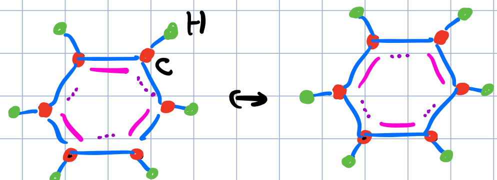
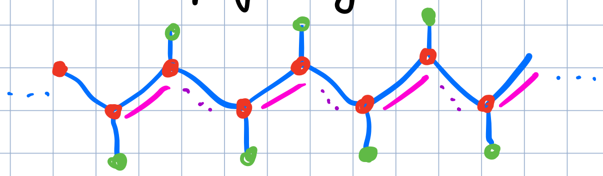
SSH model is a model proposed to explain the physics of conducting polymers like polyacetylene. (Su, Schrieffer, Heeger, PRL 1979, PRB 1980)

Recall the structure of benzene



- sp² hybridized bonds
- p^z extra electron forms "resonating valence bonds" (Pauling)

The model for polyacetylene is an extended version of benzene



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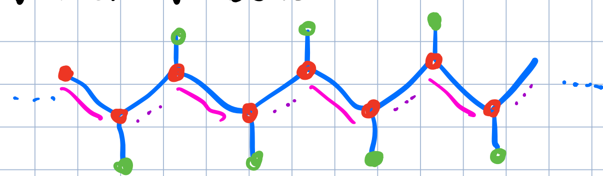
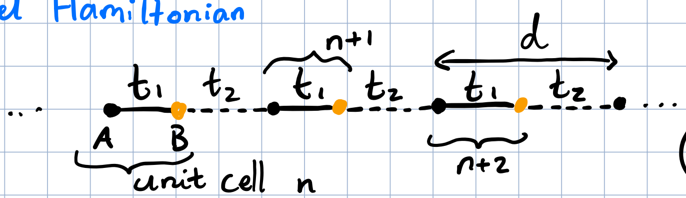


★ 1D model Hamiltonian



Alternating hopping strengths arises from spontaneous dimerization

(Let us set lattice constant $d = 1$)

$$\hat{H} = -t_1 \sum_n \left(C^+_{nA} C_{nB} + C^+_{nB} C_{nA}\right) - t_2 \sum_n \left(C^+_{n+1A} C_{nB} + C^+_{nB} C_{n+1A}\right)$$

Fourier transform: $C^+_{nA} = \frac{1}{\sqrt{N}} \sum_k C^+_{kA} e^{-ikn}$

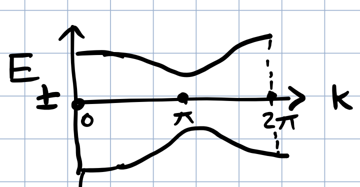
$$C^+_{nB} = \frac{1}{\sqrt{N}} \sum_k C^+_{kB} e^{-ikn}$$

$$\therefore \hat{H} = -t_1 \sum_k \left(C^+_{kA} C_{kB} + C^+_{kB} C_{kA}\right) - t_2 \sum_k \left(C^+_{kA} C_{kB} e^{-ik} + C^+_{kB} C_{kA} e^{ik}\right)$$

$$\hat{H} = \sum_k \begin{pmatrix} C_{kA} \\ C_{kB} \end{pmatrix}^+ \mathcal{H}_{Bloch}(k) \begin{pmatrix} C_{kA} \\ C_{kB} \end{pmatrix} \quad \text{where } \mathcal{H}_{Bloch}(k) \equiv \begin{pmatrix} 0 & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & 0 \end{pmatrix}$$

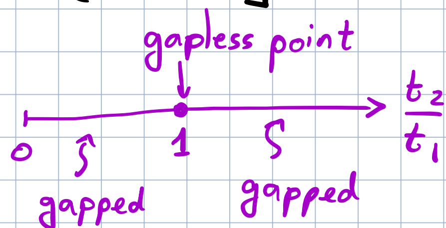
{More generally, $\mathcal{H}_{Bloch} \equiv n \times n$ matrix for n-"orbitals"}

Eigenvalues of $\mathcal{H}_{Bloch}(k)$: $E_\pm(k) = \pm\left[(t_1 + t_2 \cos k)^2 + (t_2 \sin k)^2\right]^{1/2}$



Let us set $t_1, t_2 \geq 0$

Minimum gap at $k = \pi$



Q: ▸ Are both gapped phases the same or are they distinct?

- If they are 'same', Why are they separated by a transition?
- If they are distinct phases, what distinguishes them?

▸ **Bulk distinction: geometric view**

Write $\mathcal{H}_{Bloch}(k) \equiv -\underbrace{h_\mu(k)}_{\text{real coeff}} \cdot \underbrace{\sigma_\mu}_{2\times 2 \text{ Hermitian basis}}$

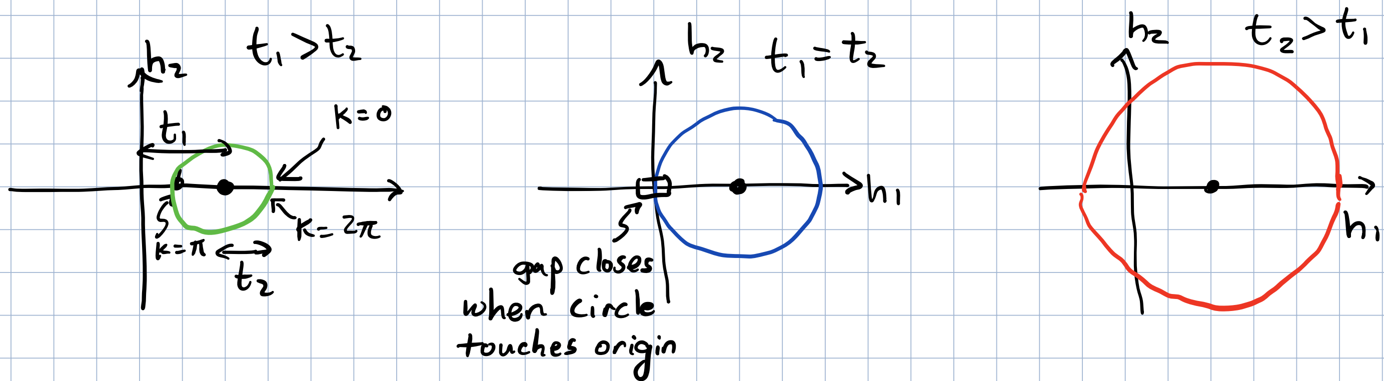
$\sigma_\mu : \sigma_0, \sigma_1, \sigma_2, \sigma_3$

$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Eigenvalues: $-h_0 \pm |\vec{h}|$ $\therefore$ gap $= 2|\vec{h}(k)|$

$$\left.\begin{aligned} h_0 &= h_3 = 0 \\ h_1 &= t_1 + t_2 \cos k \\ h_2 &= t_2 \sin k \end{aligned}\right\} \quad (h_1 - t_1)^2 + h_2^2 = t_2^2$$

{eqn for a circle in h_1, h_2 plane}



* Distinction between two insulators is in whether circle encloses the origin
* Looks like a "yes" or "no" question $\Rightarrow$ $\mathbb{Z}_2$ invariant? (revisit below)
* Looks like small deformations of "shape" (circle) shouldn't affect $\Rightarrow$ topological protection

* "Flattening the spectrum"

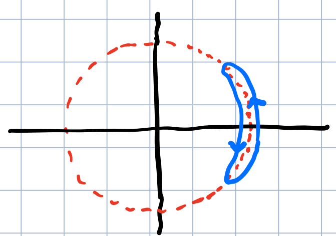
For cases where there is a gap (ie when $t_1 \neq t_2$), we can define a modified Hamiltonian by smoothly "deforming" the SSH model Bloch Hamiltonian.

Let us set : $\hat{n}(k) = \dfrac{\vec{h}(k)}{|\vec{h}(k)|}$

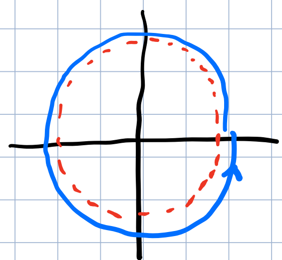
Consider $\mathcal{H}_{\text{Bloch}}(k,\alpha) = (1-\alpha)\,\vec{h}(k)\cdot\vec{\sigma} + \alpha\,\hat{n}(k)\cdot\vec{\sigma}$

$$\left.\begin{aligned} \text{When } \alpha = 0,\ & \mathcal{H}_{\text{Bloch}}(k,0) = \vec{h}(k)\cdot\vec{\sigma} \\ \alpha = 1,\ & \mathcal{H}_{\text{Bloch}}(k,1) = \hat{n}(k)\cdot\vec{\sigma} \end{aligned}\right\}$$

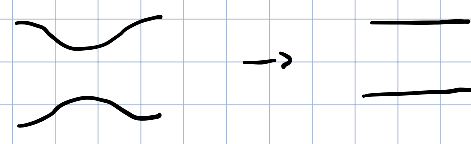
As we tune $\alpha : 0 \to 1$, spectral gap does not close



$t_1 > t_2$



$t_2 > t_1$



$\alpha = 0$ $\qquad\qquad$ $\alpha = 1$

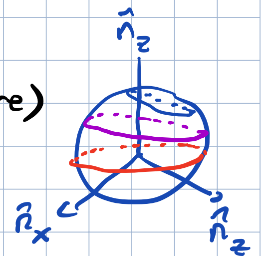
* For spectral flattened case, we can view the Bloch Hamiltonian as a map from $\underset{\text{Brillouin Zone}(k)}{\{\text{circle}\}} \longrightarrow \underset{\hat{n}(k)}{\{\text{circle}\}}$

* Maps from $S^1 \to S^1$ have topological invariant : Winding number $\in \mathbb{Z}$
  - ↳ "Homotopy theory" (David Mermin, RMP 1979 for gentle introduction)
  - ↳ Look at loops and whether they can be "contracted" into one another $\pi_1(S^1) = \mathbb{Z}$

closed loops in 1D BZ $\qquad$ $\hat{n}$ lives on circle

Q: We understand why 1D BZ is always a circle, but why does $\hat{n}$ have to be on a circle? What can we add? Why not h_0 or h_3?

- In one sense, h_0 is "harmless" ∵ only shifts both bands up or down but can impact insulator vs metal.
- h_3 has bigger impact ⇒ now $\hat{n}$ lives on S^2 (surface of sphere) and we can always contract loops without touching origin. Formally, $\pi_1(S^2) = 0$ ⇒ only trivial phase.



Q: Why $\mathbb{Z}$? Earlier we said $\mathbb{Z}_2$: "Yes" or "No". What did we miss in SSH model? Why did we only find two phases?

* Answer to both questions arises from asking what "deformations" are allowed in a general setting.

- Impose symmetries → rule out h_0, h_3
- Consider most general $\mathcal{H}$ consistent with symmetries → generalized SSH models

* What symmetries can we impose? Motivation from Random matrix theory & generalization due to AZ, (1997)

↳ Time reversal $(\mathcal{T})$: $\mathcal{U}\cdot K$
↳ Charge conjugation (C): particle-hole & Unitary
↳ Chiral symmetry $(S = C\cdot\mathcal{T})$

} symmetries ↳ 10-fold way to classify different phases

* How do different symmetries act on operators & numbers

- $\mathcal{T}$: $\quad c_n \longrightarrow \mathcal{U}_T c_n \qquad$ "n" includes spin index ⇒ $\mathcal{U}_T$ includes spin reversal

$\quad i \longrightarrow -i \qquad$ • Antiunitary

Let us construct Fourier transform (set $n = \underset{\text{site}}{l}, \underset{\text{internal}}{\alpha}$)

$$c_k = \frac{1}{\sqrt{N}} \sum_l e^{-ikl} c_n$$

$$\therefore \; c_k \longmapsto \frac{1}{\sqrt{N}} \sum_l e^{ikl}\, \mathcal{U}_T c_n = \mathcal{U}_T c_{-k}$$

$$H = \sum_k c_k^\dagger \mathcal{H}(k)\, c_k \xmapsto{T} \sum_k c_{-k}^\dagger \mathcal{U}_T^\dagger \mathcal{H}^*(k)\, \mathcal{U}_T c_{-k}$$

$$= \sum_k c_k^\dagger \mathcal{U}_T^\dagger \mathcal{H}^*(-k)\, \mathcal{U}_T c_k \qquad \therefore \mathcal{H}(k) \mapsto \mathcal{U}_T^\dagger \mathcal{H}^*(-k)\, \mathcal{U}_T$$

Is SSH Hamiltonian T-invariant?

$$\mathcal{H}(k) = -\vec{h}(k)\cdot\vec{\sigma} - h_0(k)\,\sigma_0$$

Spinless $\Rightarrow$ we can ignore $\mathcal{U}_T$

$$\therefore \mathcal{H}(k) \mapsto -h_1(-k)\,\sigma_1 - h_2(-k)\,\sigma_2^* - h_3(-k)\,\sigma_3 - h_0(-k)\,\sigma_0$$

$$= -h_1(-k)\,\sigma_1 + h_2(-k)\,\sigma_2 - h_3(-k)\,\sigma_3 - h_0(-k)\,\sigma_0$$

$\therefore$ T-invariance $\Rightarrow$ h_0, h_1, h_3 : even func^n of k & h_2 : odd func^n of k

▶ C: $\left.\begin{array}{l} i \mapsto i \\ c_n \mapsto \mathcal{U}_C^* c_n^\dagger \end{array}\right\}$ $\quad c_k \mapsto \sum_\ell e^{-ik\ell}\, \mathcal{U}_T^* c_n^\dagger$

$$\therefore c_k \mapsto \mathcal{U}_C^* c_{-k}^\dagger \qquad H \mapsto c_{-k} \mathcal{U}_C^T \mathcal{H}(k)\, \mathcal{U}_C^* c_{-k}^\dagger$$

$$\therefore H \mapsto \sum_k c_{-k\alpha} (\mathcal{U}_C^T)_{\alpha\beta}\, \mathcal{H}_{\beta\mu}(k)\, (\mathcal{U}_C^*)_{\mu\nu}\, c_{-k\nu}^\dagger$$

$$= \sum_k \left(\delta_{\alpha\nu} - c_{-k\nu}^\dagger c_{-k\alpha}\right)(\mathcal{U}_C^T)_{\alpha\beta}\, \mathcal{H}_{\beta\mu}(k)(\mathcal{U}_C^*)_{\mu\nu}$$

$$= \sum_k \mathrm{Tr}\,\mathcal{H}(k) - \sum_k c_{k\alpha}^\dagger c_{k\nu}\, \mathcal{U}_{\nu\beta}^T\, \mathcal{H}_{\beta\mu}(-k)\, \mathcal{U}_{\mu\alpha}^*$$

$$= \sum_k \mathrm{Tr}\,\mathcal{H}(k) - \sum_k c_{k\alpha}^\dagger c_{k\nu}\, (\mathcal{U}^\dagger)_{\alpha\mu}\, \mathcal{H}_{\mu\beta}^*(-k)\, (\mathcal{U})_{\beta\nu}$$

$\therefore$ C-Symmetry $\Rightarrow$

- $\mathrm{Tr}\,\mathcal{H}(k) = 0$
- $\mathcal{H}(k) \mapsto -\mathcal{U}_c^\dagger \mathcal{H}^*(-k)\, \mathcal{U}_c$

▶ S: $\quad i \to -i,\quad c_n \mapsto \mathcal{U}_S c_n^\dagger, \quad c_k \mapsto \mathcal{U}_S c_k^\dagger$ (check)

$$H \mapsto \sum_k c_{k\alpha} (\mathcal{U}_S^\dagger)_{\alpha\beta}\, \mathcal{H}_{\beta\mu}^*(k)\, (\mathcal{U}_S)_{\mu\nu}\, c_{k\nu}^\dagger = \sum_k \left(\delta_{\alpha\nu} - c_{k\nu}^\dagger c_{k\alpha}\right) \mathcal{U}_{\alpha\beta}^\dagger \mathcal{H}_{\beta\mu}^* \mathcal{U}_{\mu\nu}$$

$$= \sum_k \mathrm{Tr}\,\mathcal{H}(k) - \sum_k c_{k\alpha}^\dagger c_{k\nu}\, (\mathcal{U}^*)_{\alpha\mu}^\dagger\, \mathcal{H}_{\mu\beta}\, \mathcal{U}_{\beta\nu}^* \qquad \therefore \mathcal{H}(k) \mapsto -(\mathcal{U}_S^*)^\dagger \mathcal{H}(k)\, \mathcal{U}_S^*$$

- For SSH model, let us demand "chiral symmetry" or "sublattice" symmetry

In our case, let us choose:

$$c_{nA} \to c_{nA}^\dagger,\quad c_{nB} \to -c_{nB}^\dagger,\quad i \to -i$$

$$\therefore c_{kA} \to c_{kA}^\dagger,\quad c_{kB} \to -c_{kB}^\dagger,\quad i \to -i \Rightarrow \left\{\begin{array}{l} \mathcal{U} = \sigma_3 \\ -\sigma_3 \mathcal{H} \sigma_3 = \mathcal{H} \\ \& \ \mathrm{Tr}\,\mathcal{H} = 0 \end{array}\right\} \Rightarrow h_0 = h_3 = 0$$

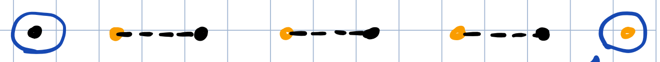
▶ Note: if we demand C-symmetry, can add $[\mathrm{odd}(k)\cdot\sigma_3] \Rightarrow \mathbb{Z}_2$ classification

▸ Edge distinction

* Consider open chains with finite # unit cells

{strong t_1 phase}

{strong t_2 phase}

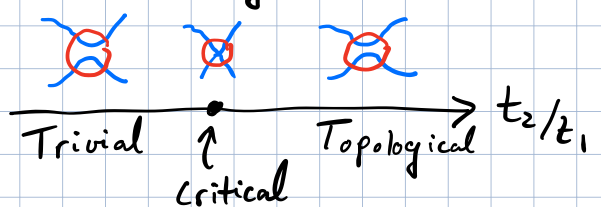


Midgap edge modes

* Field theoretic view: Jackiw, Rebbi

With S symmetry $\to \epsilon = 0$ & $L \to \infty$. Can't add $c^\dagger_{LA} c_{LA}$

Useful as toy model to think about QFTs.



Near gapless critical point, we can zoom in near $\epsilon = 0$ & $k = \pi$ & write a continuum theory

Let us call $t_1 \equiv t$, and fix $t_2/t_1 = 1 + m$ where $m = 0$ is critical point. $m < 0 \Rightarrow$ trivial & $m > 0 \Rightarrow$ topological.

Setting $k = \pi + k_x$, we get $e^{ik} = e^{i(\pi + k_x)} = -e^{ik_x} \simeq -(1 + ik_x)$

$$\therefore \mathcal{H}_{Bloch} = \begin{pmatrix} 0 & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & -t + t(1+m)(1 - ik_x) \\ -t + t(1+m)(1+ik_x) & \end{pmatrix}$$

$$\mathcal{H}_{Bloch} \cong t(k_x \sigma_2 + m\sigma_1)$$

$\hookrightarrow$ dispersion $= \pm t\sqrt{m^2 + k_x^2}$

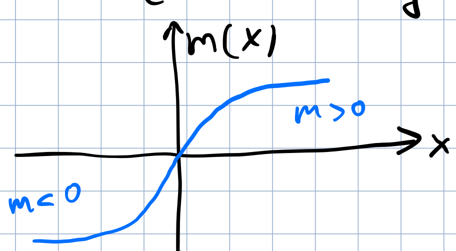
$m \equiv$ mass

"Emergent relativity"

We can go to real space & write this as

$$\mathcal{H}_{Bloch} = t\left(-i\sigma_2 \partial_x + m\sigma_1\right)$$

To make a domain wall between trivial & topological, we can let $m \equiv m(x)$. Try to solve the Dirac equation for zero energy state (motivated by pictorial viewpoint)



2 component spinor

$$-i\sigma_2 \partial_x \Psi + m(x)\sigma_1 \Psi = 0$$

$$\therefore \quad -i\partial_x \Psi - i m(x) \sigma_3 \Psi = 0$$

$$\therefore \quad \partial_x \Psi = -m(x)\sigma_3 \Psi$$

Let us pick basis states : $\Phi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ & $\Phi_- = \begin{pmatrix}0\\1\end{pmatrix}$ so $\sigma_3 \Phi_\pm = \pm \Phi_\pm$

Expand $\Psi = a_+(x)\,\Phi_+ + a_-(x)\,\Phi_-$

$$\therefore (\partial_x a_+)\,\Phi_+ + (\partial_x a_-)\,\Phi_- = -m(x)\,a_+(x)\,\Phi_+ + m(x)\,a_-(x)\,\Phi_-$$

$$\therefore \begin{cases} \partial_x a_+ = -m(x)\,a_+ \Rightarrow a_+ = A_+\, e^{-\int_0^x m(x')dx'} \\ \partial_x a_- = m(x)\,a_- \Rightarrow a_- = A_-\, e^{+\int_0^x m(x')dx'} \end{cases}$$

- Let $x \to +\infty$ & profile  $\Rightarrow$ a_- diverges

$\therefore A_- = 0$

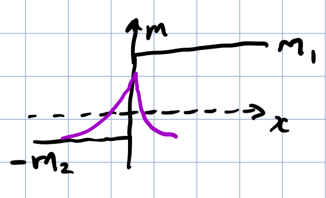
$$\therefore \Psi = A_+\, e^{-\int_0^x m(x')dx'} \begin{pmatrix}1\\0\end{pmatrix}$$

- Let $x \to +\infty$ & profile  $\Rightarrow$ a_+ diverges

$\therefore A_+ = 0$

$$\therefore \Psi = A_-\, e^{\int_0^x m(x')dx'} \begin{pmatrix}0\\1\end{pmatrix}$$

- We could pick specific profile, say



$$\Psi = \begin{pmatrix}1\\0\end{pmatrix} * \begin{cases} A_+\, e^{-m_1 x} & : x > 0 \\ A_+\, e^{m_2 x} & : x < 0 \end{cases}$$

$\star$ localized zero energy edge mode.

- General lessons: ▶ Topological phase transition ~ Dirac mass changing sign as we tune parameters

  ▶ Bulk topological phase ~ boundary localized "zero" modes.

* Exercises :

$\boxed{1}$ Consider a 1D chain with spin-polarized electrons at half-filling $\langle n_i \rangle = 1/2$ coupled to phonons.

$$H = -t\sum_n \left(c_n^+ c_{n+1} + c_{n+1}^+ c_n\right) + g\sum_n \phi_n \left(c_n^+ c_{n+1} + c_{n+1}^+ c_n\right) + u\sum_n \phi_n^2 + w\sum_n \phi_n^4$$

Assuming a variational ansatz $\phi_n = \phi(-1)^n$, calculate the variational energy and minimize it with respect to ϕ. What is the charge gap in the resulting insulator? Discuss what could happen if we dope this system with dilute electrons or holes.

$\boxed{2}$ Consider a generalized SSH model with the Hamiltonian

$$H = -t_1\sum_n \left(a_n^+ b_n + b_n^+ a_n\right) - t_2\sum_n \left(b_n^+ a_{n+1} + h.c.\right)$$
$$- t_3 \sum_n \left(b_n^+ a_{n+2} + a_{n+2}^+ b_n\right)$$

- Go to momentum space and obtain $\vec{h}(k)$. Using a geometric picture or winding #, explore the phase diagram with $t_1 = 1$ & tuning t_2, t_3.
- Diagonalize Hamiltonian on finite chain and check edge modes in different phases. Give a simple pictorial view of edge modes.
- Can we add terms at the boundary to gap out edge modes? Imagine we are in class AIII which has chiral (S) symmetry only.

$\boxed{3}$ Consider SSH model on semi-infinite chain with $t_2 > t_1$

$$H = -t_1\sum_{n=0}^{\infty} \left(a_n^+ b_n + b_n^+ a_n\right) - t_2\sum_{n=0}^{\infty} \left(b_n^+ a_{n+1} + h.c.\right)$$

Construct lattice wavefunction of localized edge mode near left edge. How does wavefunction change as $t_2 \to t_1$?