect a large class of entangled states constitutes an effective procedure to maximally detect entangled states. Construction of such witnesses is discussed in chapter 4.If a witness detects two states then it can detect all the states in their convex combination. Different possible convex combinations of entan-

gled states are discussed.Some insights on common schmidt number witnesses have also been given.
The notion of witnessing entanglement is extended to identify useful resources for teleportation in chapter 5 through hermitian operators, accordingly called teleportation witnesses. The measurability of such operators is discussed in comparison with conventional procedures.
Entanglement witnesses have been optimized in literature to bring more entangled states into its detection domain. Likewise in chapter 6 we discuss the construction of optimal teleportation witnesses. Optimality is reported for $2 \otimes 2$, $3 \otimes 3$ and generalized to $d \otimes d$ systems.
Although local operations cannot generate entanglement between separable quantum states, global unitary operations can be used for this purpose. However, absolutely separable states form a special subclass of states within separable states as they remain separable under any global unitary operation. Chapter 7 introduces the notion of hermitian operators which can recognize useful separable states from which entanglement can be created. The existence and completeness of such operators are proved. Their practical relevance and measurability is discussed through significant illustrations from arbitrary dimensions.

8.2 Future directions

This dissertation has the possibility of leading to several interesting directions of future research. The thesis focusses on the utility of hermitian operators in the detection and characterization of quantum entanglement. Separation axioms, namely the seminal Hahn-Banach theorem from functional analysis had already been utilized in the inception of entanglement witnesses to distinguish entangled states from the separable ones. Application of separation axiom was made possible by the convexity and compactness of the set of separable states.
We have probed here the detection of edge states which lie at the boundary of PPT and NPT entangled states, through hermitian operators. One may optimize such operators and bring more edge states in the detection

fold. The study may also be extended to detect edge states in multipartite systems involving many parties. Experimental realization of such operators would also be of much relevance.

One may devise other methods to construct common entanglement witnesses namely from geometric and algebraic considerations. The study may be extended to include multipartite PPT and NPT entangled states.

We have introduced the notion of teleportation witnesses here which identifies useful entangled states for quantum teleportation. We have devised several such teleportation witness and analyzed their optimality. One may thus, exercise geometrical considerations to construct teleportation witnesses and study their experimental relevance. Entanglement witnesses have been generically used to quantify entanglement . In a similar line of thought, quantification of the teleportation capability of entangled states can be probed upon through teleportation witness operators. Such quantifiers can thus be compared with the respective teleportation fidelities of the entangled states concerned.

Absolutely separable states which form a subset of the separable states, have been characterized here as convex and compact. This entailed the inception of hermitian operators to detect states which although separable, are not absolutely separable. Such states are significant as entanglement can be created from them with non-local unitary operations. An extension of the study to the optimization of such operators would be significant. One may also construct common hermitian operators which can identify separate classes of non-absolutely separable states. The study may be extended to include the detection of non-absolutely separable states in multipartite systems.

"We know that we do not know all the laws yet ....therefore things are to be learned only to be unlearned again"

Richard Feynman

List of Publications on which the thesis is based

1. *Witness for edge states and its characteristics*
   Nirman Ganguly and Satyabrata Adhikari
   Physical Review A 80, 032331 (2009).

2. *Entanglement Witness Operator for Quantum Teleportation*
   Nirman Ganguly, Satyabrata Adhikari, A. S. Majumdar and Jyotishman Chatterjee
   Physical Review Letters 107, 270501(2011).

3. *Construction of optimal teleportation witness operators from entanglement witnesses*
   Satyabrata Adhikari, Nirman Ganguly and A. S. Majumdar
   Physical Review A 86, 032315 (2012).

4. *Common entanglement witnesses and their characteristics*
   Nirman Ganguly , Satyabrata Adhikari and A. S. Majumdar
   Quantum Information Processing 12, 425 (2013).

5. *Witness of mixed separable states useful for entanglement creation*
   Nirman Ganguly, Jyotishman Chatterjee and A.S. Majumdar
   Physical Review A 89, 052304 (2014).

Total list of Publications

1. *Quantum Cloning , Bell's inequality and teleportation*
   Satyabrata Adhikari, Nirman Ganguly, Indranil Chakrabarty and B.S Choudhury
   Journal of Physics A: Math.Theor. 41, 415302 (2008).

2. *Entanglement and Mixedness of Locally cloned Non-Maximal W-State*
   Indranil Chakrabarty, Sovik Roy, Nirman Ganguly and B.S Choudhury
   International Journal of Theoretical Physics 48, 1833 (2009).

3. *Witness for edge states and its characteristics*
   Nirman Ganguly and Satyabrata Adhikari
   Physical Review A 80, 032331 (2009).

4. *Deletion, Bell's inequality, teleportation*
   Indranil Chakrabarty , Nirman Ganguly and B.S Choudhury
   Quantum Information Processing 10, 27 (2011).

5. *Entanglement Witness Operator for Quantum Teleportation*
   Nirman Ganguly, Satyabrata Adhikari, A. S. Majumdar and Jyotishman Chatterjee
   Physical Review Letters 107, 270501(2011).

6. *Construction of optimal teleportation witness operators from entanglement witnesses*
   Satyabrata Adhikari, Nirman Ganguly and A. S. Majumdar
   Physical Review A 86, 032315 (2012).

7. *Common entanglement witnesses and their characteristics*
Nirman Ganguly , Satyabrata Adhikari and A. S. Majumdar
Quantum Information Processing 12, 425 (2013).

8. *A cloned qutrit and its utility in information processing tasks*
Sovik Roy, Nirman Ganguly, Atul Kumar, Satyabrata Adhikari and A.S. Majumdar
Quantum Information Processing 13, 629 (2014).

9. *Witness of mixed separable states useful for entanglement creation*
Nirman Ganguly, Jyotishman Chatterjee and A.S. Majumdar
Physical Review A 89, 052304 (2014).

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